On the existence of skew Room frames of type t^u

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ABSTRACT. It is shown that if $t > 1$ and $u \geq 5$, then the known necessary condition for the existence of a skew Room frame of type t^u, is also sufficient with the possible exception of (u, t) where $u = 5$ and $t \in \{11, 13, 17, 19, 23, 29, 31, 41, 43\}$.

1 Introduction

Let S be a finite set, let ∞ be a "special" symbol not in S, and let $\mathcal{H}$ be a set of subsets of S. As defined in [20] a *holey Room square* (briefly HRS) having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array F, indexed by S, which satisfies the following properties:

(1) every cell of F either is empty or contains an unordered pair of symbols of $S \cup \{\infty\}$.

(2) every symbol of $S \cup \{\infty\}$ occurs at most once in any row or column of F, and every unordered pair of symbols occurs in at most one cell of F.

(3) the subarrays $H \times H$ are empty, for every $H \in \mathcal{H}$ (the subarrays are referred to as *holes*).

(4) symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{H \in \mathcal{H}}(H \times H)$; and symbol ∞ occurs in row or column t if and only if $t \in S \setminus \cup_{H \in \mathcal{H}} H$.

(5) the pair $\{s, t\}$ occurs in F if and only if $(s, t) \in (S \times S) \setminus \cup_{H \in \mathcal{H}}(H \times H)$; the pair $\{\infty, t\}$ occurs in F if and only if $t \in S \setminus \cup_{H \in \mathcal{H}} H$.

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The *order* of F is $|S|$. Note that ∞ does not occur in any cell of F if $\cup_{H\in\mathcal{H}} H = S$.

If $\mathcal{H} = \emptyset$, then an HRS($\mathcal{H}$) is called a *Room square* of side $|S|$. Also, if $\mathcal{H} = \{H\}$, then an HRS($\mathcal{H}$) is called an $(|S|, |H|)$-IRS. Here IRS stands for "incomplete Room square"

If $\mathcal{H} = \{S_1, \ldots, S_n\}$ is a partition of S, then an HRS($\mathcal{H}$) is called a *Room frame*. The *type* of F is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \le i \le k$. As is usually done in the literature, we shall refer to a Room frame as a *frame*.

If $\mathcal{H} = \{S_1, \ldots, S_n, H\}$, where $\{S_1, \ldots, S_n\}$ is a partition of S, then an HRS($\mathcal{H}$) is called an *incomplete frame* or an *I-frame*. the *type* of the I-frame is defined to be the multiset $\{(|S_i|, |S_i \cap H|) : 1 \le i \le n\}$ We may also use an "exponential" notation to describe types of I-frames: a type $(t_1, r_1)^{u_1} (t_2, r_2)^{u_2} \ldots (t_k, r_k)^{u_k}$ denotes u_i occurrences of (t_i, r_i), $1 \le i \le k$.

A holey Room square F having hole set $\mathcal{H}$ is called *skew* if, given any pair of cells (s,t) and (t,s) in $(S \times S) \cup_{H\in\mathcal{H}} (H \times H)$, precisely one is empty. Similarly we have the concepts of skew frame and skew I-frame. A skew frame of type T will be denoted by SF(T) and skew I-frame of type T by SIF(T).

Skew Room frames have bee very useful in the constructions of various combinatorial designs (see [7], [11]) such as nested m-cycle systems [12], almost resolvable cycle decompositions [9], Graph decompositions [13] and weakly 3-chromatic linear spaces [14].

It is easily seen that a skew frame of type 1^n is equivalent to a skew Room square of side n. The following result is presented in [18].

Theorem 1.1. *There exists a skew Room sguare of side n if and only if n is an odd positive integer, $n \ne 3$ or 5.*

For the existence of SF(t^u), $t > 1$, an open problem is presented by Dinitz and Stinson in [7].

Open Problem ([7, open problem 15]): Determine necessary and sufficient conditions for the existence of a skew frame of type t^u $(t > 1)$ In the case $t = 2$, prove that there exists a skew frame of type 2^u for all $u \ge 5$. In particular, prove that a skew frame of type 2^6 exists.

For the necessary conditions, the following results can be found in [6], [17].

Theorem 1.2. *There does not exist a (skew) frame of type t^u if any of the following conditions holds:*

(i) $u = 2$ or 3,

(ii) $u = 4$ and $t = 2$,

(iii) $u = 5$ and $t = 1$,

(iv) u is even and t is odd.

