The Bondage Number of a graph G can be much greater than $\Delta(G)$

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ABSTRACT. The bondage number $b(G)$ of a nonempty graph G was first introduced by Fink, Jacobson, Kinch, and Roberts [3]. In their paper they conjectured that $b(G) \leq \Delta(G)+1$ for a nonempty graph G. A counterexample for this conjecture was shown in [5]. Beyond it we show now that there doesn't exist an upper bound of the following form: $b(G) \leq \Delta(G)+c$ for any $c \in N$.

1 Introduction

Let $G=(V,E)$ be a finite, undirected graph with neither loops nor multiple edges. For $u \in V(G)$ we denote by $N(u)$ the neighborhood of u. More generally we define $N(U)=\cup_{u\in U} N(u)$ for a set $U \subseteq V$ and $N[U]=N(U)\cup U$. A set D of vertices in G is a dominating set if $N[D]=V$. A dominating set of minimum cardinality in G is called a minimum dominating set (MDS), and its cardinality is termed the domination number of G and denoted by $\gamma(G)$.

Fink, Jacobson, Kinch, and Roberts [3] defined the bondage number $b(G)$ of a nonempty graph to be the minimum cardinality among all sets of edges X for which $\gamma(G-X)>\gamma(G)$ holds. Brigham, Chinn, and Dutton [2] defined a vertex v to be critical iff $\gamma(G-v)<\gamma(G)$ and G to be a vertex domination-critical graph (from now on called '*vc*-graph') iff each vertex of G is critical.

For graph theory not presented here we follow [4].

From [1] we know that the bondage number of a graph G is bounded from above by $\Delta(G)$, when G is not a *vc*-graph. For *vc*-graphs it is more difficult to find upper bounds. *vc*-graphs in general are not even bounded from above by $\Delta+1$ [5], which had been conjectured in [3]. In [6] we condensed some new results about the bondage number, among others new sharp upper bounds like $b(G) \leq \lambda(G) + \Delta(G) - 1$, where $\lambda(G)$ is the edge-connectivity number of G. But all results which were found up to now depend on two graph-invariants, mostly including Δ, or appear to be trivial like $b(G) \leq 2\Delta(G) - 1$ which can be derived from the above result. The question whether there is an upper bound of the form $b(G) \leq \Delta(G) + c$ ($c \in N$) is still open and will be solved in this paper by constructing an infinite class of graphs G_i for which the difference between $b(G_i)$ and $\Delta(G_i)$ can be arbitrarily large.

2 The main result

Definition: Let $G_i := K_i \times K_i$ ($i \in N, i \geq 2$) be the Cartesian product of two complete graphs of order i.

Observations:

1. $\gamma(G_i) = i$.
2. G_i is vertex domination-critical.
3. $\Delta(G_i) = 2(i-1)$.

Definition: Let v be a vertex of G_i. We call $A_i(v) := \langle N_{G_i}[v] \rangle$ an i-angle and the vertex v the center of the i-angle $A_i(v)$. Let X be a set of edges. We call an i-angle with center v damaged iff $N_{G_i}[v] \neq N_{G_i - X}[v]$.

Observations: The form of an i-angle depends only on i and never on the chosen vertex v. G_i contains $i^2 = |V(G_i)|$ different i-angles. By removing one edge x out of G_i we damage two different i-angles. To damage all i-angles of G_i, we have to remove at least $\lceil \frac{i^2}{2} \rceil$ edges. We remark that

$$\lceil \frac{i^2}{2} \rceil > 3(i-1) \text{ for } i \geq 5 \tag{1}$$

Lemma 2.1. *Let $G_i = K_i \times K_i$ ($i \geq 2$), and let X be a set of edges with $|X| < 3(i-1)$, such that $\gamma(G_i - X) > \gamma(G_i)$. Then $G_i - X$ has an undamaged i-angle for $i \geq 3$.*

Proof:

i) Let $X := \{k_1, \ldots, k_5\}$ be a set of edges such that $\gamma(G_3 - X) > 3 = \gamma(G_3)$.

   Assume that $G_3 - X$ has no undamaged 3-angle anymore. Then each vertex of G_3 must be incident to an edge of X. Hence exactly one vertex v is incident to two edges of X because we have 5 edges in X but only 9 vertices in G_3. In any case the resulting graph $G_3 - X$ has either three 'parallel' vertical $K_{1,2}$'s or three 'parallel' horizontal $K_{1,2}$'s which are enough to see that $\gamma(G_3 - X) \leq 3$ which is a contradiction.

