Chvátal-Erdös Condition for Hamiltonicity in Digraphs

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Abstract

Our purpose is to determine the minimum integer $f_i(m)$ ($g_i(m)$, $h_i(m)$ respectively) for every natural m, such that every digraph D, $f_i(m)$-connected, ($g_i(m)$, $h_i(m)$-connected respectively) and $\alpha^i(D) \leq m$ is hamiltonian (D has a hamilton path, D is hamilton connected respectively), ($i = 0, 1, 2$). We give exact values of $f_i(m)$ and $g_i(m)$ for some particular values of m. We show the existence of $h_2(m)$ and that $h_2(1) = 1$, $h_2(2) = 4$ hold.

1 Introduction and Terminology

Let $G = (V(G), E(G))$ be a finite $k-$connected graph ($k \geq 2$), on n vertices, without loops or multiple edges, and with independence number $\alpha(G) = \alpha$.

In 1972, V. Chvátal and P. Erdös gave the following sufficient condition for a graph to have a hamilton cycle (path) or for every x, y in $V(G)$, G has a $xy-$ hamilton path, that is to say G is **hamilton connected**.

Theorem 1.1 ([1]) *If $\alpha \leq k$, then G is hamiltonian. If $\alpha \leq k + 1$, then G contains a hamilton path. If $\alpha \leq k - 1$, then G is hamilton connected.*

This result has given rise to many other sufficient conditions involving connectivity and independence number for graphs and digraphs to have

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various path or cycle properties, for example hamilton path (cycle), hamilton connected, pancyclic, path (cycle) covers, 2–cycle. Several related results and conjectures are presented in a well structured survey by Jackson and Ordaz [7].

We shall use $D = (V(D), E(D))$ to denote a finite digraph on n vertices, without loops or multiple arcs. Let $\alpha^0(D)$, $\alpha^1(D)$, $\alpha^2(D)$ be the size of the largest subset S of $V(D)$ such that $D[S]$, induced subdigraph by S, contains no arcs, no cycles, no cycles of length two, respectively.

Note that if G is an undirected graph and D is the symmetric digraph obtained by replacing each edge of G by a directed cycle of length two then $\alpha(G) = \alpha^0(D) = \alpha^1(D) = \alpha^2(D)$. Thus α^0, α^1, α^2 can be considered as an extension of α from graphs to digraphs. Moreover, by the definition of α^0, α^1, α^2 we have:

$$\alpha^0(D) \leq \alpha^1(D) \leq \alpha^2(D) \tag{1}$$

Hence, any upper bound on $\alpha^2(D)$ will also bound $\alpha^0(D)$ and $\alpha^1(D)$. It follows that, although α^0 is in some sense the most natural extension of α to digraphs, it will be easier to obtain "Chvátal- Erdös sufficient condition" using α^2 than α^1 or α^0.

For every natural m, we denote by $f_i(m)$(resp. $g_i(m)$, $h_i(m)$) the minimum integer such that every digraph D, $f_i(m)$–connected (resp.$g_i(m)$–connected, $h_i(m)$–connected) and $\alpha^i(D) \leq m$ is hamiltonian (resp. D has a hamilton path, D is hamilton connectd), $(i = 0, 1, 2)$. Exact values of $f_i(m)$, $g_i(m)$ are given for some particular values of m. We show the existence of h_2 and $h_2(1) = 1$, $h_2(2) = 4$ hold.

We denote by $x \xrightarrow{T} y$ a hamilton path from x to y in the graph or digraph T. For definitions and notations not given here see [1] or [7].

2 Hamilton Path and Cycle

In [2], Bondy suggested partial extending Theorem 1.1 to digraphs by conjecturing that if $\alpha^2(D) \leq k$ then either D is hamiltonian or else D belongs to a finite set of exceptional graphs.

Unfortunately, Thomassen [10] and Chakroun [3] have each constructed an infinite family of conterexamples to Bondy's conjecture for $\alpha^2(D) = k = 2$ and $\alpha^2(D) = k = 3$ respectively.

However, if we allow " well characterised " infinite families of exceptional graphs, or, if we restrict ourselves to digraphs of connectivity at least four, then it is conceivable that Bondy' conjecture is true.

In this paper we re-structure some results of survey [7] in order to present them in this context.

The following results ensure the existence of f_i, g_i and we give some exact values of them.

