On the Existence of a Stable Complementing Permutation in a *t-sc* Graph

T. Gangopadhyay
XLRI Jamshedpur
Post Box 222
Jamshedpur 831 001
India

ABSTRACT. The class of *t-sc* graphs constitutes a new generalization of self-complementary graphs. Many *t-sc* graphs exhibit a stable complementing permutation. In this paper, we prove a sufficient condition for the existence of a stable complementing permutation in a *t-sc* graph. We also construct several infinite classes of *t-sc* graphs to show the stringency of our sufficient condition.

1. Introduction and Definitions

The class of self-complementary graphs has been extensively studied by many people, among others by C.R.J. Clapham [1], R.A. Gibbs [8], S.B. Rao [11], G. Ringel [12] and H. Sachs [13]. Many problems have been solved for this class of graphs, such as the hamiltonian problem and the characterization of potentially and forcibly self-complementary degree sequences (see the references given in [11]). This interesting class has also been generalized into the class of multipartite self-complementary graphs by T. Gangopadhyay and S.P. Rao Hebbare [5]. Several important notions such as pathlength and range of diameters have already been studied for the generalized class (see [6], [7]).

In an earlier paper [4], we have presented a new generalization of self-complementary graphs called the *t-sc* graphs. Various properties of this class of graphs have been studied generalizing earlier results of Ringel [12] and Sachs [13]. In particular, stable complementing permutations, a notion associated with *t-sc* graphs have been extensively studied in [4]. In [3], we have shown the existence of a canonical stable complementing permutation for all *t-sc* graphs that have a stable complementing permutation generalizing an earlier result on self-complementary graphs (see Gibbs [8]).

In the present paper we provide a sufficient condition for the existence of stable complementing permutations in a *t-sc* graph. We also show by constructing infinite classes of *t-sc* graphs that our sufficient condition is quite stringent.

Given an integer t, the t-tuple $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ is called a ***t-sc graph*** if there exists a complete graph G such that

i) each G_i is a spanning subgraph of G,

ii) $E(G)$ is the disjoint union of $E(G_1), E(G_2), \ldots, E(G_t)$,

iii) $G_1, G_2, \ldots, G_t$ are all isomorphic graphs.

Let $(G_1, G_2, \ldots, G_t)$ be a *t-sc* graph. Let σ_i be an isomorphism from G_i to G_{i+1}, $1 \leq i \leq t-1$ and σ_t be an isomorphism from G_t to G_1. Then the t-triple $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ is called a *complementing permutation class* (cpc) for $(G_1, G_2, \ldots, G_t)$.

Let π be a cycle of σ_i. We denote by $|\pi|$ the *length* of π, i.e., the number of vertices of G_i contained in π.

Clearly if $t = 2$ then $G_2 = G_1^C$ and G_1 is a self-complementary graph in the usual sense. Also if (σ_1, σ_2) is a cpc for (G_1, G_2) then σ_1 is a *complementing permutation* for the self-complementary graph G_1 in the usual sense of the term.

Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for a *t-sc* graph. If $\sigma_1 = \sigma_2 = \cdots = \sigma_t = \sigma$ (say) then σ is called a *stable complementing permutation* (scp) for $(G_1, G_2, \ldots, G_t)$.

Figure 1.1 depicts a 5-*sc* graph on 5 points with $(u_1u_2u_3u_4u_5)$ as an scp.

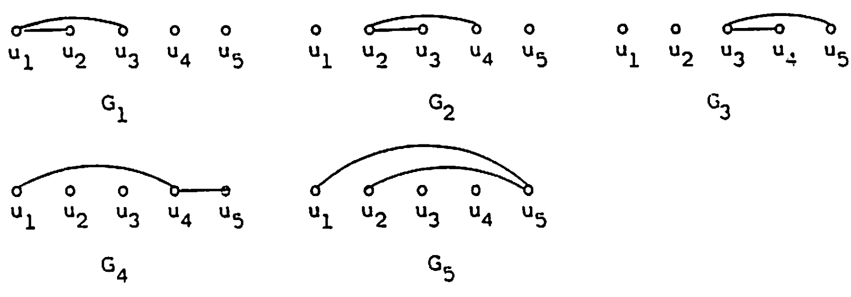

Figure 1.1

For other examples and infinite classes of connected *t-sc* graphs with scp's, please see [4].

It is not true that every *t-sc* graph $\mathcal{G}$ has an scp. Sections 3,4 and 5 of this paper give infinite classes of *t-sc* graphs without any scp.

In Section 2, we use the following lemma, proved in [4].

Lemma 1.1. *Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for a t-sc graph $(G_1, G_2, \ldots, G_t)$. If $\sigma_1 = \sigma_2 = \cdots = \sigma_{t-1} = \sigma$ (say), then σ is an scp for $(G_1, G_2, \ldots, G_t)$.*

For all undefined terms we refer to Harary [10].

2. A Sufficient Condition for an SCP

In this section we present a sufficient condition for the existence of an scp in a given *t-sc* graph. This is done in the following

Theorem 2.1. *Let $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ be t-sc. Let σ be an isomorphism from G_i to G_{i+1}, for all i, $1 \le i \le t-2$. If σ^s is the identity permutation, for some $s \ne t-1$, $1 \le s \le 2t-3$, then σ is an scp for $\mathcal{G}$.*

Proof: We first prove that σ is an isomorphism from G_{t-1} to G_t.

Let $uv \in E(G_{t-1})$. We shall show that $\sigma(u)\sigma(v) \in E(G_t)$. Clearly, $\sigma(u)\sigma(v) \in E(G_i)$ for some i, $1 \le i \le t$. Suppose first $2 \le i \le t-1$. Then $uv \in E(G_{i-1})$. So $i-1 = t-1$, a contradiction.