Other than type 1^u, the only class of skew frames of type t^u to be investigated are those of type 2^u. The following existence result is shown in [15] and [12].

Theorem 1.3. *Let $u \geq 5$, $u \neq 6, 22, 23, 24, 26, 27, 28, 30, 34$, or 38. Then there is a skew Room frame of type 2^u.*

Some small frames are also known which we collect in the following lemma.

Lemma 1.4. *Skew Room frames exist for the following types:*

(1) [16] $4^4, 4^5$,

(2) [12] $4^{12}, 4^{14}$,

(3) [5] 3^5,

(4) [1] 7^5.

In this paper, we shall first describe constructions for skew frames, both direct and recursive, in Sections 2 and 3 respectively. In Section 4, we delete the possible exceptions in Theorem 1.3 and then give an affirmative answer to the latter part of the Open Problem. In Section 5, we give an affirmative answer to the first part of the Open Problem when $u \geq 6$. For $u = 5$ a similar answer is presented in Section 6 leaving nine undecided types t^5. The main result of this paper can be summarized in the following theorem.

Theorem 1.5. *Let $t > 1$ and $u \geq 5$. A skew Room frame of type t^u does not exist if u is even and t is odd. Otherwise, a skew Room frame of type t^u always exists except possibly when $u = 5$ and $t \in \{11, 13, 17, 19, 23, 29, 31, 41, 43\}$.*

2 Direct constructions

The basic direct construction for frames is the "starter-adder" construction and its modifications (see [5], [16]). Let G be an abelian group, written additively, and let H be a subgroup of G. denote $g = |G|$, $h = |H|$ and suppose that $g - h$ is even. A *frame starter* in $G \setminus H$ is a set of unordered pairs

$$S = \{\{s_i, t_i\} : 1 \leq i \leq (g - h)/2\}$$

satisfying

(1) $\cup_{1\le i\le (g-h)/2}(\{s_i\}\cup\{t_i\}) = G\setminus H$, and

(2) $\cup_{1\le i\le (g-h)/2}\{\pm(s_i - t_i)\} = G\setminus H$.

An *adder* for S is an injection $A\colon S \to G\setminus H$, such that

$$\cup_{1\le i\le (g-h)/2}(\{s_i + a_i\}\cup\{t_i + a_i\}) = G\setminus H,$$

where $a_i = A(s_i, t_i)$, $1 \le i \le (g-h)/2$. An adder A is *skew* if, further,

$$\cup_{1\le i\le (g-h)/2}(\{a_i\}\cup\{-a_i\}) = G\setminus H.$$

We have the following construction for skew frames.

Theorem 2.1 [16, Lemma 3.1]. *Suppose there exist a frame starter S in $G\setminus H$, and a skew adder A for S. Then there is a skew frame of type $h^{g/h}$, where $g = |G|$ and $h = |H|$.*

As above, let G be an abelian group of order g and let H be a subgroup of order h, where $g-h$ is even. A *2k-intransitive frame starter* in $G\setminus H$ is defined to be a triple (S,C,R), where

$$S = \{\{s_i, t_i\}\colon 1\le i\le (g-h)/2-2k\}\cup\{\{u_i\}\colon 1\le i\le 2k\},$$
$$C = \{\{p_i, q_i\}\colon 1\le i\le k\},$$

and

$$R = \{\{p'_i, q'_i\}\colon 1\le i\le k\},$$

satisfying

(1) $\{s_i\}\cup\{t_i\}\cup\{u_i\}\cup\{p_i\}\cup\{q_i\} = G\setminus H$,

(2) $\{\pm(s_i - t_i)\}\cup\{\pm(p_i - q_i)\}\cup\{\pm(p'_i - q'_i)\} = G\setminus H$, and

(3) all $p_i - q_i$ and $p'_i - q'_i$ have even orders in G.

An *adder* for (S,C,R) is an injection $A\colon S\to G\setminus H$, satisfying

(4) $\{s_i + a_i\}\cup\{t_i + a_i\}\cup\{u_i + b_i\}\cup\{p'_i, q'_i\} = G\setminus H$

where

$$a_i = A(s_i, t_i), 1\le i\le (g-h)/2-2k,$$
$$b_i = A(u_i), 1\le i\le 2k.$$

An adder is *skew* if further

(5) $\{a_i\}\cup\{-a_i\}\cup\{b_i\}\cup\{-b_i\} = G\setminus H$,

and for each i, $1 \le i \le k$, there exists a $j \ge 1$ such that $p_i - q_i$ has order $2^j m$ and $p'_i - q'_i$ has order $2^j m'$, where m and m' are odd.