Using (1), it follows that if $f_0(m)$ exists, then $f_i(m)$ and $f_2(m)$ both exist and we have:

$$f_2(m) \leq f_1(m) \leq f_0(m) \tag{2}$$

Theorem 2.1 ([6]) *Let D be a digraph with $\alpha^2(D) \leq m$. If D is $2^m(m+2)!-$connected then D is hamiltonian.*

We deduce from Theorem 2.1 that for all $m \geq 1$, $f_2(m)$ exists and $f_2(m) \leq 2^m(m+2)!$ holds. Since $g_2 \leq f_2$ then the existence of g_2 is guaranteed.

Theorem 2.2 $m-1 \leq g_2(m)$, $(m > 2)$.

Proof. It is sufficient to consider the symmetric complete bipartite digraph $D = K_{m,m-2}$. It is clear that D is $(m-2)-$connected and $\alpha^2(D) = m$. Since there exists a α^2- stable set S with $|S| = m$ and $|N^+(S)| = m-2 < m-1 = |S|-1$ then D does not contain a hamilton path.

Theorem 2.3 $m \leq f_2(m)$

Proof. Let F be the digraph obtained from $D = K_{m,m-2}$ adding a new vertex joined by edges with all vertices in D. We have $\alpha^2(F) = m$ and F is $(m-1)-$connected. Since D does not contain a hamilton path then F is not hamiltonian. Hence $m \leq f_2(m)$. The same result can be obtained directly using $K_{m,m-1}$.

Theorem 2.4 ([5]) *Let D be a strongly connected digraph with $\alpha^0(D) \leq 2$ then D contains a hamilton path.*

Theorem 2.5 ([7]) $g_2(2) = g_1(2) = g_0(2) = 1$, *that is to say if D is strongly connected with $\alpha^i(D) = 2$ $(i = 0, 1, 2)$ then D contains a hamilton path.*

Proof. By Theorem 2.2 and Theorem 2.4 we have $g_0(2) = 1$ and since $\alpha^0 \leq \alpha^1 \leq \alpha^2$ we can conclude that $g_1(2) = g_2(2) = 1$.

Theorem 2.6 *If $\alpha^0(D) = 1$ and D is strongly connected then D is hamiltonian.*

Proof. Directly from the following Theorem due to H. Meyniel [8]: Let D be a strongly connected digraph of order n. If for every pair of non adjacent vertices x, y we have $d(x) + d(y) \geq 2n - 1$ then D is hamiltonian.

Theorem 2.7 $f_i(1) = 1$ $(i = 0, 1, 2)$

Proof. Let D be strongly connected. By Theorem 2.6, if $\alpha^0(D) = 1$ then D is hamiltonian. If $\alpha^2(D) = 1$ then D is a symmetric complete digraph hence D is hamiltonian. Moreover, since $f_2 \leq f_1 \leq f_0$ then $f_1(1) = 1$.

In [10] it is shown that every 2–connected digraph D with $\alpha^2(D) \leq 2$ except the following digraphs D_1, is hamiltonian. Let D_1 be obtained from two disjoint symmetric complete digraphs K_m^* and K_p^* by choosing distinct vertices x_1, x_2 of K_m^* and y_1, y_2 of K_p^* and then adding the 4–cycle $x_1y_1x_2y_2x_1$ to $K_m^* \cup K_n^*$.

Then we have:

Theorem 2.8 $f_1(2) = f_2(2) = 3$.

In [3] it is shown that every 3–connected digraph D with $\alpha^2(D) \leq 3$ except the following digraphs D_2 is hamiltonian. Let D_2 be obtained from three disjoint symmetric complete digraphs K_m^*, K_n^* and K_p^* by choosing distinct vertices x_1, x_2, x_3 of K_m^*, y_1, y_2, y_3 of K_n^* and z_1, z_2, z_3 of K_p^* and adding the following arcs x_1y_1, y_2x_1, x_2y_2,y_3x_2, x_3y_3, y_1x_3, y_1z_1, z_3y_1, y_2z_2, z_1y_2, y_3z_3, z_2y_3 to $K_m^* \cup K_n^* \cup K_p^*$

Then we have;

Theorem 2.9 $f_2(3) = 4$.

Theorem 2.10 $g_2(3) = 2$.

