Domination in Regular Graphs

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ABSTRACT. A two-valued function f defined on the vertices of a graph $G = (V, E)$, $f : V \to \{-1, 1\}$, is a signed dominating function if the sum of its function values over any closed neighborhoods is at least one. That is, for every $v \in V$, $f(N[v]) \geq 1$, where $N[v]$ consists of v and every vertex adjacent to v. The function f is a majority dominating function if for at least half the vertices $v \in V, f(N[v]) \geq 1$. The weight of a signed (majority) dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The signed (majority) domination number of a graph G, denoted $\gamma_s(G)$ ($\gamma_{maj}(G)$, respectively), equals the minimum weight of a signed (majority, respectively) dominating function of G. In this paper, we establish an upper bound on $\gamma_s(G)$ and a lower bound on $\gamma_{maj}(G)$ for regular graphs G.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let v be a vertex in V. If $v \in V$, the degree of v in G is written as *deg* v. The graph G is r-regular if $deg\, v = r$ for all $v \in V$. In particular, if $r = 3$, then we call G a cubic graph. For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest u-v path. If S is a set of vertices of G and v is a vertex of G, then the *distance from v to S*, denoted by $d_G(v, S)$, is the shortest distance from v to a vertex of S. For graph theory terminology not presented here we follow [2].

The *open neighborhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \bigcup \{v\}$. For a set S of vertices, we define the open neighborhood $N(S) = \bigcup N(v)$ over all v in S, and the closed neighborhood $N[S] =$

$N(S) \bigcup S$. A set S of vertices is a dominating set if $N[S] = V$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G.

For any real valued function $g : V \to R$ and $S \subseteq V$, let $g(S) = \sum g(u)$ over all $u \in S$. Let $g : V \to \{0,1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0,1\}$. We say g is a *dominating function* if for every $v \in V$, $g(N[v]) \geq 1$. We say g is a minimal dominating function if there does not exist a dominating function $h : V \to \{0,1\}$, $h \neq g$, for which $h(v) \leq g(v)$ for every $v \in V$. This is equivalent to saying that a dominating function g is minimal if for every vertex v such that $g(v) > 0$, there exists a vertex $u \in N[v]$ for which $g(N[u]) = 1$. The domination number of a graph G can be defined as $\gamma(G) = min\{g(V) \mid g \text{ is a dominating function on } G\}$.

A *signed dominating function* is defined in [4] as a function $g : V \to \{-1,1\}$ such that for every $v \in V$, $g(N[v]) \geq 1$. The *signed domination number* for a graph G is $\gamma_s(G) = \min\{g(V) \mid g \text{ is a signed dominating function on } G\}$.

A *majority dominating function* has been defined by Hedetniemi [5] as a function $g : V \to \{-1,1\}$ such that for at least half the vertices $v \in V$, $g(N[v]) \geq 1$. The *majority domination number* for a graph G is $\gamma_{maj}(G) = min\{g(V) \mid g \text{ is a majority dominating function on } G\}$. The majority dominating function was studied in [1].

2 An upper bound on $\gamma_s(G)$ for regular graphs G.

We begin by stating a useful result from [4].

Proposition A. *A signed dominating function g on a graph $G = (V,E)$ is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exists a vertex $u \in N[v]$ with $g(N[u]) \in \{1,2\}$.*

In [4] and [6] the following lower bounds on $\gamma_s(G)$ for r-regular graphs G of order n for r even and odd, respectively, are established.

Theorem A. *For every r-regular ($r \geq 2$) graph G of order n,*

$$\gamma_s(G) \geq \begin{cases} \frac{2n}{r+1} & \text{for } r \text{ odd} \\ \frac{n}{r+1} & \text{for } r \text{ even.} \end{cases}$$

Zelinka [7] established the following upper bound on $\gamma_s(G)$ for a cubic graph G.

Theorem B. *For every cubic graph G of order n, $\gamma_s(G) \leq \frac{4}{5}n$.*

In this section we generalize the result of Theorem B to r-regular graphs.