The following result is known.

Theorem 2.2 [16, Lemma 3.4]. *If there is a $2k$-intransitive frame starter and a skew adder in $G \setminus H$, where $g = |G|$ and $h = |H|$, then there is a skew frame of type $h^{g/h}(2k)^1$.*

3 Recursive constructions

In this section, we describe recursive constructions for skew frames. We need some design-theoretic terminology. For those not mentioned in this paper we refer the reader to [2].

A *pairwise balanced design* (v, K)-PBD is a pair $(X, \mathcal{A})$ where X is a v-set (of *points*) and $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality at least two, such that every unordered pair of points is contained in a unique block and that for every block A, $|A| \in K$.

A *holey group divisible design* (briefly HGDD) is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{A})$ where X is a finite set of elements (called *points*), $\mathcal{G}$ is a set of disjoint subsets of X (called *groups*) whose union is X, $\mathcal{H}$ is a set of some subsets of X (called *holes*), and $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that any two points from same group or same hole are contained in no blocks, any two points from different groups and different holes are contained in a unique block.

If $\mathcal{H} = \emptyset$, then an HGDD is the usual group divisible design (GDD) The *type* of the GDD is defined to be the multiset $\{|G| : G \in \mathcal{G}\}$. If $\mathcal{H} = \{H\}$, then an HGDD is usually called an incomplete GDD, denoted by IGDD. The type of the IGDD is the multiset $\{|G|, |G \cap H| : G \in \mathcal{G}\}$. A *transversal design* (or TD(k, t)) is a GDD with group type t^k and all blocks having size k.

Let S be a set and let $\mathcal{H}$ be a set of disjoint subsets of S. A *holey Latin square* having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

(1) every cell of L either is empty or contains a symbol of S.

(2) every symbol of S occurs at most once in any row or column of L.

(3) the subarrays $H \times H$ are empty, for every $H \in \mathcal{H}$ (these subarrays are referred to as *holes*).

(4) symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{H \in \mathcal{H}} (H \times H)$.

Two holey Latin squares on symbol set S and hole set $\mathcal{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S\times S)\setminus\cup_{H\in\mathcal{H}}(H\times H)$. We shall use the notation IMOLS$(s,h_1,\ldots,h_n)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathcal{H}=\{H_1,\ldots,H_n\}$, where $s=|S|$ and $h_i=|H_i|$ for $1\le i\le n$.

If $\mathcal{H}=\emptyset$, then a holey Latin square is the usual Latin square. It is well known that the existence of a TD(k,t) is equivalent to that of $k-2$ mutually orthogonal Latin squares (MOLS) of order t. For a list of lower bounds on the number of MOLS for all orders up to 10000, we refer the reader to Brouwer [3] and [4].

The following two constructions for skew frames can be found in [16].

Theorem 3.1 (Inflation Construction). *Suppose there is a skew Room frame of type $t_1^{u_1}t_2^{u_2}\ldots t_k^{u_k}$, and suppose also that $m\neq 2$ or 6. Then there exists a skew Room frame of type $(mt_1)^{u_1}(mt_2)^{u_2}\ldots(mt_k)^{u_k}$.*

Theorem 3.2 (FFC). *Let $(X,\mathcal{G},\mathcal{A})$ be a GDD, and let $w\colon X\to Z^+\cup\{0\}$ (we say that w is a* weighting*). For every $A\in\mathcal{A}$, suppose there is a skew frame of type $\{w(x)\colon x\in A\}$. Then there is a skew frame of type $\{\Sigma_{x\in G}w(x)\colon G\in\mathcal{G}\}$.*

Denote $SF_t=\{n\colon$ there exists a skew frame of type $t^n\}$. Then we have the following corollary to Theorem 3.2, which means that the set SF_t is PBD-closed.

Lemma 3.3. *Suppose there is an (n,SF_t)-PBD. Then $n\in SF_t$.*

Proof: The hypothesized PBD can be thought of as a GDD in which every group has size 1. Give every point weight t and apply Theorem 3.2.