Proof. Let D be a 2–connected digraph with $\alpha^2(D) = 3$ and H be the digraph obtained from D adding a new vertex u joined by edges with all vertices in D. We have $\alpha^2(H) = \alpha^2(D) = 3$ and H is 3– connected. Since H is different from digraph D_2 then H contains a hamilton cycle C, consequently $C - \{u\}$ is a hamilton path of D. Then $g_2(3) \leq 2$ and by Theorem 2.2 we conclude that $g_2(3) = 2$.

3 Hamilton Connected Digraphs

This section is devoted to show the existence of h_2 and we stablish that $h_2(1) = 1$, $h_2(2) = 4$ hold.

Theorem 3.1 *For every natural m there exists $h_2(m)$ and we have $h_2(m) \leq 2^{m+1}(m+3)! + 2$.*

Proof. Let D be a $(f_2(m+1)+2)-$ connected digraph with $\alpha^2(D) \leq m$. The existence of $f_2(m+1)$ is ensured by Theorem 2.1. Let us considered two vertices u, v in D. Let H be the digraph obtained from D as follows: $V(H) = V(D) - \{u, v\} + w$ with w a new vertex outside of D. $E(H)$ it is constitued by the arcs in $E(D)$ which are not incident with vertices u, v and arc xw belongs to $E(H)$ if $xv \in E(D)$; arc wy belongs to $E(H)$ if $uy \in E(D)$.

We will show that H is $f_2(m+1)-$ connected and $\alpha^2(H) \leq m+1$.

In fact, if x, y are vertices in H different from w, there are in D $f_2(m+1)+2$ internally disjoint paths from x to y. Hence there are at least $f_2(m+1)$ internally disjoint paths in H.

If $x = w$ then we consider in D $f_2(m+1)$ internally disjoint paths from u to y no incident to v and first arc ux_i from every one these paths is replaced by wx_i. So we obtain in H $f_2(m+1)$ internally disjoint paths from w to y.

If $y = w$, we select in D $f_2(m+1)$ internally disjoint paths from x to v no incident to u and last arc y_iv from every one of these paths is replaced by y_iw. So we obtain in H $f_2(m+1)$ internally disjoint paths from x to w.

In this way we obtain in H $f_2(m+1)$ internally disjoint paths from x to y.

Now, we will see that $\alpha^2(H) \leq m+1$.

Let S be a α^2-stable set in H. If $w \notin S$ then $S \subseteq V(D)$ is a α^2-stable set in D then $|S| \leq m$. If $w \in S$ then $S_1 = S - \{w\}$ is a α^2- stable set in D and $|S_1| \leq m$, therefore $|S| \leq m+1$.

Consequently, by definition of $f_2(m+1)$, H is hamiltonian. If C is a hamilton cycle in H and w_1, w_2 are vertices in C incident with w then, since w_1w is an arc in H then arc w_1v is in D. In the same way we can see that there exists the arc uw_2.

Hence $uw_2 \overset{C}{\rightarrow} w_1v$ is a hamilton path in D from u to v.

Consequently, we have shown that D is hamilton connected and $h_2(m) \leq f_2(m+1)+2$ holds. Since $f_2(m+1) \leq 2^{(m+1)}(m+3)!$ by Theorem 2.1, we have $h_2(m) \leq 2^{(m+1)}(m+3)!+2$.

Theorem 3.2 $h_2(m) \geq m+1$, $m \geq 2$.

Proof. Let $D = K_{m,m}$ be the bipartite digraph with $m \geq 2$. We have that D is $m-$connected and $\alpha^2(D) = m$. But an $xy-$hamilton path with x, y in the same independent set in bipartition of D does not exist.

Theorem 3.3 $h_2(1) = 1$.

Proof. Let D be a digraph with $\alpha^2(D) = 1$ then D is symmetric complete. Hence D is hamilton connected.

Lemma 3.4 *Let D be a strongly connected digraph. Then any two vertices of D are joined by a path of length at most $2\alpha^1(D) - 1$.*

Since $\alpha^1 \leq \alpha^2$, we can deduce the following result:

Corollary 3.5 *Let D be a strongly connected digraph. Then any two vertices of D are joined by a path of length at most $2\alpha^2(D) - 1$.*

Theorem 3.6 *Let D be a $k-$connected digraph with $\alpha^2(D) \leq m$. If T is any set of q vertex disjoint paths of D of total length at most t and $k \geq h_2(m) + 2(t + qm - m - q)$, then T is contained in a hamilton cycle of D.*

Proof. If $q = 1$ then $T = \{x_1, x_2, ..., x_w\}$. It is clear that $H = D - \{x_2, ..., x_w\}$ is $h_2(m)-$connected. By definition, there exists a $x_w x_1-$hamilton path Q in H. Then $x_1 x_2 ... x_w Q$ is a hamilton cycle.