Theorem 1. *For every r-regular ($r \geq 2$) graph $G = (V, E)$ of order n,*

$$\gamma_s(G) \leq \begin{cases} \left(\frac{(r+1)^2}{r^2+4r-1}\right) n & \text{for } r \text{ odd} \\ \left(\frac{r+1}{r+3}\right) n & \text{for } r \text{ even.} \end{cases}$$

Proof: Let $f : V \to \{-1, 1\}$ be any signed dominating function on G for which $f(V) = \gamma_s(G)$. Let P and M (standing for "positive" and "minus") be the sets of vertices in G that are assigned the values +1 and −1, respectively, under f. Then $|P| + |M| = n$, and $\gamma_s(G) = |P| - |M| = n - 2|M|$. Each vertex of M must be adjacent to at least $\lceil \frac{r}{2} \rceil + 1$ vertices of P, and each vertex of P is adjacent to at most $\lfloor \frac{r}{2} \rfloor$ vertices of M. Before proceeding further, we introduce the following notation. Let M_1 be the set of all vertices of M that are adjacent to exactly $\lceil \frac{r}{2} \rceil + 1$ vertices of P, and let $M_2 = M - M_1$. Hence $M = M_1 \cup M_2$. Note that if $r = 2$ or 3, then $M_2 = \emptyset$. For $i = 1, 2$, let $P_i = \{v \in P \mid d(v, M) = i\}$. Let A be the set of all vertices of P_1 that are adjacent to exactly $\lfloor \frac{r}{2} \rfloor$ vertices of M. Let B be the set of all vertices of $P_1 - A$ that are adjacent to at least one vertex of M_1, and let $C = P_1 - (A \cup B)$. Hence $P_1 = A \cup B \cup C$. Note that if $r = 2$ or 3, then $B = \emptyset$ and $C = \emptyset$. Further, let $D = P - P_1$.

Claim 1: $d(v, M) \leq 2$ for all $v \in V$.

Proof: If $d(v, M) \geq 3$ for some $v \in V$, then $v \in P$ and the function $g : V \to \{-1, 1\}$ defined by $g(v) = -1$ and $g(w) = f(w)$ if $w \in V - \{v\}$ is a signed dominating function on G with $g(V) = f(V) - 2$, which contradicts the minimality of f. □

By Claim 1, $D = P_2$ and the sets M_1, M_2, A, B, C and D are pairwise disjoint and their union is V.

Claim 2: Each $v \in C \cup D$ is adjacent to at least one vertex of A.

Proof: By Proposition A, there must exist a vertex $u \in N[v]$ with $f(N[u]) \in \{1, 2\}$. Each vertex $w \in D$ satisfies $f(N[w]) = r + 1 \geq 3$. Each vertex of $P_1 - A$ is adjacent to at most $\lfloor \frac{r}{2} \rfloor - 1$ vertices of M and therefore to at least $\lceil \frac{r}{2} \rceil + 1$ other vertices of P. Hence $f(N[w]) \geq (\lceil \frac{r}{2} \rceil + 2) - (\lfloor \frac{r}{2} \rfloor - 1) \geq 3$ for each $w \in P_1 - A$. However, each vertex $w \in A$ satisfies $f(N[w]) = (\lceil \frac{r}{2} \rceil + 1) - \lfloor \frac{r}{2} \rfloor \in \{1, 2\}$. Hence, since each $v \in D$ is adjacent only to vertices in P, it follows from Proposition A that each $v \in D$ is adjacent to at least one vertex of A. Furthermore, each vertex of M_2 is adjacent to at least $\lceil \frac{r}{2} \rceil + 2$ vertices of P, so $f(N[w]) \geq (\lceil \frac{r}{2} \rceil + 2) - (\lfloor \frac{r}{2} \rfloor - 2) \geq 4$ for each $w \in M_2$. Hence, since each vertex $v \in C$ is adjacent only to vertices in $P \cup M_2$, it follows from Proposition A that each $v \in C$ is adjacent to at least one vertex of A. □