Thus we know that $G_3 - X$ must have an undamaged 3-angle.

ii) Let $X: = \{k_1, \ldots, k_8\}$ be a set of edges such that $\gamma(G_4 - X) > 4 = \gamma(G_4)$. Assume that $G_4 - X$ has no undamaged 4-angle anymore. Then each vertex of G_4 must be incident to an edge of X, hence X is a perfect matching of G_4, which means each vertex of G_4 is incident to exactly one edge in X. Let D consist of the four vertices of one of the 'main diagonals' of G_4. Then each vertex of G_4 is either in D or adjacent to two vertices of D. Hence each vertex of $G_4 - X$ is either in D or adjacent to at least one vertex of D, which means, $D \in MDS(G_4 - X)$, but since $|D| = 4$ we have a contradiction.

Thus we know that $G_4 - X$ must have an undamaged 4-angle.

iii) From (1) we know that $\lceil \frac{i^2}{2} \rceil > 3(i-1)$ for $i \geq 5$. Hence there must be an undamaged i-angle, such that the proof is complete.

□

Theorem 2.2. *If $G_i = K_i \times K_i$ then $b(G_i) = 3(i-1)$ for $i \geq 2$.*

Proof: We will prove the theorem by the method of induction.

Since $b(G_2) = b(C_4) = 3$ the induction hypothesis is true for $i = 2$.

Lemma 2.7 of [6] says that

$$b(G) \leq \min\{\deg u + \deg v - t + 1; u \text{ and } v \text{ belong to the same } K_t \subseteq G\} \tag{2}$$

Thus we conclude that $b(G_i) \leq 2(i-1) + 2(i-1) - i + 1 = 3(i-1)$. It remains to show that $b(G_i) \geq 3(i-1)$.

Induction step: In the following we show that

$$b(G_i) < 3(i-1) \Longrightarrow b(G_{i-1}) < 3(i-2) \tag{3}$$

holds for $i \geq 3$. This will violate the induction hypothesis and provide the necessary contradiction.

Let $X: = \{k_1, \ldots, k_t\}$, $t < 3(i-1)$ be a set of edges such that $\gamma(G_i - X) > i = \gamma(G_i)$. Now remove $N[v]$ from $G_i - X$, where v is the center of an undamaged i-angle. There must be such a vertex v by Lemma 2.1.

Let $H_{i-1}: = \langle G_i - X - N[v] \rangle$. Then $H_{i-1} \subseteq G_{i-1} = G_i - N[v]$.

$\gamma(H_{i-1}) > i - 1 = \gamma(G_{i-1})$ (assume that $\gamma(H_{i-1}) \leq i - 1$; then we conclude that $\gamma(G_i - X) \leq i$, which is a contradiction). That means that

$$b(G_{i-1}) \leq |Y| \text{ with } Y: = X \cap E(G_{i-1})$$

So we must ask: How many edges of X don't belong to Y?

Let $k := |X - Y|$. If $k \geq 3$ the proof of (3) is finished, because then we have $b(G_{i-1}) \leq |X| - k < 3(i-1) - 3 = 3(i-2)$. It remains to show that

$$k \geq 3 \tag{4}$$

Let $x_1, \ldots, x_{i-1}$ be the 'vertical' neighbors of v and $y_1, \ldots, y_{i-1}$ be the 'horizontal' neighbors of v (see Figure 1).