Suppose $q \geq 2$. In this case we can inductively contruct a path that cointains T.

Let $P_1 P_2 ... P_q$ the vertex disjoint paths in T.

First Step. Let D_1 be the digraph D without the vertices of T excepting the endvertex a_1 of P_1 and the initialvertex b_1 of P_2, then D_1 is of connectivity at least $h_2(m) + (q-1)(2m-2) \geq h_2(m)$. By Corollary 3.5 there exists a $a_1 b_1-$path Q_1 of length at most $2m - 1$.

Second Step. ($q \geq 3$). Let D_2 be the digraph D_1 without the vertices of Q_1 and adding the endvertex a_2 of P_2 and initialvertex b_2 of P_3, then D_2 is of connectivity at least $h_2(m) + (q-1)(2m-2) - (2m-2) = h_2(m) = (q-2)(2m-2) \geq h_2(m)$. By Corollary 3.5 there exists a a_2b_2-path Q_2 of length at most $2m-1$.

Suppose now that we are in (i-1)-Step. If $q \geq i+1$ the following Step is:

i-ith. Step. Let D_i the digraph D_{i-1} without the vertices of Q_{i-1} and adding the endvertex a_i of P_i and initialvertex b_i of P_{i+1}. Then D_i is of connectivity at least $h_2(m) - (q-i)(2m-2)$ and since $(q-i) \geq 1$ D_i is $h_2(m)-$connected. Consequently there exists a a_ib_i-path Q_i of length at most $2m-1$.

When $i = q-1$ we have the path P :

$P_1Q_1P_2Q_2...P_{q-1}Q_{q-1}P_q$. Let b_0 the initial vertex of P, a_q the endvertex of P and D_q the digraph D without vertices of P but safe b_0 and a_q. Since $|V(P) - \{b_0, a_q\}| = t + (q-1)(2m-1) + 1 - 2$ the connectivity of D_q is at least $h_2(m) + 2(t + qm - m - q) - (t + (q-1)(2m-1) - 1) = h_2(m) + t - q$ and since $q \leq t$ then D_q is $h_2(m)-$connected. Therefore there exists an a_qb_0-hamilton path Q in D_q. The paths P and Q conform a hamilton cycle in D that contains T.

Corollary 3.7 *For every pair of natural number m and t, there exists a natural number $h_2(m,t)$ such that if D is a digraph $h_2(m,t)-$connected and $\alpha^2(D) \leq m$. Then every set of vertices disjoint paths T of total length at least t can be extended to a hamilton cycle in D.*

Proof. In before theorem $q \leq t$. If we have $q = t$ we obtain $h_2(m,t) \leq h_2(m) + 2m(t-1)$.

The following lemmas and remarks will be used to proof Theorem 3.11: Every 3$-$connected digraph with $\alpha^2(D) = 2$ is hamilton connected except for all 3$-$connected digraphs member of three families D_3, D_4, D_5. As a corollary of this theorem we will show that $h_2(2) = 4$.

Lemma 3.8 *Let G be a graph such that $\alpha(G) = 2$. Then we have one of the following alternatives:*

3.8.1 *G is hamilton connected,*

3.8.2 *G has a spanning subgraph consisting of two disjoint complete graphs H_1 and H_2.*

3.8.3 *G has a spanning subgraph consisting of two disjoint graphs one of which is a complete graph H_2 and the other, H_1, is a complete graph with at least three vertices and a missing edge (a_1, b_1). Furthermore, there exist two independent edges (a_1, a_2), (b_1, b_2) between H_1 to H_2 and no edges from $H_1 - \{a_1, b_1\}$ to H_2 and every vertex of H_2 is adjacent to a_1 or b_1.*

Proof. If $k(G) \geq 3$ then by Theorem 1.1 G is hamilton connected. Then G has structure 3.8.1.