Suppose F is a Room frame with hole set $\{S_1,\ldots,S_n\}$, where $S=\cup S_i$. A *complete transversal* is a set T of $|S|$ filled cells in F such that every symbol is contained in exactly two cells of T. If the pairs in the cells of T are ordered so that every symbol occurs once as a first coordinate and once as a second coordinate in a cell of T, then T is said to be an *ordered transversal.* Note that any transversal can be ordered, since the union of all the edges in a transversal forms a disjoint union of cycles. If these cycles are arbitrarily oriented, then the direction of each edge provides an ordering for the transversal. A complete ordered transversal will be referred to as a *CO transversal.*

Suppose L_1 and L_2 are IMOLS$(m+u;u)$ on symbol set S and hole set $\mathcal{H}=\{H\}$, where m is an even positive integer. A *holey* row (or column) of L_1 or L_2 is one that meets the hole. A holey row (or column), T, is said to be *partitionable* if the superposition of row (or column) T of L_1 and L_2 can be partitioned into two subsets T_1 and T_2 of $m/2$ cells, so that every symbol of $S\setminus H$ is contained in one cell in each of T_1 and T_2. An IMOLS$(m+u;u)$ is said to be *partitionable* if every holey row and column is partitionable.

We use the notation IMOLS($m+u;u$) to denote IMOLS($m+u;u$) that are transpose of each other.

Example 3.4 There is a partitionable ISOLS(4+1;1). It is shown in Table 1.

1	x	4	3	2
3	2	1	x	4
x	4	3	2	1
2	1	x	4	3
4	3	2	1	

Table 1. Apartitionable ISOLS(4+1;1)

In [20], there is a construction to obtain frames from a frame having disjoint CO transversals. To obtain skew Room frames it suffices to start with a skew frame and replace the partitionable IMOLS($m+u_i;u_i$) by partitionable ISOLS($m+u_i;u_i$). We state the modified construction as follows.

Theorem 3.5. *Suppose there is a skew Room frame of type t^g having l disjoint CO transversals. For $1 \leq i \leq l$, let $u_i \geq 0$ be an integer. Let m be an even positive integer, $m \neq 2$ or 6. Suppose there exist partitionable ISOLS($m+u_i;u_i$), for $1 \leq i \leq l$. Then there is a skew Room frame of type $(mt)^g(2u)^1$, where $u = \Sigma u_i$.*

Theorem 3.6 (Filling in holes). *Suppose there is a skew I-frame of type $\{(s_i,h_i): 1 \leq i \leq n\}$. If there is a skew frame of type $\{h_i: 1 \leq i \leq n\}$, then there ia a skew frame of type $\{s_i: 1 \leq i \leq n\}$.*

Proof: Suppose F is the skew I-frame based on a set S having hole set $\{S_1,\ldots,S_n,H\}$, where $|S_i| = s_i$ and $|S_i \cap H| = h_i$ for $1 \leq i \leq n$. Suppose F_H is a skew frame based on H having hole set $\{S_i \cap H : 1 \leq i \leq n\}$. Define an $|S| \times |S|$ array $F'(s,t)$ as follows:

$$F'(s,t) = \begin{cases} F(s,t), & \text{if } (s,t) \in (S \times S) \setminus (H \times H), \\ F_H(s,t), & \text{if } (s,t) \in H \times H. \end{cases}$$

It is easy to see that F' is a skew frame of type $\{s_i: 1 \leq i \leq n\}$.

The known Construction 2.2 in [8] can be generalized to obtain skew frames. It suffices to replace frame and I-frames by skew onces. We state the modified construction as follows.

Theorem 3.7. *Suppose there is a TD($k+1,t$) and let $e_i \geq 0$, $1 \leq i \leq t$. If there is a skew I-frame of type $(m+e_i,e_i)^k$ for any i, $1 \leq i \leq t$, then there is a skew I-frame of type $(mt+e,e)^k$, where $e = \Sigma e_i$. Further, if there is a skew frame of type e^k, then there ia a skew frame of type $(mt+e)^k$.*

4 Skew frames of type 2^n

In this section we shall construct skew frames of types left in Theorom 1.3.

Lemma 4.1. *There exists a skew frame of type* 2^6.

Proof: Take $G = Z_{10}$ and $H = \{0, 5\}$. Construct a 2-intransitive starter as follows:

$$S = \{\{1,3\},\{2,6\}\} \cup \{8,9\},$$
$$C = \{4,7\},$$
$$R = \{8,9\}.$$

A skew adder is as follows:
$a_1 = 3$, $a_2 = 1$, $A(8) = 4$ and $A(9) = 2$.
Applying Theorem 2.2 yields an SF(2^6).