Suppose next $i = 1$. We then consider two ranges of s separately. Suppose first $1 \le s \le t-2$. Then $\sigma(u)\sigma(v) \in E(G_1)$ which implies that $uv = \sigma^s(u)\sigma^s(v) = \sigma^{s-1}(\sigma(u))\sigma^{s-1}(\sigma(v)) \in E(G_{1+s-1}) = E(G_s)$, a contradiction since $s \le t-2$ and $uv \in E(G_{t-1})$.

Suppose next $t \le s \le 2t-3$. Then $\sigma(u)\sigma(v) \in E(G_1) \to \sigma^2(u)\sigma^2(v) \in E(G_2) \to \cdots \to \sigma^{s-t+2}(u)\sigma^{s-t+2}(v) \in E(G_{s-t+2})$ (since $s-t+2 \le t-1$). But $uv \in E(G_{t-1}) \to \sigma^{-1}(u)\sigma^{-1})(v) \in E(G_{t-2}) \to \sigma^{s-1}(u)\sigma^{s-1}(v) \in E(G_{t-2}) \to \sigma^{s-2}(u)\sigma^{s-2}(v) \in E(G_{t-3}) \to \cdots \to \sigma^{s-t+2}(u)\sigma^{s-t+2}(v) \in E(G_1)$. Hence $s-t+2 = 1 \to s = t-1$, a contradiction.

Thus $i = 1$ also leads to a contradiction. Hence $i = t$ and we have proved that $\sigma(u)\sigma(v) \in E(G_t)$.

Conversely let $uv \notin E(G_{t-1})$. We shall prove that $\sigma(u)\sigma(v) \notin E(G_t)$. Suppose $\sigma(u)\sigma(v) \in E(G_t)$. Let $E^* = \{\sigma(w)\sigma(x) \mid wx \in E(G_{t-1})\}$. Then by our earlier reasoning $E^* \subseteq E(G_t)$. Also $\sigma(u)\sigma(v) \in E(G_t) - E^*$. So $|E(G_t)| > |E^*| = |E(G_{t-1})|$. This is a contradiction since G_{t-1} is isomorphic to G_t.

This proves that σ is an iosmorphism from G_{t-1} to G_t also. The theorem now follows from Lemma 1.1.

Corollary 2.2. *Let $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ be t-sc. Let σ be an isomorphism from G_i to G_{i+1}, for all i, $1 \le i \le t-2$. If every cycle of σ is of the same length s for some $s \ne t-1$, $1 \le s \le 2t-3$, then σ is an scp for $\mathcal{G}$.*

Proof: Clearly $\sigma^s =$ identity for some $s \ne t-1$, $1 \le s \le 2t-3$. The proofs now follows from Theorem 2.1.

Corollary 2.3. *Let $\mathcal{G} = (G_1, G_2, G_3)$. If there is an isomorphism σ from G_1 to G_2 such that all cycles of σ have length 3 then σ is an scp for $\mathcal{G}$.*

In Figure 2.1, we illustrate Theorem 2.1 for $s = t = 3$. Here $\sigma = (124)(365)$ is an scp for $\mathcal{G} = (G_1, G_2, G_3)$ since it is an isomorphism from G_1 to G_2.

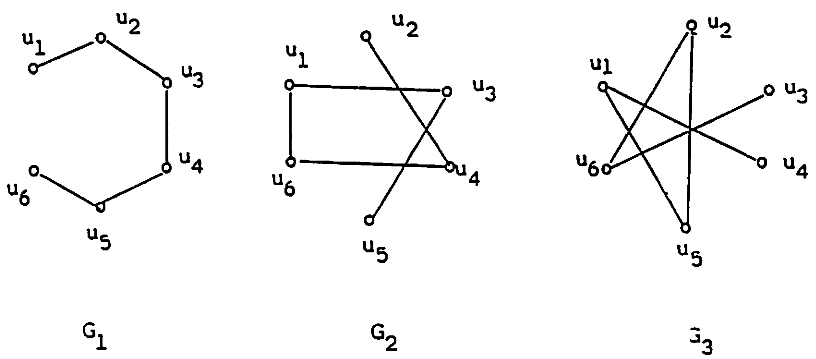

Figure 2.1

3. On the Stringency of the Sufficient Condition

In this section, for every odd $t \geq 7$, we construct a *t-sc* graph without an scp, thereby showing that in Theorem 2.1, the maximum value of i cannot be reduced from $t-2$ in general. This in a way restricts the scope of any improved version of the theorem. Our construction is as follows:

Construction 3.1: Let $n \geq 3$ and $t = 2n+1$. Define $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ where $V(G_i) = \{u_1, u_2, \ldots, u_{2n+1}\}$ for all i, $1 \leq i \leq t$ and

$$E(G_i) = \begin{cases} \{u_i u_{i+j} \mid j = 2, 3, \ldots, n\} \cup \{u_{i+1} u_{i+2}\} & \text{if } 1 \leq i \leq t-2 \\ \{u_{2n} u_j \mid 1 \leq j \leq n-1\} \cup \{u_{2n+1} u_2\} & \text{if } i = t-1 \\ \{u_{2n+1} u_j \mid 3 \leq j \leq n\} \cup \{u_{2n+1} u_1, u_1 u_2\} & \text{if } i = t \end{cases}$$

where all u-subscripts are taken modulo $(2n+1)$.

Figure 3.1 illustrates the construction for $n = 4$.