If $k(G) = 0$ and since $\alpha(G) = 2$ then G consists of exactly two components both of which have stability number one, hence G has structure 3.8.2. If $k(G) = 1$, we may choose a vertex v such that $G - \{v\}$ is disconnected. Since $\alpha(G) = 2$, $G - \{v\}$ consists of exactly two components H_1 and H_2 both of which have stability number one, and hence are complete. Moreover v is adjacent either to all vertices of H_1 or to all vertices of H_2. Thus $Cl(G) = 2$, and it can easily be seen that G satisfies 3.8.2.

If $k(G) = 2$, there exists a separating set $\{p, q\}$ of G. Since $\alpha(G) = 2$, $G - \{p, q\}$ consists of exactly two components G_1 and G_2 both of which have stability number one. Moreover since $\alpha(G) = 2$ we have either:

Case a. p and q are adjacents to all vertices of G_1(or to all vertices of G_2). Putting $H_1 = G_1 \cup \{p, q\}$ and $H_2 = G_2$, we have the following subcase: If $(p, q) \in E(G)$ then G has structure 3.8.2 and if $(p, q) \notin E(G)$ then, putting, $p = a_1$, $q = b_1$, G has structure 3.8.3.

Case b. p is adjacent to all vertices of G_1 and q is adjacent to all vertices of G_2. Then putting, $H_1 = G_1 \cup \{p\}$ and $H_2 = G_2 \cup \{q\}$, G has structure 3.8.2.

Remark 3.9 *Let $D = (V, A, E)$ be a digraph such that $G(D) = (V, A)$ has structure 3.8.2. Let $x, y \in H_1$. If there exist arcs xu_2 with $x \in H_1$, $u_2 \in H_2$ and t_2t_1 with $t_2 \in H_2 - \{u_2\}$, $t_1 \in H_1 - \{x, y\}$, then $xu_2 \xrightarrow{H_2} t_2t_1 \xrightarrow{H_1 - \{x\}} y$ is a $xy-$hamilton path.*

Remark 3.10 *Let $D = (V, A, E)$ be a digraph such that $G(D) = (V, A)$ has structure 3.8.2. Let $x, y \in H_1$. If there exist arcs: u_1u_2 with $u_1 \in H_1 - \{y, x\}$, $u_2 \in H_2$ and t_2t_1 with $t_2 \in H_2 - \{u_2\}$, $t_1 \in H_1 - \{x, u_1\}$ then*

$x \xrightarrow{H_1 - \{y, t_1\}} u_1u_2 \xrightarrow{H_2} t_2t_1y$ *is an xy hamilton path.*

Lemma 3.11 *Let $D = (V, E, A)$ be a 3−connected digraph with $\alpha^2(D) = 2$. If $G(D) = (V, E)$ has structure 3.8.2 of Lemma 3.8 then D is hamilton connected or isomorphic to a digraph of the family D_3 or the family D_4.*

Proof. Let x, y be two different vertices of D. We shall see that there exists an xy−hamilton path except that D is isomorphic to a member of families D_3 or D_4.

Let $x \in H_1$, $y \in H_2$. Since $D - \{x, y\}$ is strongly connected there exists an arc t_1t_2 from $H_1 - \{x\} \neq \emptyset$ (if $|H_1| = 1$ it is not necessary to eliminate x from H_1) to $H_2 - \{y\} \neq \emptyset$ (if $|H_2| = 1$ it is not necessary to eliminate y from H_2). Hence the path $x \xrightarrow{H_1} t_1t_2 \xrightarrow{H_2} y$ is a hamilton path.

If $|V(D)| < 6$ and since $d^+(x) \geq 3$, $d^-(x) \geq 3$ we can apply Overbeck-Larisch Theorem [9] in order to shown that D is hamilton connected. If $|H_1| = 1$, we can apply the same theorem in order to show that D is hamilton connected.

Now, let us considere $|V(D)| > 5$. Let $x, y \in H_1$.

If $|H_1| = 2$ there exist two arcs xu_2, t_2y with $u_2 \neq t_2$ $(u_2, t_2 \in H_2)$. Hence

$xu_2 \xrightarrow{H_2} t_2y$ is a hamilton path.

Now, we suppose that $|H_1| > 2$.

Case a $N(x) \cap H_2 = \emptyset$.

Let u_1u_2 be an arc from $H_1 - \{y\}$ to H_2. Let t_2t_1 be an arc with $t_2 \in H_2 - \{u_2\}$,$t_1 \in H_1 - \{u_1\}$. By Remark 3.10 D has a xy−hamilton path.