Lemma 4.2. *There exists a skew frame of type* 2^{22}.

Proof: $G = Z_{32}$, $H = \{0, 16\}$. Take

$$S = \{\{2,6\},\{5,13\},\{11,23\}\} \cup \{14,18,20,25,3,31,27,30,21,15,26,22\},$$
$$C = \{\{9,19\},\{10,24\},\{1,12\},\{4,7\},\{8,17\},\{28,29\}\},$$
$$R = \{\{23,25\},\{30,24\},\{31,26\},\{4,29\},\{15,28\},\{16,21\}\}.$$

and the skew adder:
$a_1 = 31$, $a_2 = 30$, $a_3 = 29$, $A(14) = 28$, $A(18) = 27$, $A(20) = 26$, $A(25) = 25$, $A(3) = 24$, $A(31) = 23$, $A(27) = 22$, $A(30) = 21$, $A(21) = 20$, $A(15) = 19$, $A(26) = 18$, $A(22) = 17$.
We obtain a skew frame of type $2^{16}12^1$. Filling in the hole of size 12 with a skew frame of type 2^6 gives an SF(2^{22}).

Lemma 4.3. *There exists a skew frame of type* 2^{23}.

Proof: We need only to give a skew frame of type $2^{18}10^1$ by starter-adder techniques. Let $G = Z_{36}$ and let $H = \{0, 18\}$. Take

$$S = \{\{2,4\},\{6,10\},\{5,13\},\{11,23\},\{14,30\},\{21,35\},\{15,32\}\}$$
$$\cup \{26,7,22,9,17,24,29,31,33,3\},$$
$$C = \{\{8,34\},\{12,25\},\{1,16\},\{19,28\},\{20,27\}\},$$
$$R = \{\{33,27\},\{30,31\},\{32,29\},\{21,26\},\{13,24\}\}.$$

and the skew adder:
$a_1 = 35$, $a_2 = 34$, $a_3 = 33$, $a_4 = 32$, $a_5 = 31$, $a_6 = 29$, $a_7 = 27$, $A(26) = 30$, $A(7) = 28$, $A(22) = 26$, $A(9) = 25$, $A(17) = 24$, $A(24) = 23$, $A(29) = 22$, $A(31) = 21$, $A(33) = 20$, $A(3) = 19$.

Lemma 4.4. *There exists a skew frame of type 2^{27}.*

Proof: A skew frame of type $2^{20}14^1$ is shown below.

$$\begin{aligned}
&G = Z_{40}, H = \{0,20\},\\
&S = \{\{6,10\},\{5,13\},\{11,23\},\{14,30\},\{12,34\}\}\\
&\quad \cup \{2,19,21,36,32,37,35,24,28,9,7,15,33,3\},\\
&C = \{\{22,16\},\{27,1\},\{38,31\},\{18,29\},\{26,17\},\{4,39\},\{8,25\}\},\\
&R = \{\{39,29\},\{35,37\},\{32,33\},\{16,3\},\{30,5\},\{18,21\},\{36,17\}\}.
\end{aligned}$$

The skew adder is as follows:
$a_i = 39 - i$, $1 \le i \le 5$, and $A(2) = 39$, $A(19) = 33$, $A(21) = 32$, $A(36) = 31$, $A(32) = 30$, $A(37) = 29$, $A(35) = 28$, $A(24) = 27$, $A(28) = 26$, $A(9) = 25$, $A(7) = 24$, $A(15) = 23$, $A(33) = 22$, $A(3) = 21$.

Lemma 4.5. *There exists a skew frame of type 2^{28}.*

Proof: A skew frame of type $2^{22}12^1$ is shown below

$$\begin{aligned}
&G = Z_{44}, H = \{0,22\},\\
&S = \{\{2,6\},\{4,12\},\{7,19\},\{11,27\},\{18,38\},\{1,35\},\{3,21\},\\
&\quad \{26,40\},\{24,43\}\} \cup \{29,31,42,34,5,9,14,20,17,25,28,8\},\\
&C = \{\{15,13\},\{36,33\},\{30,23\},\{32,41\},\{10,37\},\{16,39\}\},\\
&R = \{\{11,17\},\{27,28\},\{36,41\},\{26,37\},\{25,12\},\{24,9\}\}.
\end{aligned}$$

The skew adder is as follows
$a_i = 44 - i$, $1 \le i \le 9$, $A(29) = 34$, $A(31) = 33$, $A(42) = 32$, $A(34) = 31$, $A(5) = 30$, $A(9) = 29$, $A(14) = 28$, $A(20) = 27$, $A(17) = 26$, $A(25) = 25$, $A(28) = 24$, $A(8) = 23$.