For the rest of this section, $\mathcal{G}$, G_i will always be as in Construction 3.1. Further, we shall denote by σ the permutation $(u_1 u_2 \ldots u_t)$. We now prove several properties of $\mathcal{G}$.

Lemma 3.1. *The permutation σ is an isomorphism from G_i to G_{i+1}, $1 \leq i \leq t-3$.*

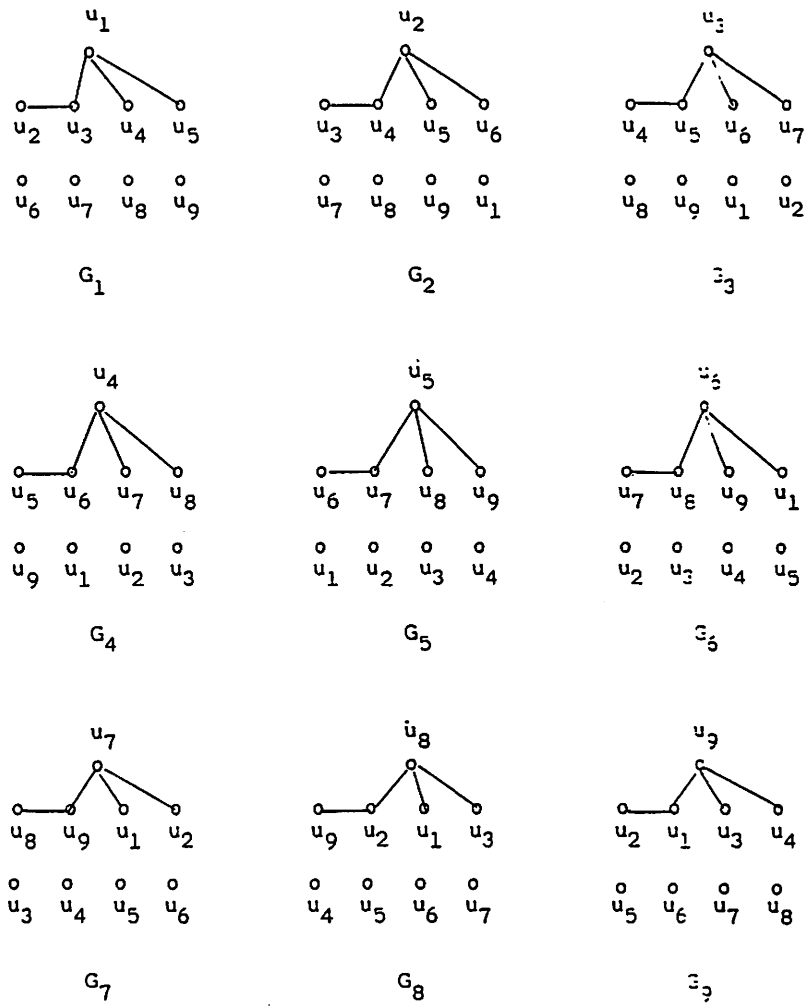

Figure 3.1: An illustration of Construction 3.1 for $n = 4$.

Proof: Let $1 \leq i \leq t-3$ and $w, x \in V(G_i)$. Now

$$\begin{aligned} wx \in E(G_i) &\Leftrightarrow \text{either } wx = u_i u_{i+j} \text{ for some } j, 2 \leq j \leq n \\ &\qquad \text{or } wx = u_{i+1} u_{i+2} \\ &\Leftrightarrow \text{either } \sigma(w)\sigma(x) = u_{i+1} u_{i+1+j} \text{ for some } j, \\ &\qquad 2 \leq j \leq n \text{ or } \sigma(w)\sigma(x) = u_{i+1+1} u_{i+1+2} \\ &\Leftrightarrow \sigma(w)\sigma(x) \in E(G_{i+1}) \end{aligned}$$

This proves the lemma.

Lemma 3.2. *G_{t-2}, G_{t-1} and G_t are isomorphic.*

Proof: This follows since

$$\sigma_{t-2} = (1)(2n-1\ 2n\ 2n+1\ 2\ 3\ 4 \ldots n-1) \prod_{i=n}^{2n-2} (i)$$

is an isormophism from G_{t-2} to G_{t-1} and

$$\sigma_{t-1} = (2n\ 2n+1\ 2\ 1\ 3\ 4\ 5 \ldots n) \prod_{i=n+1}^{2n-1} (i)$$

is an isomorphism from G_{t-1} to G_t. Note that σ_{t-2} and σ_{t-1} are both well defined since $n \geq 3$.

Lemma 3.3. *The graph $\mathcal{G}$ is t-sc.*

Proof: Note that

$$\begin{aligned}\bigcup_{i=1}^{t-2} E(G_i) &= \{u_i u_{i+j} \mid 2 \leq j \leq n, 1 \leq i \leq t-2\} \cup \{u_{i+1}u_{i+2} \mid 1 \leq i \leq t-2\} \\ &= \left(\bigcup_{j=2}^{n} \{u_i u_{i+j} \mid 1 \leq i \leq t-2\}\right) \cup \{u_{i+1}u_{i+2} \mid 1 \leq i \leq t-2\}\end{aligned}$$

and

$$\begin{aligned}E(G_{t-1}) \cup E(G_t) &= \{u_{t-1}u_{t-1+j} \mid 2 \leq j \leq n\} \cup \{u_t u_{t+j} \mid 3 \leq j \leq n\} \cup \\ &\qquad \{u_t u_1, u_t u_2, u_1 u_2\} \\ &= \left(\bigcup_{j=2}^{n} \{u_i u_{i+j} \mid t-1 \leq i \leq t\}\right) \cup \{u_t u_1, u_1 u_2\}.\end{aligned}$$

