Total Domination In Graphs

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Abstract. Let G be a graph of order p such that both G and $\overline{G}$ have no isolated vertices. Let γ_t and $\overline{\gamma_t}$ denote respectively the total domination number of G and $\overline{G}$. In this paper we obtain a characterization of all graphs for which (i) $\gamma_t + \overline{\gamma_t} = p + 1$ and (ii) $\gamma_t + \overline{\gamma_t} = p$.

1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [2].

Let $G = (V, E)$ be a (p,q) graph without isolated vertices. A set $S \subseteq V$ is called a total dominating set if every vertex in V is adjacent to some vertex of S. The total domination number γ_t of G is the minimum cardinality taken over all minimal total dominating sets in G. The total domination number of the complement $\overline{G}$ of G is denoted by $\overline{\gamma_t}$.

We denote by P_n and C_n respectively the path and cycle on n vertices. If G is any connected graph and m is any positive integer, then mG stands for the graph with m components, each isomorphic to G. Also $G + x$ denotes the graph obtained by adding an edge x to G.

In [1] Cockayne et al have established the following.

Theorem 1.1 [1]. If G has p vertices, no isolates and $\Delta < p - 1$, then $\gamma_t + \overline{\gamma_t} \leq p + 2$ with equality if and only if G or $\overline{G}$ is mK_2.

In this paper we obtain a characterization of all graphs for which (i) $\gamma_t + \overline{\gamma_t} = p + 1$ and (ii) $\gamma_t + \overline{\gamma_t} = p$.

We need the following theorems.

Theorem 1.2 [1]. If G is connected and $\Delta < p - 1$, then $\gamma_t \leq p - \Delta$.

Theorem 1.3 [1] For a graph G without isolated vertices, $\gamma_t = p$ if and only if $G = mK_2$.

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2. Main results

Theorem 2.1 *Let G be a graph of order p such that G and $\overline{G}$ have no isolated vertices. Then* $\gamma_t + \overline{\gamma_t} = p + 1$ *if and only if G or $\overline{G}$ is isomorphic to* $P_3 \cup mK_2$, $C_3 \cup mK_2$ or C_5, $m \geq 1$.

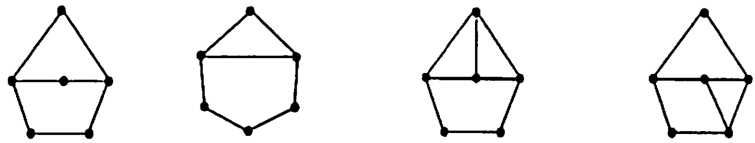
Proof: Obviously $\gamma_t + \overline{\gamma_t} = p + 1$ for all graphs stated in the theorem.