Lemma 4.6. *There exists a skew frame of type $2^8 4^1$.*

Proof: Take $G = Z_{16}$, $H = \{0,8\}$, and let

$$\begin{aligned}
&S = \{\{6,4\},\{11,15\},\{7,1\}\} \cup \{5,12,2,13\},\\
&C = \{\{10,3\},\{9,14\}\},\\
&R = \{\{15,2\},\{10,11\}\}.
\end{aligned}$$

The skew adder is as follows:
$a_1 = 15$, $a_2 = 14$, $a_3 = 13$, $A(5) = 12$, $A(12) = 11$, $A(2) = 10$, $A(13) = 9$.

We are now in a position to state the main result of this section.

Theorem 4.7. *There exists a skew frame of type 2^u for all $u \ge 5$.*

Proof: By Theorem 1.3 and the above lemmas, we need only to show the existence for $u = 24, 26, 30, 34$, and 38.

Start with a skew frame of type 4^4 from Lemma 1.4(1) and apply Theorem 3.1 with $m = 3$, we get a skew frame of type 12^4. Filling in the size 12 holes with an SF(2^6) produces an SF(2^{24}).

Take a block in a (31,6,1)-BIBD and delete one or five points from the block, we know that $26, 30 \in B(\{5,6\})$. Take a block in a TD(8,7) and delete all points from the last three groups but keep those points in the block. We have $38 \in B(\{5,6,7,8\})$. By Lemma 3.3 we obtain an SF(2^u) for $u = 26$, 30 and 38.

For a skew frame of type 2^{34} we apply Theorem 3.1 with $m = 4$ and an initial skew frame of type 4^4 to get a skew frame of type 16^4. Filling in the holes but one with a skew frame of type $2^8 4^1$ shown in Lemma 4.6, we have a skew frame of type $2^{24}20^1$. Further filling in the size 20 hole with an SF(2^{10}), we obtain an SF(2^{34}).

This completes the proof.

5 Skew frames of type t^u, $t > 1$, $u \geq 6$

In this section we shall investigate the existence of a skew frame of type t^u for $t > 1$ and $u \geq 6$.

Lemma 5.1. *There exists a skew frame of type $(2h)^u$, $u \geq 5$, $h \neq 2$ or 6.*

Proof: By Theorem 4.7 an SF(2^u) exists for $u \geq 5$. Apply Theorem 3.1.

Theorem 5.2. *There exists a skew frame of type t^u, u odd ≥ 7.*

Proof: For any odd positive integer $u \geq 7$, by Thoorem 1.1, there exists a skew frame of type 1^u. Apply Theorem 3.1 with any positive integer $m \neq 2$ or 6, we get a skew frame of type t^u. Skew frames of type 2^u and 6^u can be obtained by taking $h = 1, 3$ in Lemma 5.1 respectively. The proof is complete.

If u is an even integer and $u \geq 6$, by Theorem 1.2, t must also be even for the existence of a skew frame of type t^u. From Lemma 5.1 what we need to discuss is the type 4^u and 12^u. Denote

$$K_4 = \{4,5,6,7,8,9,10,11,12,14,15,18,19,23\}.$$

We shall apply Lemma 3.3.

Lemma 5.3. *There exists a skew frame of type 4^6.*

Proof: Take $G = Z_{24}$, $H = \{0,6,12,18\}$. Construct a starter as follows

$$S = \{\{1,2\},\{3,5\},\{4,8\},\{7,16\},\{9,19\},\{10,21\},$$
$$\{11,14\},\{13,20\},\{15,23\},\{17,22\}\}.$$

A skew adder is as follows:
$a_1 = 1$, $a_2 = 2$, $a_3 = 9$, $a_4 = 4$, $a_5 = 19$, $a_6 = 11$, $a_7 = 8$, $a_8 = 3$, $a_9 = 10$, $a_{10} = 17$.
Apply Theorem 2.1, the desired result is obtained.

Lemma 5.4. *There exists a skew frame of type* 4^8.