So

$$\begin{aligned}\bigcup_{i=1}^{t} E(G_i) &= \left(\bigcup_{j=2}^{n} \{u_i u_{i+j} \mid 1 \leq i \leq t\}\right) \cup \{u_i u_{i+1} \mid 1 \leq i \leq t\} \\ &= \bigcup_{j=1}^{n} \{u_i u_{i+j} \mid 1 \leq i \leq t\} = \bigcup_{i=1}^{t} \{u_i u_{i+1} \mid 1 \leq j \leq n\} \\ &= \bigcup_{i=1}^{t} \{u_i u_j \mid i+1 \leq j \leq i+n\} = \bigcup_{i=1}^{t-1} \{u_i u_j \mid i+1 \leq j \leq t\}\end{aligned}$$

This proves that K_t is a disjoint union of the graphs $G_1, G_2, \ldots, G_t$. The lemma now follows from Lemmas 3.1 and 3.2.

Theorem 3.4. *The graph $\mathcal{G}$ is t-sc without any scp, although σ is an isomorphism from G_i to G_{i+1}, $1 \le i \le t-3$.*

Proof: By Lemmas 3.1 and 3.3, it is enough to prove that $\mathcal{G}$ has no scp. Suppose π is an scp of $\mathcal{G}$. We consider two cases:

Case 1: $n \ge 4$. Then u_i is the only point of degree $n-1$ in G_i, $1 \le i \le t$. So $\pi = (u_1 u_2 u_3 \ldots u_t)$. But then $u_t u_1 \in E(G_t)$ whereas $\pi(u_t)\pi(u_1) = u_1 u_2 \notin E(G_1)$, a contradiction.

Case 2: $n = 3$. By Construction 3.1, either $\pi = (u_1 u_2 \ldots u_7)$ and we get a contradiction as in Case 1, or, $\pi(u_4) = u_3$, $\pi(u_3) = u_2$, and then although $u_3 u_4 \in E(G_2)$, $\pi(u_3)\pi(u_4) = u_2 u_3 \notin E(G_3)$, also a contradiction.

Thus, in each case, we obtain a contradiction, and hence, $\mathcal{G}$ has no scp. This proves the theorem.

We conclude the section by constructing a 5-*sc* graph $\mathcal{G} = (G_1, G_2, G_3, G_4, G_5)$ which has no scp although there is an isomorphism σ from G_i to G_{i+1} for $i = 1, 2$. This is given in Figure 3.2.

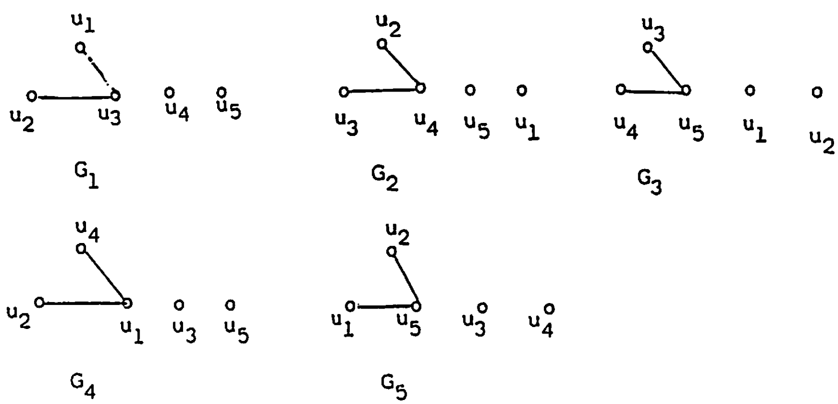

Figure 3.2

The graph $\mathcal{G} = (G_1, G_2, \ldots, G_5)$ has no scp since if π was an scp for $\mathcal{G}$ then by construction $\pi(u_4) = \pi(u_1) = u_5$ a contradiction, even though $\sigma = (u_1 u_2 u_3 u_4 u_5)$ is an isomorphism from G_i to G_{i+1}, $i = 1, 2$.

4. An Infinite Class of *t-sc* Graphs with no SCP-*t* Odd

In this section, for every odd integer $t \ge 3$, we construct a graph with certain properties, thereby showing that the range of s in Theorem 2.1 cannot be extended to include certain values. This in a way restricts the scope of any improved version of the theorem. Our construction is as follows:

Construction 4.1: Let $t = 2n + 1$. Define $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ where $V(G_i) = \{u_1, u_2, \ldots, u_{t-1}, v\}$ and

$$E(G_i) = \begin{cases} \{u_{2n+i-1}v\} \cup \{u_{i+j}u_{n+i-j-1}, u_{n+i+j}u_{2n+i-j-2} \,|\, j = 0, 1, \ldots, m-1\} \\ \quad \text{if } n = 2m+1 \text{ and } 1 \le i \le t-1 \\ \{u_{2n+i-1}v\} \cup \{u_{i+j}u_{n+i-j-1} \mid j = 0, 1, \ldots, m-1\} \\ \quad \{u_{n+i+j}u_{2n+i-j-2} \mid j = 0, 1, \ldots, m-2\} \\ \quad \text{if } n = 2m \text{ and } 1 \le i \le t-1 \\ \{u_j u_{n+j} \; j = 1, 2, \ldots, n\} \text{ if } i = t. \end{cases}$$

where all u-subscripts are taken modulo $(2n)$.

For the rest of this section, $\mathcal{G}$, G_i will always be as in Construction 4.1. Further, we shall use the graph G_i' repeatedly, where $G_i' = G_i - v$, $1 \le i \le t-1$. Also σ' will denote the permutation $(u_1 u_2 \ldots u_{t-1})$ and σ the permutation $(u_1 u_2 \ldots u_{t-1})(v)$.