Conversely let G be any graph for which $\gamma_t + \overline{\gamma_t} = p + 1$. If G is disconnected, then $\overline{\gamma_t} = 2$ so that $\gamma_t = p - 1$. Hence exactly one component of G is of order three and all the remaining components are of order two so that G is isomorphic to $P_3 \cup mK_2$ or $C_3 \cup mK_2$, $m \geq 1$. Now suppose that G and $\overline{G}$ are connected. If $diam(G) \geq 3$, then $\overline{\gamma_t} = 2$ and by Theorem 1.2, $\gamma_t \leq p - 2$ so that $\gamma_t + \overline{\gamma_t} \leq p$. Hence $diam(G) = diam(\overline{G}) = 2$. It follows from Theorem 1.2 that $\gamma_t + \overline{\gamma_t} \leq p + 1 - \Delta + \delta$ and hence $\Delta = \delta$ so that G is regular of degree r, say. Then $\overline{G}$ is regular of degree $\overline{r} = p - 1 - r$. We claim that $r \leq \lfloor p/2 \rfloor$. Suppose $r \geq \lfloor p/2 \rfloor + 1$ so that $\overline{r} \leq \lfloor p/2 \rfloor - 1$. It follows from Theorem 1.2, that $\gamma_t \leq \lfloor p/2 \rfloor$ and since $diam(\overline{G}) = 2$, $\overline{\gamma_t} \leq 1 + \overline{r} \leq \lfloor p/2 \rfloor$ so that $\gamma_t + \overline{\gamma_t} \leq p$, which is a contradiction. Hence $r \leq \lfloor p/2 \rfloor$. In a similar way it can be shown that $\overline{r} \leq \lfloor p/2 \rfloor$. If p is odd, equality follows in both cases and if p is even, equality follows in one of the inequalities and hence we may assume without loss of generality that $r = \lfloor p/2 \rfloor$. We now claim that $r = 2$. Suppose $r \geq 3$. Let u be any vertex of G. Let T be a spanning tree of G with $deg_T(u) = r$; $N(u) = \{u_1, u_2, \ldots, u_r\}$ and $V - N[u] = \{v_1, v_2, \ldots, v_m\}$. Since $r = \lfloor p/2 \rfloor$, $m \leq r$. Also if $m < r$, then $deg_T(u_i) = 1$ for some i and $N[u] - \{u_i\}$ is a total dominating set so that $r_t \leq r$. Thus $\gamma_t + \overline{\gamma_t} \leq p$. Hence $m = r$ and $p = 2r + 1$. Now if u_i is not adjacent to v_i in T for all $i = 1, 2, \ldots, r$, then $deg_T(u_i) = 1$. Also if u_i is adjacent to v_j and v_k in T where $1 \leq j < k \leq i$, then $deg_T(u_m) = 1$ for some m and hence $\gamma_t \leq r$. Hence it follows that each u_i is adjacent to exactly one v_j in T and without loss of generality, we may assume that u_i is adjacent to v_i in T for all $i = 1, 2, \ldots, r$. Now if v_i is adjacent to u_j for some $j \neq i$ in G, then $N[u] - \{u_i\}$ is a total dominating set. If v_i is adjacent to v_j and v_k in G, then $(N[u] - \{u_j, u_k\}) \cup \{v_i\}$ is a total dominating set. Thus $\gamma_t \leq r$ and $\gamma_t + \overline{\gamma_t} \leq p$. Hence $r = 2$ so that $p = 5$ and $G = C_5$. ■

Theorem 2.2 *Let G be a graph of order p such that G and $\overline{G}$ have no isolated vertices. Then* $\gamma_t + \overline{\gamma_t} = p$ *if and only if G or $\overline{G}$ is isomorphic to* $2P_3 \cup mK_2$, $2C_3 \cup mK_2$, $P_3 \cup C_3 \cup mK_2$, $P_4 \cup mK_2$, $P_5 \cup mK_2$, $P_6 \cup mK_2$, $C_6 \cup mK_2$, $m \geq 0$; $K_4 \cup mK_2$, $C_4 \cup mK_2$, $(C_4 + x) \cup mK_2$, $K_{1,3} \cup mK_2$, $(K_{1,3} + x) \cup mK_2$, $C_5 \cup mK_2$, $m \geq 1$ *or any one of the graphs given in figure 1.*

Proof: It can be easily verified that for all graphs stated in the theorem, $\gamma_t + \overline{\gamma_t} = p$.

Conversely, let G be any graph for which $\gamma_t + \overline{\gamma_t} = p$.



G_1 G_2 $\overline{G_1}$ $\overline{G_2}$

Fig. 1

Case (i) G is disconnected.

Then $\overline{\gamma_t} = 2$ so that $\gamma_t = p - 2$. Hence two components of G have three vertices and the remaining components have two vertices or one component of G has four or five or six vertices and the remaining components have two vertices. If two components of G have three vertices each, then G is isomorphic to $2P_3 \cup mK_2$, $2C_3 \cup mK_2$ or $P_3 \cup C_3 \cup mK_2$, $m \geq 0$. If one component of G has four or five or six vertices, then G is isomorphic to one of the graphs $K_4 \cup mK_2$, $P_4 \cup mK_2$, $C_4 \cup mK_2$, $K_{1,3} \cup mK_2$, $(K_{1,3} + x) \cup mK_2$, $(C_4 + x) \cup mK_2$, $C_5 \cup mK_2$, $P_5 \cup mK_2$, $C_6 \cup mK_2$ or $P_6 \cup mK_2$, $m \geq 1$.

Case (ii) G and $\overline{G}$ are connected and $diam\,(G) \geq 3$.