Let $|M_1| = m_1$, $|M_2| = m_2$, $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. Further let ℓ be the number of edges joining a vertex of M_1 and a vertex of B. Since G is r-regular, each vertex of M_2 is adjacent to at most r vertices of P. By definition of M_1, each vertex of M_1 is adjacent to $\lceil \frac{r}{2} \rceil + 1$ vertices of P. Hence there are at most $(\lceil \frac{r}{2} \rceil + 1)m_1 - \ell + rm_2$ edges joining a vertex of M and a vertex of A. On the other hand, since each vertex of A is adjacent to $\lfloor \frac{r}{2} \rfloor$ vertices of M, there are $\lfloor \frac{r}{2} \rfloor a$ edges joining a vertex of M and a vertex of A. Consequently,

$$a \leq \frac{(\lceil \frac{r}{2} \rceil + 1)m_1 - \ell + rm_2}{\lfloor \frac{r}{2} \rfloor} \tag{1}$$

Since each vertex of B is adjacent to at least one vertex of M_1, we have

$$b \leq \ell. \tag{2}$$

Furthermore, since each vertex of A is adjacent to $\lceil \frac{r}{2} \rceil$ other vertices of P, it follows from Claim 2 that

$$c + d \leq \left\lceil \frac{r}{2} \right\rceil a. \tag{3}$$

Hence, by (1), (2) and (3), it follows that

$$\begin{aligned}
n &= m_1 + m_2 + a + b + c + d \\
&\leq m_1 + m_2 + \left(\lceil \tfrac{r}{2} \rceil + 1\right) a + \ell \\
&= m_1 + m_2 + \tfrac{\lceil \frac{r}{2} \rceil + 1}{\lfloor \frac{r}{2} \rfloor} \left((\lceil \tfrac{r}{2} \rceil + 1)m_1 - \ell + rm_2\right) + \ell \\
&= m_1 + m_2 + \tfrac{\lceil \frac{r}{2} \rceil + 1}{\lfloor \frac{r}{2} \rfloor} \left((\lceil \tfrac{r}{2} \rceil + 1)m_1 + rm_2\right) + \left(1 - \tfrac{\lceil \frac{r}{2} \rceil + 1}{\lfloor \frac{r}{2} \rfloor}\right) \ell.
\end{aligned}$$

However, since $\ell \geq 0$ and $\left(1 - \frac{\lceil \frac{r}{2} \rceil + 1}{\lfloor \frac{r}{2} \rfloor}\right) < 0$, it follows that

$$\left(1 - \frac{\lceil \frac{r}{2} \rceil + 1}{\lfloor \frac{r}{2} \rfloor}\right) \ell \leq 0.$$

Thus

$$
\begin{aligned}
n &\le m_1 + m_2 + \tfrac{\lceil \frac{r}{2}\rceil + 1}{\lfloor \frac{r}{2}\rfloor}\left((\lceil \tfrac{r}{2}\rceil + 1)m_1 + rm_2\right) \\
&= \left(\tfrac{(\lceil \frac{r}{2}\rceil+1)^2}{\lfloor \frac{r}{2}\rfloor} + 1\right) m_1 + \left(\tfrac{r(\lceil \frac{r}{2}\rceil+1)}{\lfloor \frac{r}{2}\rfloor} + 1\right) m_2 \\
&\le \left(\tfrac{r(\lceil \frac{r}{2}\rceil+1)}{\lfloor \frac{r}{2}\rfloor} + 1\right) m_1 + \left(\tfrac{r(\lceil \frac{r}{2}\rceil+1)}{\lfloor \frac{r}{2}\rfloor} + 1\right) m_2 \\
&= \left(\tfrac{r(\lceil \frac{r}{2}\rceil+1)}{\lfloor \frac{r}{2}\rfloor} + 1\right) |M|,
\end{aligned}
$$

since

$$\left(\left\lceil \frac{r}{2}\right\rceil + 1\right)^2 \le r\left(\left\lceil \frac{r}{2}\right\rceil + 1\right).$$

Hence

$$|M| \ge \left(\frac{\lfloor \frac{r}{2}\rfloor}{r\lceil \frac{r}{2}\rceil + r + \lfloor \frac{r}{2}\rfloor}\right) n. \tag{4}$$

Thus, by (4),

$$
\begin{aligned}
\gamma_s(G) &= n - 2|M| \\
&\le n - \left(\tfrac{2\lfloor \frac{r}{2}\rfloor}{r\lceil \frac{r}{2}\rceil + r + \lfloor \frac{r}{2}\rfloor}\right) n \\
&= \left(\tfrac{r\lceil \frac{r}{2}\rceil + r - \lfloor \frac{r}{2}\rfloor}{r\lceil \frac{r}{2}\rceil + r + \lfloor \frac{r}{2}\rfloor}\right) n \\
&= \begin{cases} \left(\frac{(r+1)^2}{r^2+4r-1}\right) n & \text{for } r \text{ odd} \\ \left(\frac{r+1}{r+3}\right) n & \text{for } r \text{ even.} \end{cases}
\end{aligned}
$$

This completes the proof of the upper bound of Theorem 1. □

It remains an open problem to establish whether the upper bounds of Theorem 1 are sharp.