Below, in Figures 4.1 and 4.2, we illustrate Construction 4.1 for $t = 7$ and $t = 9$ respectively.

We say that an edge $u_k u_\ell$ of G_j is an *i-pair* in G_j if k, $\ell < t$, and $|k - \ell| = i$ (mod $t-1$), $1 \le j \le t$, $1 \le i \le n$.

In order to prove that $\mathcal{G}$ has certain properties we make use of the notion of 'elegant numberings of a graph'. This is defined below.

A *numbering* N of a graph G, which has p points and q edges is an assignment of nonnegative integers to the points and edges of G so that each point v receives a distinct number $N(v)$ and each edge uv receives the number $N(uv) = |N(u) - N(v)|$. Further N is called an *elegant* numbering of G if the edges of G receive the numbers $1, 2, \ldots, q$ and the maximum number assigned by N to a point of G is $\max(p, q)$. We call G *elegant* if G has an elegant numbering.

The notion of elegant numbering is clearly a generalization of the notion of graceful numbers (Golumb [9]). It is evident that if $q \ge p$, then there is no difference between an elegant numbering and a graceful numbering. Thus in particular, all connected graphs are graceful if and only if they are elegant. Further results on elegant numberings are presented in [2]. We now state a series of results that lead to Theorem 4.9, the main result of this section. Except for Lemma 4.7, all others require straightforward verifications and are thus stated without proof.

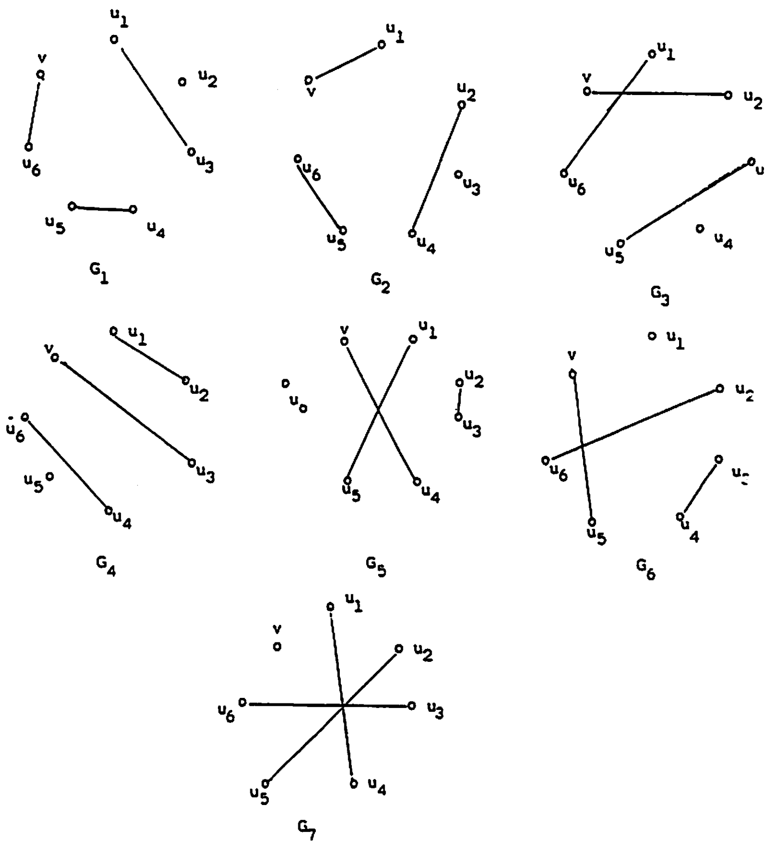

Figure 4.1: An illustration of Construction 4.1 for $t = 7$.

Theorem 4.2. *The numbering N defined by $N(u_i) = i$ for all $i = 1, 2, \ldots, t-1$, is an elegant numbering of G_1'.*

Lemma 4.3. *Let $1 \leq i \leq t-1$. The graph G_i consists of $n+1$ components, n of which are K_2's and the $(n+1)$st component consists of the single point u_{m+i} if $n = 2m+1$ and consists of the single point $u_{n+m+i-1}$ if $n = 2m$. Further G_t is also isomorphic to G_{t-1}, and v is of degree zero in G_t.*

Corollary 4.4. *The graph G_i has exactly n edges, for all i, $1 \leq i \leq t$ and the graph G_i' has exactly $n-1$ edges; for all i, $1 \leq i \leq t-1$.*

Lemma 4.5. *σ is an isomoprhism from G_i to G_{i+1} for all i, $1 \leq i \leq t-2$.*