Then $\overline{\gamma_t} = 2$ so that $\gamma_t = p - 2$. Since $\gamma_t \leq p - \Delta$, it follows that $\Delta = 2$ and hence G is isomorphic to P_4, P_5, P_6 or C_6.

Case (iii) G and $\overline{G}$ are connected and $diam\,(G) = diam\,(\overline{G}) = 2$.

Then $\overline{\gamma_t} \leq 1 + \overline{\delta} = 1 + p - 1 - \Delta = p - \Delta$. Also $\gamma_t \leq p - \Delta$ and hence $\Delta \leq \lfloor p/2 \rfloor$. Similarly $\overline{\Delta} \leq \lfloor p/2 \rfloor$ and hence we may assume that $\Delta = \lfloor p/2 \rfloor$. Further it follows that $\Delta = \delta$ or $\Delta = \delta + 1$. We now claim that p is even. Suppose $p = 2m + 1$. Then $\Delta = \delta = \overline{\Delta} = \overline{\delta} = m$. Let u be any vertex of G. Let T be a spanning tree of G such that $deg_T(u) = m$. Let $N(u) = \{u_1, u_2, \ldots, u_m\}$ and $V - N[u] = \{v_1, v_2, \ldots, v_m\}$. Hence v_i is adjacent to u_i for all $i = 1, 2, \ldots, m$. If v_i is adjacent to u_j, $j \neq i$, then $\gamma_t \leq m$. Otherwise all the v_i's are mutually adjacent so that $\gamma_t = 3$. Hence $\gamma_t \leq m$ or $\gamma_t = 3$. Similarly $\overline{\gamma_t} \leq m$ or $\overline{\gamma_t} = 3$. If $\gamma_t \leq m$ and $\overline{\gamma_t} \leq m$, then $\gamma_t + \overline{\gamma_t} < p$. If $\gamma_t = \overline{\gamma_t} = 3$, then $\gamma_t + \overline{\gamma_t} = 6 \neq p$ since p is odd. If $\gamma_t \leq m$ and $\overline{\gamma_t} = 3$, then $\gamma_t = 2$ and hence $p = 5$ so that $G = C_5$ and in this case $\gamma_t + \overline{\gamma_t} = 6 \neq p$. Hence p is even. Let $p = 2m$. We now claim that $\Delta = \delta + 1$. Suppose $\Delta = \delta = m$. Let T be a spanning tree of G such that $deg_T(u) = m$. Let $N(u) = \{u_1, u_2, \ldots, u_m\}$ and $V - N[u] = \{v_1, v_2, \ldots, v_{m-1}\}$. At least one vertex of $N(u)$, say u_m is a pendant vertex in T. If two vertices of $N(u)$ are pendant vertices in T, then $\gamma_t \leq \Delta - 1$ so that $\gamma_t + \overline{\gamma_t} \leq \Delta - 1 + (p - \Delta) = p - 1$, which is a contradiction. Hence we may assume that u_i is adjacent to v_i for all $i = 1, 2, \ldots, m - 1$. Since v_1 has degree m in G, v_1 must be adjacent to some u_j, $j \neq 1$. Hence $N[u] - \{u_m, u_1\}$ is a total dominating set so that $\gamma_t \leq m - 1$ Further $\overline{\gamma_t} \leq m$ so that $\gamma_t + \overline{\gamma_t} < p$, which is a contradiction. Hence $\Delta = \delta + 1$. Now let u be any vertex of degree δ. Let $N(u) = \{u_1, u_2, \ldots\ldots, u_{m-1}\}$ and $V - N[u] = \{v_1, v_2, \ldots\ldots, v_m\}$. We may assume that v_i is adjacent to u_i for all $i = 1, 2, \ldots\ldots, m - 1$ and v_m is adjacent to u_1.

We now claim that $\delta = 2$.

Suppose $\delta \geq 4$. If v_2 is adjacent to some u_i, $i \neq 2$, then $N[u] - \{u_2\}$ is a total dominating set of cardinality δ. Otherwise v_2 is adjacent to at least three vertices v_i, v_j and v_k. If v_i, v_j and v_k are different from v_1 and v_m, then $(N[u] - \{u_i, u_j, u_k\}) \cup \{v_2\}$ is a total dominating set. If $v_i = v_1$ or v_m and v_j, v_k are different from v_1 and v_m, then $(N[u] - \{u_j, u_k\}) \cup \{v_2\}$ is a total dominating set. If $v_i = v_1$ and $v_j = v_m$, then $(N[u] - \{u_1, u_k\}) \cup \{v_2\}$ is a total dominating set. Thus in all cases $\gamma_t \leq \delta$ so that $\gamma_t + \overline{\gamma_t} < p$, which is a contradiction.