3 A lower bound on $\gamma_{maj}(G)$ for regular graphs G.

Zelinka [7] established the following lower bound on $\gamma_{maj}(G)$ for a cubic graph G.

Theorem C. *For every cubic graph G of order n, $\gamma_{maj}(G) \geq -\frac{n}{4}$ and this bound is sharp.*

In this section we generalize the result of Theorem C to r-regular graphs.

Theorem 2. *For every r-regular ($r \geq 2$) graph $G = (V, E)$ of order n,*

$$\gamma_{maj}(G) \geq \begin{cases} \left(\frac{1-r}{2(r+1)}\right) n & \text{for } r \text{ odd} \\ \left(\frac{-r}{2(r+1)}\right) n & \text{for } r \text{ even,} \end{cases}$$

and these bounds are sharp.

Proof: Let $f : V \to \{-1, 1\}$ be any majority dominating function on G for which $f(V) = \gamma_{maj}(G)$. Let P and M (standing for "positive" and "minus") be the sets of vertices in G that are assigned the values $+1$ and -1, respectively, under f. Then $|P| + |M| = n$. Further, let P^+ and P^- be the sets of vertices in P whose closed neighborhood sum under f is positive and nonpositive, respectively. Define M^+ and M^- analogously. Then $P = P^+ \cup P^-$ and $M = M^+ \cup M^-$. Further, let $|M^+| = a$, $|P^+| = b$ and $|P^-| = c$. Then, since f is a majority dominating function, $a+b \geq n/2$. We consider two possibilities.

Case 1. $a < \left(\frac{\lfloor \frac{r}{2} \rfloor}{2(r+1)}\right) n$.

Then, since $|P| = b + c \geq b \geq n/2 - a$, it follows that

$$|P| > \frac{n}{2} - \left(\frac{\lfloor \frac{r}{2} \rfloor}{2(r+1)}\right) n = \left(\frac{r+1-\lfloor \frac{r}{2} \rfloor}{2(r+1)}\right) n.$$

Hence,

$$\begin{aligned} \gamma_{maj}(G) &= |P| - |M| \\ &= 2|P| - n \\ &> \left(\frac{r+1-\lfloor \frac{r}{2} \rfloor}{r+1}\right) n - n \\ &= \left(\frac{-\lfloor \frac{r}{2} \rfloor}{r+1}\right) n, \end{aligned}$$

which yields the desired result.

Case 2. $a \geq \left(\frac{\lfloor \frac{r}{2} \rfloor}{2(r+1)}\right) n$.

Let ℓ be the number of edges joining a vertex of M^+ and a vertex of P. Then, since each vertex of M^+ must be adjacent to at least $\lceil \frac{r}{2} \rceil + 1$ vertices of P, we have that $\ell \geq \left(\lceil \frac{r}{2} \rceil + 1\right) a$. On the other hand, although a vertex of P^- may be adjacent to even r vertices of M, each vertex of P^+ is adjacent to at most $\lfloor \frac{r}{2} \rfloor$ vertices of M. It follows that $\ell \leq \lfloor \frac{r}{2} \rfloor b + rc$. Consequently,

$$\left(\left\lceil \frac{r}{2} \right\rceil + 1\right) a \leq \left\lfloor \frac{r}{2} \right\rfloor b + rc.$$