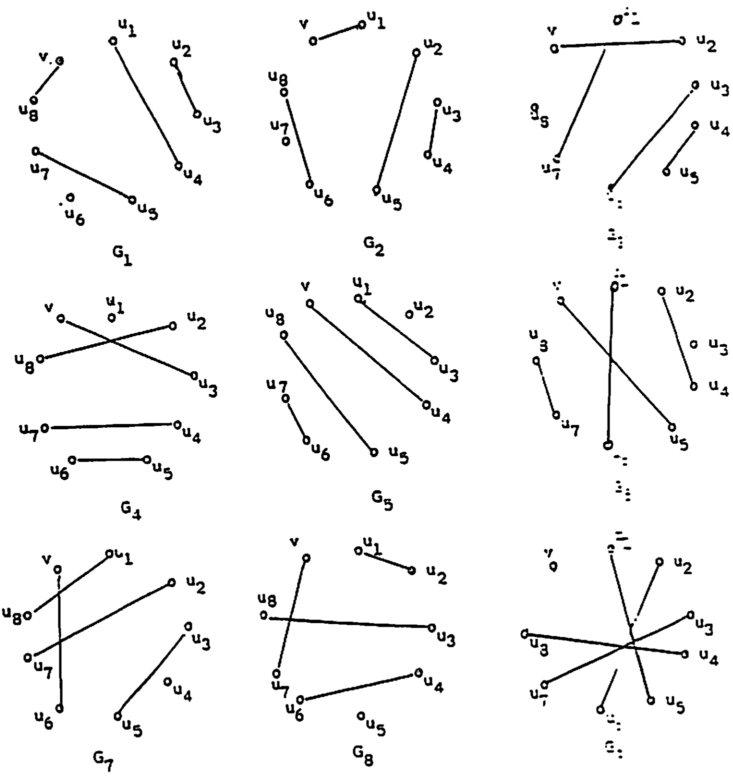

Figure 4.2: An illustration of Construction 4.1 for $t = 9$.

Corollary 4.6. *σ' is an isomorphism from G'_i to G'_{i+1} for all i, $1 \leq i \leq t-2$.*

Lemma 4.7. *For all j, $1 \leq j \leq t-1$, the edge set of G_j consists of exactly one edge incident with v and exactly one i-pair for each i, $1 \leq i \leq n-1$.*

Proof: Let $1 \leq j \leq t-1$. By Construction 4.1, each G_j contains exactly one edge incident with v, namely $u_{2n+j-1}v$. Thus we have to show that $E(G'_j)$ consists of exactly one i-pair for each i, $1 \leq i \leq n-1$. For $j = 1$, this follows since by Theorem 4.2, G'_1 is elegant with $N(u_k) = k$ for all k, $1 \leq k \leq t-1$. By Corollary 4.6, $\sigma' = (12\ldots t-1)$ is an isomorphism from G'_j to G'_{j+1} for all j, $1 \leq j \leq t-2$. Now suppose $u_k u_\ell$ is an i-pair in G_j. Then $\sigma'(u_k) = u_{k+1}$ and $\sigma'(u_\ell) = u_{\ell+1}$ where subscripts are modulo $t-1$ and so $\sigma'(u_k)\sigma'(u_\ell)$ is an i-pair in G'_{j+1}. It now follows by induction on j

that $E(G'_j)$ consists of exactly one i-pair for each i, $1 \le i \le n-1$, since by Corollary 4.4, $|E(G'_j)| = n-1$. This proves the lemma.

Lemma 4.8. *Let $k, \ell < t$. If for some $i, 1 \le i \le n-1$, e_1 is an i-pair in G_k and e_2 is an i-pair in G_ℓ then $e_1 \ne e_2$.*

We are now ready to prove the main theorem of this section.

Theorem 4.9. *Let $\mathcal{G}$ be as in Construction 4.1. Then $\mathcal{G}$ is t-sc without any scp, although for each i, $1 \le i \le t-2$, $\sigma = (u_1u_2 \dots u_{t-1})(v)$ is an isomorphism from G_i to G_{i+1}.*

Proof: We prove the theorem through a series of claims stated below.

1. The graphs $G_1, G_2, \dots, G_t$ are all isomorphic. This follows from Lemma 4.3.

2. G_k and G_ℓ have no edge in common, $1 \le k < \ell \le t$. We prove number 2 in two cases.

   Case 1: $\ell = t$. Clearly G_t has no edge in common with G_k since every edge in G_t is an n-pair and no edge is incident with v, whereas by Lemma 4.7, an edge in G_k is either incident with v, or is an i-pair, $1 \le i \le n-1$.

   Case 2: $\ell < t$. Suppose e is an edge common to both G_k and G_ℓ. Then if e is incident with v then by Construction 4.1, $u_{2n+k-1} = u_{2n+\ell-1} \to k = \ell$, a contradiction. So e is not incident with v. By Lemma 4.7, e is an i-pair for some i, $1 \le i \le n-1$, in both G_ℓ and G_k. But then by Lemma 4.8, $e \ne e$, a contradiction. This proves 2.

3. $\bigcup_{i=1}^{t} G_i = K_t$. By Corollary 4.4, $|E(G_i)| = n$ for all i, $1 \le i \le t$. By 2, it now follows that $\bigcup_{i=1}^{t} G_i$ is a graph with t points and $nt = \frac{t(t-1)}{2}$ edges. This proves 3.

4. $\mathcal{G}$ has no scp.

   If possible let σ^* be an scp. We consider the following two cases:

   Case 1: $n = 2m+1$. By Lemma 4.3, u_{m+i} is the only point of degree 0 in G_i, $1 \le i \le t-1$ and v is the only point of degree 0 in G_t. So $\sigma^* = (u_{m+1}u_{m+2} \dots u_m v)$, a contradiction since $u_1u_{n+1} \in E(G_t)$, but $\sigma^*(u_1)\sigma^*(u_{n+1}) = u_2u_{n+2} \notin E(G_i)$.

   Case 2: $n = 2m$. By Lemma 4.3, $u_{n+m+i-1}$ is the only point of degree 0 in G_i, $1 \le i \le t-1$ and v is the only point of degree 0 in G_t. So $\sigma^* = (u_{n+m}u_{n+m+1} \dots u_{n+m-1}v)$, a contradiction since $u_1u_{n+1} \in E(G_t)$, but $\sigma^*(u_1)\sigma^*(u_{n+1}) = u_2u_{n+2} \notin E(G_1)$.

   Thus we get a contradiction in either case and hence 4 is proved.

From 1, 2, and 3 it follows that $\mathcal{G}$ is t-sc and 4 shows that $\mathcal{G}$ has no scp. The theorem now follows from Lemma 4.5.