Suppose $\delta = 3$. Then $\Delta = 4$ so that $p = 8$. If v_2 is adjacent to some $u_i, i \neq 2$ then $N[u] - \{u_2\}$ is a total dominating set of cardinality δ, which is a contradiction. Hence v_2 is adjacent to two of the vertices of $\{v_1, v_3, v_4\}$. Similarly v_3 is adjacent to two of the vertices of $\{v_1, v_2, v_4\}$. Further v_2 cannot be adjacent to both v_1 and v_4. Similarly v_3 cannot be adjacent to both v_1 and v_4. Hence $deg(v_2) = deg(v_3) = 3$ and we may assume that v_2 is adjacent to v_1 ; v_3 is adjacent to v_4 and v_2 is adjacent to v_3. Now v_1 and v_4 cannot both be adjacent to u_2 or u_3. Without loss of generality, we may assume that v_1 is adjacent to u_2 and v_4 is adjacent to u_3. Now $\{u, u_2, u_3\}$ is a total dominating set of cardinality δ, which is a contradiction. Hence $\delta = 2$, $\Delta = 3$, $p = 6$ and $\gamma_t = \overline{\gamma_t} = 3$. Let u be a vertex of degree 3 and let $N(u) = \{u_1, u_2, u_3\}$. Let $S = V - N[u] = \{v_1, v_2\}$. If S is independent, then v_1 and v_2 are adjacent to some u_i so that $\{u, u_i\}$ is a total dominating set, which is a contradiction. Hence v_1 and v_2 are adjacent and without loss of generality we may assume that $v_1u_1 ; v_2u_2 \in E(G)$. Clearly $q = 7$ or 8. When $q = 7$, G is isomorphic to G_1 or G_2 and when $q = 8$, G is isomorphic to $\overline{G_1}$ or $\overline{G_2}$. ■

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A Semigroup Operation Using Powers Of Two

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ABSTRACT. A new numerical monoid based on the arithmetic function related to powers of 2 called 'delta', is used first to define a noncommutative monoid on the odd integers with five submonoids, all infinite: the associativity of the structure is used to define a relationship between the delta functions of two arbitrary odd numbers.

Introduction

This note describes a very interesting monoid on odd integers greater than 3 and which contains at least three infinite submonoids.

Detailed Definitions

Define $\delta(x) = 2^k$ where $2^k < x < 2^{k+1}$.

Let $r(N)$ be $N-\delta(N)$: let the semigroup operation be $m(x, y) = \frac{1}{2}\delta(x)(y-3) + x$, for odd integers $x, y > 1$: this awkward definition hides some surprises since in fact there is unexpectedly an identity .

The Main Work

Theorem. *Let S be the set of odd numbers greater than one; then (S, m) is a monoid which is not commutative.*

Proof: 3 has the role of identity in this system (this can easily be checked), the more significant part of the proof is associativity: first we require two lemmas:

Lemma 1. *$m(5, x) = 2x - 1$ and $m(7, y) = 2y + 1$:*

Proof:

$$\delta(5) = 4 \text{ thus } m(5,x) = (4/2)(x-3) + 5 = 2x - 1$$
$$\delta(7) = 4 \text{ thus } m(7,y) = (4/2)(y-3) + 7 = 2y + 1$$

Lemma 2. *Let n be odd and $n > 3$, then applying the functions $x| \rightarrow 2x-1$ and $y| \rightarrow 2y+1$ to 5, and seven generates all odd numbers.*

Proof: The Lemma is true for 5,7,9,11,13,15; let c be the smallest exception to the Lemma: then either $c = 2n-1$ where n is odd or $c = 2n+1$ where n is odd, thus from this contradiction on the inductive argument we can therefore generate all odd numbers greater than unity by the action of $m(5,x)$ and $m(7,y)$ on the set $\{3,5,7\}$, this concludes the proof of our lemma.