Case b. $N(x) \cap H_2 \neq \emptyset$

Subcase b.1. $N^+(x) \cap H_2 = \emptyset$

Let us suppose there are no xy−hamilton paths. It is clear that $|H_1| \geq 4$. Since $D - \{y\}$ is 2−connected there exist two independent arcs from $H_1 - \{y\}$ to H_2, saying u_1u_2 and t_1t_2. Moreover, since there are no xy−hamilton paths the following conditions are satisfied:

Condition 1. The only arcs from H_2 to $H_1 - \{x\}$ are u_2t_1 and t_2u_1. Moreover if $|H_2| > 2$ then arc v_2x exist with $v_2 \neq t_2, u_2, v_2 \in H_2$.

Condition 2. All arcs from $H_1 - \{u_1, t_1\}$ to H_2 have y as initial vertex.

Consequently we are in presence of a digraph member of the family described below:

FAMILY D_3

- Constitued by two complete graphs H_1 ($|H_1| \geq 4$) and H_2 ($|H_2| \geq 2$).

- There are four distinguished vertices $x, y, u_1, t_1 \in H_1$ and two distinguished vertices $u_2, t_2 \in H_2$.

- The only arcs from H_2 to $H_1 - \{x\}$ are u_2t_1 and t_2u_1.

-All arcs from $H_2 - \{u_2, t_2\}$ to H_1 have x as ending vertex and if $|H_2| > 2$ there exists at least one arc from $H_2 - \{u_2, t_2\}$ to H_1, saying v_2x.

-All arcs from $H_1 - \{u_1, t_1\}$ to H_2 have y as initial vertex and there exists at least one arc from $H_1 - \{u_1, t_1\}$ to H_2, saying yw_2.

Subcase B.2 $N^+(x) \cap H_2 = \emptyset$

In this case we assume that there does not exist an $xy-$hamilton path.

Let xu_2 be an arc with $u_2 \in H_2$. There exists an arc t_2t_1 from $H_2 - \{u_2\}$ to $H_1 - \{x\}$. If $t_1 \neq y$, by Remark 3.5 there exists an $xy-$hamilton path. Contradiction. Therefore:

Condition 1. Every arc from $H_2 - \{u_2\}$ to $H_1 - \{x\}$ have y as ending vertex and there exists the arc t_2y.

Let w_1w_2 be arc from $H_1 - \{x\}$ to $H_2 - \{t_2\}$. If $w_1 \neq y$, by Remark 3.10 D contains a $xy-$hamilton path. Contradiction. Therefore:

Condition 2. All arcs from $H_1 - \{x\}$ to $H_2 - \{t_2\}$ have y as initial vertex.

Let z_1z_2 be an arc from $H_1 - \{x, y\}$ to H_2. So, $z_2 = t_2$. There exists an arc v_2v_1 from $H_2 - \{t_2\}$ to $H_1 - \{z_1\}$. If $v_1 = x$, by Remark 3.9. D contains an $xy-$hamilton path. Contradiction. Therefore:

Condition 3. All arcs from $H_2 - \{t_2\}$ to $H_1 - \{z_1\}$ has x as terminal vertex.

Then, there exists an arc p_2p_1 from $H_2 - \{t_2\}$ to $H_1 - \{x\}$. By Condition 3 $p_1 = z_1$ and by Condition 1 $p_2 = u_2$. Hence $u_2z_1 \in E(D)$.

Consequently we are in presence of a digraph member of family described below:

FAMILY D_4

- It is 3-connected but not 4-connected.

- Constitue by two complete graphs H_1 ($|H_1| \geq 3$) and H_2 ($|H_2| \geq 2$).

- There are three distinguished vertices $x, y, z_1 \in H_1$ and two distinguished vertices $u_2, t_2 \in H_2$.

- Arcs xu_2, t_2y and z_1t_2 are present.

- All arcs from $H_2 - \{u_2\}$ to $H_1 - \{x\}$ have y as ending vertex.
- All arcs from $H_1 - \{x\}$ to $H_2 - \{t_2\}$ have y as ending vertex.
- All arcs from $H_2 - \{t_2\}$ to $H_1 - \{z_1\}$ have x as ending vertex.