5. An Infinite Class of t-sc graphs with no SCP-t Even

In this section, for every even integer $t \geq 4$, we construct a t-sc graph with certain properties, therby showing that the range of s in Theorem 2.1 cannot be extended to include certain values. This in a way restricts the scope of any improved version of the theorem. Our construction is as follows:

Construction 5.1: Let $n \geq 2$ and $t = 2n$. For all i, $1 \leq i \leq t-1$, let G_i' be the graph with $V(G_i') = \{u_1, u_2, \ldots, u_{2t-2}\}$, and

$$E(G_i') = \{u_i u_{i+j}, u_{i+t-1} u_{i+t+j-1} \mid j = 1, 2, \ldots, t-2\} \cup \{u_i u_{i+t-1}\},$$

where all subscripts are taken modulo $2(t-1)$.

Then define $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ where

$$V(G_i) = \{u_1, u_2, \ldots, u_{2(t-1)}, v_1, v_2\}, 1 \leq i \leq t \text{ and}$$

$$E(G_i) = \begin{cases} E(G_i') \cup \{u_i v_{2-\delta_i}, u_{i+t-1} v_{1+\delta_i}\} \text{ if } 1 \leq i \leq t-1, \\ \quad \text{where } \delta_i = 1 \text{ for odd } i \text{ and } \delta_1 = 0 \text{ for even } i, \text{ all} \\ \quad u\text{-subscripts being taken modulo } 2(t-1). \\ \quad \{u_{2t-1} u_{2j}, u_{2t} u_{2j-1} \mid j = 1, 2, \ldots, t-1\} \cup \{v_1 v_2\} \text{ if } i = t. \end{cases}$$

We illustrate Construction 5.1 for $t = 6$ in Figure 5.1.

For the rest of this section the symbols $\mathcal{G}$, G_i, u_i, v_1, v_2 will be as in Construction 5.1. Further, σ will denote the permutation $(u_1 u_2 \ldots u_{2(t-2)})(v_1 v_2)$.

We now state without proof, a series of Lemmas that lead directly to Theorem 5.5, the main result of this section.

Lemma 5.2. *The permutation σ is an isomorphism from G_i to G_{i+1}, $1 \leq i \leq t-2$.*

Lemma 5.3. *Let π be a function from $V(G_{t-1})$ to $V(G_t)$ defined by*

$$\begin{array}{rlrl} \pi(u_{t-1}) = v_1 & & \pi(u_{2t-2}) = v_2 \\ \pi(u_{2t-1}) = u_{2t-2} & & \pi(u_{2t}) = u_{2t-3} \\ \pi(u_i) = u_{2i-1}, & \textit{for all} & i = 1, 2, \ldots, t-2 \\ \pi(u_{t+i}) = u_{2i+2}, & \textit{for all} & i = 0, 1, \ldots, t-3 \end{array}$$

Then π is an isomorphism from G_{t-1} to G_t.

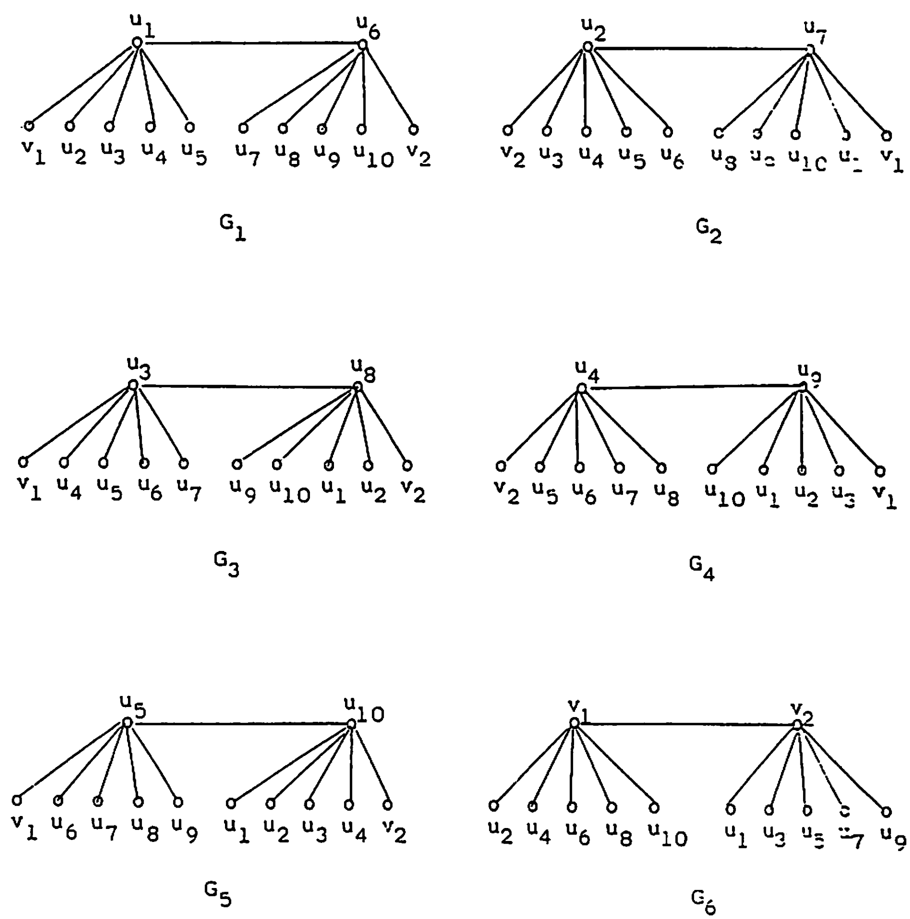

Figure 5.1: An illustration of Construction 5.1 for $t = 6$.