Proof: Take $G = Z_{32}$ and $H = \{0, 8, 16, 24\}$. Construct a starter as follows

$$S = \{\{11,22\},\{12,18\},\{13,20\},\{15,25\},\{17,26\},\{1,6\},\{2,30\},\\ \{3,5\},\{4,21\},\{7,10\},\{9,23\},\{14,27\},\{19,31\},\{28,29\}\}.$$

A skew adder is as follows:
$a_1 = 11$, $a_2 = 7$, $a_3 = 23$, $a_4 = 13$, $a_5 = 3$, $a_6 = 20$, $a_7 = 15$, $a_8 = 2$, $a_9 = 10$, $a_{10} = 5$, $a_{11} = 18$, $a_{12} = 28$, $a_{13} = 31$, $a_{14} = 6$.
Apply Theorem 2.1, a skew frame of type 4^8 is obtained.

Lemma 5.5. *There exists a skew frame of type* 4^{10}.

Proof: Take $G = Z_{40}$ and $H = \{0, 10, 20, 30\}$. Construct a starter as follows

$$S = \{\{14,15\},\{16,28\},\{17,35\},\{18,39\},\{19,33\},\{22,37\},\\ \{13,21\},\{9,15\},\{1,6\},\{2,38\},\{3,4\},\{5,12\},\{7,23\},\\ \{8,31\},\{11,24\},\{26,29\},\{27,36\},\{32,34\}\}.$$

A skew adder is as follows:
$a_1 = 14$, $a_2 = 18$, $a_3 = 21$, $a_4 = 13$, $a_5 = 8$, $a_6 = 7$, $a_7 = 11$, $a_8 = 28$, $a_9 = 16$, $a_{10} = 17$, $a_{11} = 4$, $a_{12} = 9$, $a_{13} = 2$, $a_{14} = 5$, $a_{15} = 34$, $a_{16} = 37$, $a_{17} = 15$, $a_{18} = 1$.
Apply Theorem 2.1, a skew frame of type 4^{10} is obtained.

Lemma 5.6. *There exists a skew frame of type* 4^{18}.

Proof: Take $G = Z_{17}$ and $H = \{0\}$. Construct a starter as follows

$$S = \{\{1,2\},\{3,5\},\{4,10\},\{6,13\},\{7,11\},\{8,16\},\{9,14\},\{12,15\}\}.$$

A skew adder is as follows:
$a_1 = 1$, $a_2 = 5$, $a_3 = 3$, $a_4 = 9$, $a_5 = 7$, $a_6 = 13$, $a_7 = 2$, $a_8 = 11$.
By Theorem 2.1, we obtain a skew frame of type 1^{17}. It is easy to see that the skew frame has two disjoint CO transversals $T_1 = \{(1+i, 2+i) \bmod 17 | 0 \le i \le 16\}$ and $T_2 = \{(3+i, 5+i) \bmod 17 | 0 \le i \le 16\}$. Since there is a partitionable ISOLS(4+1;1) in Example 3.4, by Theorem 3.5, we get a skew frame of type 4^{18}.

Theorem 5.7. *There exists a skew frame of type* 4^u, $u \ge 4$

Proof: Since $\{u \geq 4: u \text{ is an integer }\} = B(K_4)$ (see [2]), by Lemma 3.3 one needs only to construct an SF(4^u) for $u \in K_4$. Theorem 5.2 takes care of all odd $u \geq 7$ in K_4. The values $u = 4, 5, 12, 14$, are provided by Lemma 1.4. The remaining four values $u = 6, 8, 10, 18$ are shown in Lemmas 5.3 - 5.6.

Corollary 5.8. *There exists a skew frame of type* 12^u, $u \geq 4$.

Proof: Apply Inflation Construction with an SF(4^u) and MOLS(3).

We conclude this section with the following theorem.

Theorem 5.9. *Let* $t > 1$ *and* $u \geq 6$. *A skew frame of type* t^u *exists if, and only if,* $t(u-1) \equiv 0 \bmod 2$.

Proof: The second assertion comes from Theorem 1.2 and the first from Theorem 5.2 for u odd and from Lemma 5.1,. Theorem 5.7 and Corollary 5.8 for u even.

6 Skew frames of type t^5, $t > 1$

Lemma 6.1. *There exists a skew frame of type* $(2h)^5$ *for all* $h \geq 1$.

Proof: The conclusion comes from Lemma 5.1, Theorem 5.7 and Corollary 5.8.

By Lemma 6.1 we need only to investigate the existence of skew frames of type t^5 for odd $t > 1$. We shall mainly use Theorom 3.7. For "input" designs we have the following.