Lemma 3.12 *Let $D = (V, E, A)$ be a 3−connected digraph with $\alpha^2(D) = 2$. If $G(D) = (V, E)$ has structure 3.8.3 but not 3.8.2 of Lemma 3.8 then D is hamilton connected or isomorphic to a digraph of the family D_5.*

Proof. Let x, y two different vertices of D. We shall see that there exists an xy−hamilton connected path except that D is isomorphic to a member of families D_5.

We have the following cases:

Case a $|H_1| \geq 4$.

Subcase a.1 $x \in H_1$, $y \in H_2$.

Subcase a.1.1 $x \notin \{a_1, b_1\}$

Without loss of generality, we suppose that $y \neq a_2$.

Then $x \xrightarrow{H_1} a_1a_2 \xrightarrow{H_2} y$ is a hamilton path.

Subcase a.1.2 $x = a_1$, $y \neq b_2$ (or $x = b_1$, $y \neq a_2$)

In this case $x \xrightarrow{H_1} b_1b_2 \xrightarrow{H_2} y$ is a hamilton path.

Subcase a.1.3 $x = a_1$, $y = b_2$ (or $x = b_1$, $y = a_2$).

Since $D - \{a_1, b_2\}$ is strongly connected then there exists an arc t_1t_2 with $t_1 \in H_1 - \{a_1\}$, $t_2 \in H_2 - \{b_2\}$. Therefore the path $x \xrightarrow{H_1} t_1t_2 \xrightarrow{H_2} y$. is a hamilton path.

Subcase a.2 $x \in H_2$, $y \in H_1$.

Subcase a.2.1 $y \notin \{a_1, b_1\}$.

We can suppose without loss of generality that $x \neq a_2$, then

$x \xrightarrow{H_2} a_2a_1 \xrightarrow{H_1} y$ is a hamilton path.

Subcase a.2.2 $y = a_1$, $x \neq b_2$ ($y = b_1$, $x \neq a_2$).

In this case $x \xrightarrow{H_2} b_2b_1 \xrightarrow{H_1} y$ is a hamilton path.

Subcase a.2.3 $y = a_1$, $x = b_2$.

Since $D - \{a_1, b_2\}$ is strongly connected there exists $t_2 t_1 \in E(D)$ with $t_1 \in H_1 - \{a_1\}$, $t_2 \in H_2 - \{b_2\}$, then the path $x \xrightarrow{H_2} t_2 t_1 \xrightarrow{H_1} y$ is a hamilton path.

Subcase a.3 $x, y \in H_1$.

Subcase a.3.1 $x, y \notin \{a_1, b_1\}$.

In this case we have that the following path:

$x \xrightarrow{H-\{y,b_1\}} a_1 a_2 \xrightarrow{H_2} b_2 b_1 y$ is a hamilton path.

Subcase a.3.2 $x = a_1$, $y \neq b_1$ ($x = b_1$, $y \neq a_1$).

Then we have that $x a_2 \xrightarrow{H_2} b_2 b_1 \xrightarrow{H_1-\{x\}} y$ is a hamilton path.

Subcase a.3.3 $x = a_1$, $y = b_1$ ($x = b_1$, $y = a_1$).

In this case, it is clear that the problem of finding an xy–hamilton path it is not altered adding edge (x, y). Therefore the problem is reduced to Lemma 3.11 when $x, y \in H_1$. In this lemma we showed that always exists an xy–hamilton path, except for cases D_3, D_4. Our original digraph D cannot generate an element of family D_3, or an element of family D_4 with $|H_2| > 2$. But it can generate one of family D_4 with $|H_2| = 2$ this happens when D is a member of described below family D_5.

FAMILY D_5.

The description of family D_5 is exactly the same as that of family D_4 with $|H_2| = 2$ and edge (x, y) missing.

Subcase a.4 $x, y \in H_2$.

If $|H_2| = 2$ is trivial that there exists an xy–hamilton path.

We suppose that $|H_2| > 2$.

Subcase a.4.1 x or $y \notin \{a_2, b_2\}$.

Without loss of generality, let us suppose that $x \notin \{a_2, b_2\}$ then

$x \xrightarrow{H_2-\{b_2,y\}} a_2 a_1 \xrightarrow{H_1} b_1 b_2 y$ is a hamilton path.

Subcase a.4.2 $x = a_2$, $y = b_2$ ($x = b_2$, $y = a_2$).