Hence it follows that,

$$\begin{aligned}
|P| &= b + c \\
&\geq b + \left((\lceil \tfrac{r}{2} \rceil + 1)a - \lfloor \tfrac{r}{2} \rfloor b\right)/r \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) b + \left(\lceil \tfrac{r}{2} \rceil + 1\right) \tfrac{a}{r} \\
&\geq \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) \left(\tfrac{n}{2} - a\right) + \left(\lceil \tfrac{r}{2} \rceil + 1\right) \tfrac{a}{r} \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) \tfrac{n}{2} + \tfrac{a}{r} \left(\lceil \tfrac{r}{2} \rceil + \lfloor \tfrac{r}{2} \rfloor + 1 - r\right) \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) \tfrac{n}{2} + \tfrac{a}{r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma_{maj}(G) &= 2|P| - n \\
&\geq \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) n + \tfrac{2a}{r} - n \\
&= \tfrac{2a}{r} - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor n \\
&\geq \tfrac{1}{r}\left(\tfrac{\lfloor \frac{r}{2} \rfloor}{r+1}\right) n - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor n \\
&= \left(\tfrac{-\lfloor \frac{r}{2} \rfloor}{r+1}\right) n.
\end{aligned}$$

This completes the proof of the lower bound of Theorem 2.

That the lower bounds of Theorem 2 are sharp, may be seen as follows. For positive integers n, x, s where n is divisible by $2(r+1)$,

$$x = \left(\frac{\lfloor \frac{r}{2} \rfloor}{2(r+1)}\right) n \quad \text{and} \quad s = \left(\frac{\lceil \frac{r}{2} \rceil + 1}{2(r+1)}\right) n,$$

we define a graph $G(m, x, s)$ of order $n/2$ as follows. The graph has vertex set $M \cup P$ where $M = \{u_0, u_1, \ldots, u_{x-1}\}$ and $P = \{v_1, v_2, \ldots, v_s\}$. The vertices in M induce a $(\lfloor\frac{r}{2}\rfloor - 1)$-regular graph and the vertices in P induce a $\lceil\frac{r}{2}\rceil$-regular graph. Then for $1 \leq i \leq s$ join v_i to the $\lfloor\frac{r}{2}\rfloor$ vertices u_j for $\lfloor\frac{r}{2}\rfloor(i-1) \leq j \leq \lfloor\frac{r}{2}\rfloor i - 1$, where subscripts are read modulo x. Hence there are

$$\left(\frac{\lfloor\frac{r}{2}\rfloor\left(\lceil\frac{r}{2}\rceil + 1\right)}{2(r+1)}\right) n$$

edges with one end in M and the other end in P, and these are distributed evenly amongst M. (The above definition is merely one way to ensure an even distribution.) Hence each vertex of M is adjacent to $\lceil\frac{r}{2}\rceil + 1$ vertices of P, and each vertex of P is adjacent to $\lfloor\frac{r}{2}\rfloor$ vertices of M. It follows that $G(n, x, s)$ is an r-regular graph. Now let F_n be any r-regular graph on $n/2$ vertices, and let G_n be the graph obtained from the (disjoint) union of F_n and $G(n, x, s)$. Then G_n is an r-regular graph of order n. Furthermore, the function $f : V(G_n) \to \{-1, 1\}$ defined by $f(v) = 1$ for $v \in P$ and $f(v) = -1$ for $v \in M \cup V(F_n)$, is a majority dominating function on G_n in which every vertex of $G(n, x, s)$ has positive neighborhood sum under f. Hence

$$\begin{aligned}
\gamma_{maj}(G_n) &\leq f(V) \\
&= |P| - |M| - \frac{n}{2} \\
&= \left(\frac{\lceil\frac{r}{2}\rceil + 1}{2(r+1)} - \frac{\lfloor\frac{r}{2}\rfloor}{2(r+1)} - \frac{1}{2}\right) n \\
&= \left(\frac{\lceil\frac{r}{2}\rceil - \lfloor\frac{r}{2}\rfloor - r}{2(r+1)}\right) n \\
&= \begin{cases} \left(\frac{1-r}{2(r+1)}\right) n & \text{for } r \text{ odd} \\ \left(\frac{-r}{2(r+1)}\right) n & \text{for } r \text{ even.} \end{cases}
\end{aligned}$$

This, together with the lower bound of Theorem 2, establishes that

$$\gamma_{maj}(G_n) = \begin{cases} \left(\frac{1-r}{2(r+1)}\right) n & \text{for } r \text{ odd} \\ \left(\frac{-r}{2(r+1)}\right) n & \text{for } r \text{ even.} \end{cases}$$

This completes the proof of Theorem 2. □

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