Uniqueness is evident for the same reasoning clearly.

Remark: Following a suggestion of the referee, instead of writing $m(r,u) = v$, $m(s,v) = z$ let us for convenience write $r * u = v$, $s * v = z$ so that $s * (r * v) = z$: putting $r = s = 5$ we obtain

$$5 * (5 * v) = z = 5 * (2v-1) = 2(2v-1) - 1 = 4v - 3$$

and $(5*5)*v = (9)*v = (8/2)(v-3)+9 = 4v-3$ so that $5*(5*v) = (5*5)*v$ similarly

$$(7 * (7 * v)) = 2(2v+1) + 1 = 4v + 3$$
$$((7 * 7) * v) = 15 * v = (8/2)(v-3) + 15 = 4v + 3$$
$$((5 * 7) * v) = (13) * v = (8/2)(v-3) + 13 = 4v + 1$$
$$(5 * (7 * v)) = 5 * (2v+1) = (4/2)(2v - 3 + 1) + 5 = 4v + 1$$
$$((7 * 5) * v) = (11) * v = (8/2)(v-3) + 11 = 4v - 1 = (7 * (5 * v))$$

Main Proof Continuation: All odd numbers except three and unity can be replaced by equivalent unique strings of 5's and 7's, and the monoid multiplication function by concatenation or juxtaposition of these strings. This is proved when we can show that $m(x,y) = x * y$ corresponds to the concatenations of the strings for x and y. To prove this we will adopt quote notation for the strings representing each odd number, thus 11 is written '75' and 25 = '557'. Now we can start to use an inductive proof on the length of the string representing the left variable in $x * y$. The start of the induction is easy; let N be the number for which the correspondence breaks down which has the shortest equivalent string. Then $N * y = c$ is the operation in question say; let N' be the $cdr(N)$ with the first 5 or seven missing, then $N' * y = c'$ *does* obey the correspondence and the isomorphism works: then '$5' * y = c$, or '$7' * y = c$ which follows from the remark, the meaning of the quote-notation and the initial hypothesis, thus we have a contradiction and the inductive argument is established as required. From this, since concatenation is associative, the associativity of m is also established easily.

Thoughts Having established that this mapping is associative and has identity, we have the fact that with the appendation of 3 as identity to the structure we have a monoid. The next step is the discovery of congruences and submonoids: the first fact is that our monoid (S, m) is generated entirely by 5 and 7 with 3 which then lead us to determine what $\langle 5 \rangle$ and $\langle 7 \rangle$ are . In fact

$$\langle 3, 5 \rangle \text{ consists of all numbers of the form } 2^n + 1$$
$$\langle 3, 7 \rangle \text{ consists of all numbers of the form } 2^n - 1 \text{ (less unity) :}$$

there is also the intriguing fact that multiples of 3 form a closed structure so that $\langle 3n \colon n = 1, 2, 3, 4, \dots \rangle$ is also a submonoid, whit has a non-trivial intersection with the first two.

We have therefore respectively, submonoids A, B, C with $F = A \cap C$ and $D = B \cap C$: five submonoids,all together .

The referee noted that m also has the left and right cancelletion property, related to the abbreviation of strings; however the odd numbers do not contain the inverse operations.

From this theorem we can deduce another though somewhat obscure theorem.

Theorem. *For any odd numbers x, y: $\delta(y)/2 = \delta((y-3) \cdot \delta(x)/2 + x)/\delta(x)$.*

Proof: The definition of m and associativity leads to $\delta((y-3) \cdot \delta(x)/2 + x)(z-3) = \delta((z-3) \cdot \delta(y)/2)$, putting $x = z$ leads to the result.

Corollary. *For any odd integers x, y: $\delta(x) \cdot \delta(y) = \delta((y-3) \cdot \delta(x)/2 + x) + \delta((x-3) \cdot \delta(y) + y)$*

Proof: Swap x and y in the theorem and add the two results.

Future Material We can work on whether the same results apply for delta defined in terms of powers of any odd prime, and whether there are analogous formulae for any prime above two. If not the we can split off the behaviour of powers of 2 from other powers in this context; also the representation of nunbers by these strings (rewritten in binary) may produce results, naturally.

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