Let $c_2 \neq a_2$, b_2 be in H_2. Since $\{a_1, b_1, c_2\}$ it is not α^2–independent then edges (c_2, a_1) or (c_2, b_1) exist. In both cases we interchange c_2 with x or y and we are in subcase a.4.1.

Case b $|H_1| = 3$, $H_1 = \{a_1, b_1, c_1\}$.

Subcase b.1 $|H_2| = 2$.

We can apply Overbeck-Larisch Theorem [9], in order to conclude that D is hamilton connected.

Subcase b.2 $|H_2| = 3$, $|H_2| = \{a_2, b_2, c_2\}$.

There exist edges (a_1, c_2) or (b_1, c_2). Suppose that (a_1, c_2) is an edge. Then putting $H_1^1 = H_2 \cup \{a_1\}$, $H_2^1 = H_1 - \{c_2\}$ we have $|H_1^1| \geq 4$ and must be $\alpha(H_1^1) = 2$. Hence we are in case a.

Subcase b.3 $|H_2| > 3$ and we suppose there exists a pair of vertices $c_2, d_2 \in H_2 - \{a_2, b_2\}$ such that $a_1c_2 \notin E(D)$ and $b_1d_2 \in E(D)$.

In this case there must exist edges (b_1, c_2) and (a_1, d_2).

Subcase b.3.1 $x = a_1$, $y \in H_2$.

Without loss of genarality we can suppose that $y \neq c_2$.

In this case the $xy-$hamilton path is: $x \xrightarrow{H_1} b_1c_2 \xrightarrow{H_2} y$.

Subcase b.3.2 $x = a_1$, $y = c_1$.

In this case the $xy-$hamilton path is: $xa_2 \xrightarrow{H_2} b_2b_1c_1$.

Subcase b.3.3 $x = a_1$, $y = b_1$.

Since $D - \{a_1, b_1\}$ is strongly connected there exists edge c_1t_2. If $t_2 \neq b_2$ we have that $xc_1t_2 \xrightarrow{H_2} b_2b_1$ is an $xy-$hamilton path.

If $t_2 \neq c_2$ then $xc_1t_2 \xrightarrow{H_2} c_2b_2b_1$ is an $xy-$hamilton path.

Subcase b.3.4 $x \in H_2$, $y = a_1$.

In this case $x \xrightarrow{H_2} b_2b_1c_1a_1$ is an $xy-$hamilton path.

Subcase b.3.5 $x \in H_2$, $y = b_1$.

In this case $x \xrightarrow{H_2} a_2a_1c_1b_1$ is an $xy-$hamilton path.

Subcase b.3.6 $x \in H_2$, $y = c_1$.

Since $\{x, a_1, b_1\}$ is not α^2-independent edges (x, a_1) or (x, b_1) exist. Without loss of generality, let us suppose that (x, a_1) exist. Then the $xy-$hamilton path is $xa_1a_2 \xrightarrow{H_2 - \{x\}} b_2b_1c_1$.

Subcase b.4

Let us suppose we don't have the condition of b.3. Then for every $c_2 \neq a_2, b_2$, (a_1, c_2) is an edge. Then putting $H_1^1 = H_2 \cup \{a_1\}$, $H_2^1 = H_1 - \{a_1\}$, we are in case a, because $|H_1^1| \geq 4$ and there is only one edge missing from H_1^1 and also H_2^1 ia a complete graph.

Theorem 3.13 *Let $D = (V, E, A)$ be a 3−connected digraph with $\alpha^2(D) = 2$. Then D is hamilton connected or isomorphic to a digraph of the family D_3, D_4 or D_5.*

Proof. Let $G(D) = (V, E)$. It is clear that $\alpha(G(D)) = 2$. If $G(D)$ has structure 3.8.1 of Lemma 3.8 then $G(D)$ and D are hamilton connected. If $G(D)$ has structures 3.8.2 or 3.8.3 then by Lemma 3.11 and Lemma 3.12 D is hamilton connected except that D is isomorphic to member of D_3, D_4 or D_5.

Corollary 3.14 $h_2(2) = 4$.

Proof. By Theorem 3.13 every 3−connected digraph D is hamilton connected except for all 3−connected isomorphic digraph members of the family D_3, D_4 or D_5. Since these members are not 4−connected we can conclude that $h_2(2) = 4$.

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