Lemma 6.2. *There exists an SF(t^5) for* $t = 3, 5, 7$.

Proof: An SF(5^5) is constructed by Theoreem 2.1. Take $G = Z_{25}$ and $H = \{0, 5, 10, 15, 20\}$. The starter is

$$S = \{\{1,2\},\{3,6\},\{4,22\},\{7,21\},\{8,16\},\{9,13\},\{11,24\},\\ \{12,18\},\{14,23\},\{17,19\}\}$$

The skew adder is as follows:
$a_1 = 1$, $a_2 = 8$, $a_3 = 22$, $a_4 = 16$, $a_5 = 13$, $a_6 = 4$, $a_7 = 23$, $a_8 = 6$, $a_9 = 18$, $a_{10} = 14$.
The other two skew frames come from Lemma 1.4.

Lemma 6.3. *If there is an SF(t^u), then there exists a skew I-frame of type* $(ht, t)^u$, $h \neq 2$ *or 6.*

Proof: Apply Theorem 3.1 with an initial SF(t^u) and MOLS(h). We obtain an SF($(ht)^u$) with a subdesign SF(t^u). Removing the subdesign produces a skew I-frame of type $(ht, t)^u$.

Lemma 6.4. *There exists an SIF($(v,t)^5$) for (v,t) = (8,2), (9,3), (14,2) and (15,3).*

Proof: Apply Lemma 6.3 with an SF(t^5) for $t = 2,3$ and MOLS(h) for $h = 4,3,7,5$.

Lemma 6.5. *If there is a TD($6,v$) and there exists a skew frame of type $(3a+2b)^5$, where $0 \le a+b \le v$, then there exists a skew frame of type $(6v+3a+2b)^5$ or type $(12v+3a+2b)^5$.*

Proof: Take $k = 5$, $m = 6$ or 12, $e_1 = \cdots = e_a = 3$, $e_{a+1} = \cdots = e_{a+b} = 2$, and $a_{a+b+1} = \cdots = e_v = 0$ in Theorem 3.7. The required skew I-frames come from Lemma 6.4.

For the existence of a TD($6,v$) we have from [19] the following

Lemma 6.6. *There exists a TD($6,v$) if $v \ge 5$, $v \ne$ 6,10,14,18,22,26, 30,34 or 42.*

Theorem 6.7. *An SF(t^5) exists if t odd ≥ 261.*

Proof: For any t odd ≥ 261 there is an integer $v \ge 43$ such that $t = 6v+r$, where r = 3, 5, or 7. By Lemma 6.6 we may apply Lemma 6.5 with $a = 1$ and $b \in \{0,1,2\}$.

Lemma 6.8. *An SF(t^5) exists if $45 \le t \le 259$ and t is odd.*

Proof: A similar proof as in Theorem 6.7 will leave the intervals $6v+3 \le t \le 6v+7$ for v =10,14,18,22,26,30,34,42. Let $v' = v/2$. By Lemma 6.6 there is a TD($6,v'$). Therefore, these intervals can be done by applying Lemma 6.5 and writing $6v+r = 12v'+r$, where r = 3,5,7.

Theorem 6.9. *A skew frame of type t^5 exists if $t > 1$ and $t \notin \{11,13,17,19,23,29,31,41,43\}$.*

Proof: Combine Lemma 6.1, Theorem 6.7 and Lemma 6.8, we need only to consider the values of odd $t \le 43$. All but one are multiples of 3, 5, or 7, which can be taken care of by Inflation Construction. The remaining value $t = 37$ can be done by writing $37 = 6 \cdot 5 + 7$ and applying Lemma 6.5.

7 Concludlng remarks

Combining Sections 4, 5, and 6 we have obtained our main result shown in Theorem 1.5. For $t > 1$ and $u \ge 5$,we have solved the existence of skew Room frames of type t^u, leaving nine undecided cases. The only remaining class is the types t^4, t even. Using an SF(4^4) in Lemma 1.4 and the Inflation Construction one obtains an SF(t^4) for $t \equiv 0 \pmod 4$ leaving two undecided cases t = 8, 24. For $t \equiv 2 \pmod 4$ and $t > 1$, other than the nonexistence of an SF(2^4) nothing is known in this class.

For the existence of frames of type t^u without the skew property, an almost complete solution has recently been obtained in [8] and [21], leaving only one possible exception of type 14^4.

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