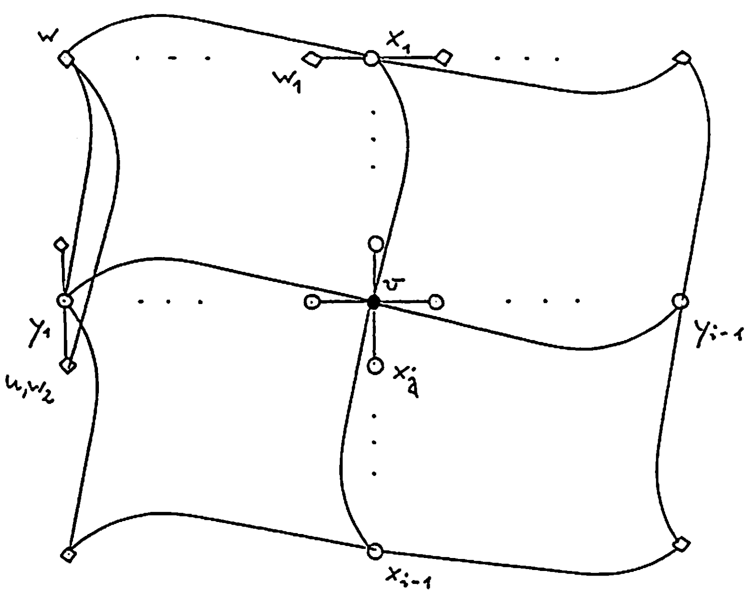

Figure 1: The graph G_i

Assume that none of the 'horizontal' edges incident to $x_1, \ldots, x_{i-1}$ belongs to X. Then $\{x_1, \ldots, x_{i-1}, v\}$ is a MDS$(G_i - X)$ which is a contradiction to $\gamma(G_i - X) > i$. Analogously one of the 'vertical' edges incident to $y_1, \ldots, y_{i-1}$ must belong to X. Hence we know that $k \geq 2$. Call these two edges z_1 and z_2.

Assumption: No further edge incident to a neighbor of v belongs to X. That means especially that no edge of $\langle N[v] \rangle$ belongs to X.

If the assumption were not true there would be a third edge of X not belonging to Y and (4) would be shown.

Case 1: z_1 and z_2 are not adjacent.

W.l.o.g. let $z_1 = x_1w_1$ and $z_2 = y_1w_2$, where $w_1, w_2 \in V(H_{i-1})$, $w_1 \neq w_2$ (see Figure 1). Let $x_j \in N(w_2)$. Then $\{y_1, \ldots, y_{i-1}, x_j\}$ is a MDS$(G_i - X)$ which is a contradiction. Hence our assumption is wrong, we have a third edge and (4) is shown.

Case 2: z_1 and z_2 are adjacent (w.l.o.g. $z_1 = x_1w$, $z_2 = y_1w$)

Case 2.1: w is not isolated in $G_i - X$, say w.l.o.g., w has the 'vertical' neighbor u in $G_i - X$. Let $x_j \in N(u)$, $j \neq 1$. Because of our assumption x_j is the center of an undamaged i-angle, but x_j and w have a common neighbor, namely u.

Now let $H'_{i-1} :=\langle G_i - X - N[x_j]\rangle$. Then $H'_{i-1} \subseteq G'_{i-1} = G_i - N[x_j]$. Analogously

$$b(G'_{i-1}) \leq |Y'| \text{ with } Y' := X \cap E(G'_{i-1})$$

Let $k' := |X - Y'|$. It remains to show that $k' \geq 3$. Analogously to the original proof we easily get $k' \geq 2$ with edges z'_1 and z'_2. But $z'_1 = z_1 = x_1w$ whereas z'_2 can't be incident to w because the new center x_j and w have the common neighbor u. z'_2 can't be incident to x_1 as well because the i-angle with center x_j is undamaged. Hence z'_1 and z'_2 are not adjacent and we get a contradiction analogous to case 1.

Case 2.2: w is isolated in $G_i - X$.

Then we know already $2(i-1)$ edges of X, namely the edges incident to w (call them E_w). Let $\tilde{H}_i := G_i - E_w$. Then $\gamma(\tilde{H}_i) = i$ is immediate. It remains to show that $b(\tilde{H}_i) \geq i - 1$ to get a contradiction to (3), which will be done separately in the following lemma:

Lemma 2.3. $b(\tilde{H}_i) \geq i - 1$.

Proof: We use the same idea as in the proof of the main theorem. The truth of the lemma for $i = 2$ and $i = 3$ is obvious.

Induction step: In the following we show that

$$b(\tilde{H}_i) < i - 1 \Longrightarrow b(\tilde{H}_{i-1}) < i - 2 \tag{5}$$

holds for $i \geq 4$. This will violate the induction hypothesis and provide the necessary contradiction.

Let $\tilde{X} := \{l_1, \ldots, l_s\}$, $s < i - 1$ be a set of edges such that $\gamma(\tilde{H}_i - \tilde{X}) > i = \gamma(\tilde{H}_i)$. Now remove a vertex $\tilde{v}$ with its neighborhood out of $\tilde{H}_i - \tilde{X}$, where $\tilde{v}$ is the center of an undamaged i-angle. Since $\lceil\frac{(i-1)^2}{2}\rceil$ edges had to be removed to damage all i-angles in $\tilde{H}_i$, there must be such a vertex $\tilde{v}$ for $i \geq 4$.