Lemma 5.4. $|\bigcup_{k=1}^{t}(E(G_k))| = 2t_{c_2}$.

Theorem 5.5. *Let $\mathcal{G} = (G_1, G_2, \ldots, G_t)$. Then $\mathcal{G}$ is t-sc and $\mathcal{G}$ has no scp although σ is an automorphism from G_i to G_{i+1}, $1 \le i \le t-2$.*

Proof: By Lemma 5.2 and Lemma 5.3, $G_1, G_2, \ldots, G_t$ are all isomorphic. By Lemma 5.4, $|\bigcup_{i=1}^{t}(E(G_i))| = |E(K_{2t})|$. From Construction 5.1, it follows that $|E(G_i)| = 2t - 1$ for all $i = 1, 2, \ldots, t$. Thus

$$|\sum_{i=1}^{t} E(G_i)| = t(2t-2) = |\bigcup_{i=1}^{t}(E(G_i))|.$$

Hence $E(G_1), E(G_2), \ldots, E(G_t)$ are disjoint and thus K_{2t} is a disjoint union of the graphs $G_1, G_2, \ldots, G_t$. This proves that $\mathcal{G}$ is *t-sc*.

Suppose now $\mathcal{G}$ has an scp π. Since $t \geq 4$, clearly G_1, G_2, G_3 and G_t are all distinct. Now since π is an isomorphism from G_t to G_1 and since v_1 is of degree t in G_t and u_1, u_t are the only two vertices of degree t in G_1 it follows that either $\pi(v_1) = u_1$ or $\pi(v_1) = u_t$.

We consider two cases accordingly:

Case 1: $\pi(v_1) = u_1$. So $\pi(v_2) = u_t$. Now π is an isomorphism from G_1 to G_2. Since $u_1u_{t+1}, u_2u_t \in E(G_2)$, it follows that $v_1\pi^{-1}(u_{t+1})$, $v_2\pi^{-1}(u_2) \in E(G_1)$. From Construction 5.1, it follows that $\pi^{-1}(u_{t+1}) = u_1$ and $\pi^{-1}(u_2) = u_t$. But π is also an isomorphism from G_2 to G_3. However, u_{t+1} is adjacent to both v_1 and u_1 in G_2 whereas in G_3 there is no point adjacent to both $\pi(v_1) = u_1$ and $\pi(u_1) = u_{t+1}$. This gives a contradiction in Case 1.

Case 2: $\pi(v_1) = u_t$. So $\pi(v_2) = u_1$. Now π is an isomorphism from G_1 to G_2. Since $u_1u_{t+1}, u_2u_t \in E(G_2)$, it follows that $v_2\pi^{-1}(u_{t+1})$, $v_1\pi^{-1}(u_2) \in E(G_1)$. From Construction 5.1, it now follows that $\pi^{-1}(u_{t+1}) = u_t$ and $\pi^{-1}(u_2) = u_1$. But π is also an isomorphism from G_2 to G_3. However, u_{t+1} is adjacent to both v_1 and u_1 in G_2, whereas, in G_3 there is no point adjacent to both $\pi(v_1) = u_t$ and $\pi(u_1) = u_2$. This gives a contradiction in Case 2.

Thus in either case we reach a contradiction and this proves that the *t-sc* graph $\mathcal{G}$ has no scp. The theorem now follows from Lemma 5.2.

6. Conclusion

The examples in Sections 4 and 5 show that the range of s in Theorem 2.1 cannot be extended to include the values $t-1$ and $2(t-1)$ in general. We have also shown in Section 3 that the maximum value of i in Theorem 2.1 cannot be reduced from $t-2$ in general. These restrictions limit the scope of any improved version of Theorem 2.1.

References

[1] C.R.J. Clapham, Hamiltonian arcs in self-complementary graphs, *Discrete math.*, 8 (1974), 251–255.

[2] T. Gangopadhyay, Elegant Numberings of a Graph, in process.

[3] T. Gangopadhyay, On the Existence of a Canonical Stable Complementing Permutation for *t-sc* Graphs, submitted.

[4] T. Gangopadhyay, The Class of *t-sc* Graphs and Their Stable Copmlementing Permutations, submitted.

[5] T. Gangopadhyay and S.P. Rao Hebbare, Multipartite self-complementary graphs, *Ars Combinatoria* 13 (1982), 87–114.

[6] T. Gangopadhyay S.P. Rao Hebbare, Paths in r-partite self-complementary Graphs, *Discrete Math.*, **32** (1980), 229–244.

[7] T. Gangopadhyay and S.P. Rao Hebbare, r-partite self-complementary Graphs—diameters, *Discrete Math.*, **32** (1980), 245–256.

[8] R.A. Gibbs, Self-complementary Graphs, *J. Comb. Theory* **16** (1974), 106–123.

[9] S.W. Golomb, "How to number a graph", *Graph Theory and Computing* (R.C. Read, ed.), Academic Press, New York, 1972, pp. 23–37.

[10] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[11] S.B. Rao, Explored, Semi-explored, and unexplored territories in the structure theory of self-complementary graphs and digraphs, *Proceedings of the Symposium on Graph Theory, I.S.I. Lecture Notes* **4**, Macmillan Co. (Ed. A.R. Rao).

[12] G. Ringel, Selbs Komplementäre Graphen, *Arch. Math.*, **14** (1963), 354–358.

[13] H. Sachs, Über Selbstkoplementäre Graphen, *Pub. Math. Debrecen* **9** (1962), 270–288.