Let $I_{i-1} := \langle\tilde{H}_i - \tilde{X} - N[\tilde{v}]\rangle$. Then $I_{i-1} \subseteq \tilde{H}_{i-1} = \tilde{H}_i - N[\tilde{v}]$.

$\gamma(I_{i-1}) > i - 1 = \gamma(\tilde{H}_{i-1})$ (assume that $\gamma(I_{i-1}) \leq i - 1$, then again we conclude that $\gamma(\tilde{H}_i - \tilde{X}) \leq i$, which is a contradiction). That means that

$$b(\tilde{H}_{i-1}) \leq |\tilde{Y}| \text{ with } \tilde{Y}: = \tilde{X} \cap E(\tilde{H}_{i-1})$$

So we ask again: How many edges of $\tilde{X}$ don't belong to $\tilde{Y}$?

If $\tilde{k}: = |\tilde{X} - \tilde{Y}| \geq 1$ we are done because then we have shown the truth of (5).

Let again $x_1, \ldots, x_{i-1}$ be the 'vertical' neighbors of $\tilde{v}$ and $y_1, \ldots, y_{i-1}$ be the 'horizontal' neighbors of $\tilde{v}$ (see Figure 2).

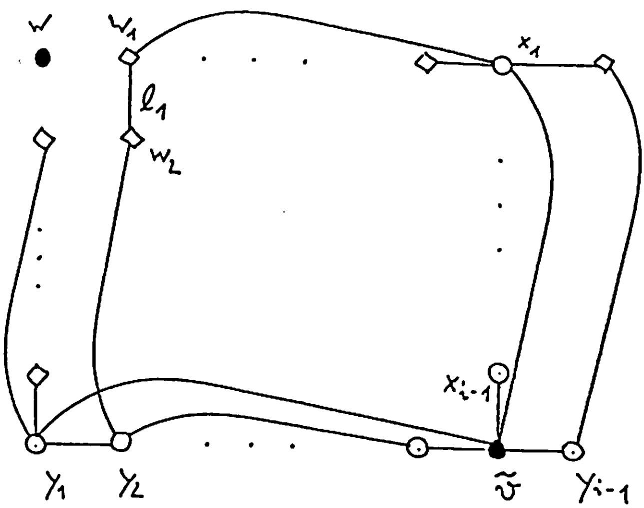

Figure 2: The graph $\tilde{H}_i$

Assume that none of the edges incident to $x_1, \ldots, x_{i-1}$ and none of the edges incident to $y_1, \ldots, y_{i-1}$ belong to $\tilde{X}$. Otherwise we would have $\tilde{k} \geq 1$ and would be done.

W.l.o.g. let l_1 be an edge of $\tilde{X}$ with arbitrary position in $\tilde{H}_i$. Let $l_1 = w_1w_2$ and let x_{j1} and x_{j2} be the 'x-neighbors' of w_1 and w_2 (where $x_{j1} = x_{j2}$ is possible). If $j_1 = 1$ we take the 'y-neighbors' y_{j3} and y_{j4} of w_1 and w_2 (where $y_{j3} = y_{j4}$ is possible). If $j_1 = 1$, j_3, $j_4 \geq 2$ since the isolated vertex w is the common neighbor of x_1 and y_1.

Now let $I'_{i-1}: = \langle \tilde{H}_i - \tilde{X} - N[x_{j1}] \rangle \subseteq \tilde{H}'_{i-1} = \tilde{H}_i - N[x_{j1}]$ (resp. y_{j3} instead of x_{j1} if $j_1 = 1$) . The i-angle with center x_{j1} was undamaged (assumption). And with the same argument as in case 2.1 of the main theorem we get a new $\tilde{Y}': = \{l_j; 1 \leq j \leq s, l_j \in E(\tilde{H}'_{i-1})\}$ and $\tilde{k}': =$

$|\tilde{X} - \tilde{Y}'| \geq 1$ because $l_1 \notin \tilde{Y}'$. Hence we have shown (5), and the lemma is proved. □

The proof of the lemma was the missing link in the main proof, so the proof of the theorem is complete. □

Corollary 2.4. *There is no upper bound of the form $b(G) \leq \Delta(G) + c$, $c \in N$.*

Proof: Take the class G_i of graphs. $b(G_i) - \Delta(G_i) = i - 1$. Let c be an arbitrary natural number. Then for the graph G_{c+2} we have $b(G_{c+2}) = \Delta(G_{c+2}) + c + 1 > \Delta(G_{c+2}) + c$. □

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