Ramsey Numbers of K_3 versus (p, q)-Graphs

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ABSTRACT. In this note we complete the table of Ramsey numbers for K_3 versus the family of all (p,q)-graphs for $p \leq 8$. Moreover, some results are obtained for the general case.

1 Introduction

If p and q are natural numbers with $q \leq \binom{p}{2}$ the set Ramsey number $r(K_3, \langle p, q \rangle)$ is defined to be the smallest natural number n such that in every 2-coloring of the edges of the complete graph K_n (with green and red) there is a green K_3 or a red (p,q)-graph, i. e. a graph with p vertices and q edges. The values of $r(K_3, \langle p, q\rangle)$ for $p \leq 7$ are given in [2] (see Table 1). Moreover, the values are known for $8 \leq p \leq 9$ in case of $\binom{p}{2} - 1 \leq q \leq \binom{p}{2}$ (see [3], [6] and [7], note that $r(K_3, \langle p, \binom{p}{2} - 1\rangle) = r(K_3, K_p - e)$ and $r(K_3, \langle p, \binom{p}{2}\rangle) = r(K_3, K_p)$). In this note, $r(K_3, \langle p, q\rangle)$ will be evaluated when q is not too large relative to p, and the still missing values for $p = 8$ will be determined. More general set Ramsey numbers are discussed in [1] and [4].

p \ q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
3	3	5	6																									
4	4	4	5	7	7	9																						
5	5	5	5	5	7	7	9	9	11	14																		
6	6	6	6	6	6	6	7	9	9	9	11	11	12	17	18													
7	7	7	7	7	7	7	7	7	7	9	9	11	11	11	11	13	13	14	17	21	23							
8	8	8	8	8	8	8	8	8	8	8	8	8	9	11	11	11	13	13	13	13	13	15	16	16	18	21	25	28

Table 1. $r(K_3, \langle p, q\rangle)$ for $p \leq 8$

Some specialized notation will be used here. A 2-coloring of K_n is called a $(K_3, \langle p, q\rangle)$-coloring, if it contains neither a green K_3 nor a red (p,q)-graph. We use V to denote the vertex set of K_n and define G_v and R_v, for all $v \in V$, to be the sets of green and red neighbors of v. The number of green edges incident to v is denoted by $g(v)$, and $g(S,T)$ denotes the number of green edges between two disjoint vertex sets S and T. If S consists of a single vertex w, we write $g(w,T)$.

2 General results

An immediate consequence of Turán's theorem ([8]) is

Theorem 1.

$$r(K_3, \langle p,q\rangle) = p \text{ for } p \geq 1 \text{ and } q \leq \binom{p}{2} - \lfloor p^2/4 \rfloor. \tag{1}$$

The next theorem gives a lower bound for $r(K_3, \langle p,q\rangle)$ in case of $q > \binom{p}{2} - \lfloor p^2/4 \rfloor$. This bound is sharp if q is close to $\binom{p}{2} - \lfloor p^2/4 \rfloor$.

Theorem 2. *Let $p \geq 2$ and let s be an integer such that $1 \leq s \leq \lfloor p/2 \rfloor$. Put $c_p = \binom{p}{2} - \lfloor p^2/4 \rfloor$. Then the following assertions hold.*

(i) If p is odd and $c_p + (s-1)s + 1 \leq q \leq c_p + s(s+1)$ then

$$r(K_3, \langle p,q\rangle) \geq p + 2s. \tag{2}$$

Equality in (2) holds for $p \geq 5$, $s \leq \max\{1, \sqrt{p - 19/4} - 1/2\}$.

(ii) If p is even and $c_p + (s-1)^2 + 1 \leq q \leq c_p + s^2$ then

$$r(K_3, \langle p,q\rangle) \geq p + 2s - 1. \tag{3}$$

Equality in (3) holds for $p \leq 4$, $s = 1$ and for $p \geq 6$, $s \leq \max\{2, \sqrt{p-5}\}$.

Proof: First we will verify inequalities (2) and (3). Consider the 2-coloring of $K_{2\lfloor (p-1)/2 \rfloor + 2s}$ consisting of two disjoint red $K_{\lfloor (p-1)/2 \rfloor + s}$ with all edges between them green. It contains no green K_3. Moreover, in every p-subgraph (i.e. subgraph with p vertices) there are at most $c_p + (s-1)s$ red edges for p odd and at most $c_p + (s-1)^2$ for p even. This yields the required inequalities.

To complete the proof of (i) we have to show that equality holds in (2) in certain cases. Suppose that equality does not hold for some p, s and q with p odd, $p \geq 5$, $s \leq \max\{1, \sqrt{p - 19/4} - 1/2\}$ and $c_p + (s-1)s + 1 \leq q \leq c_p + s(s+1)$. Then there exists a $r(K_3, \langle p,q\rangle)$-coloring χ of K_{p+2s}. Denote the vertex set of K_{p+2s} by V and put $k = \lfloor p^2/4 \rfloor - s(s+1) + 1$. Note that in χ every p-subgraph must have at least k green edges. Let H_1 be a red complete subgraph with maximum number of vertices in χ. Denote the vertex set of H_1 by V_1 and put $l = |V_1|$. Then $l \leq p-2$, because otherwise H_1 would yield a red (p,q)-graph. Furthermore, p+2$s \geq 7$ and $r(K_3, K_3) = 6$ imply that $l \geq 3$. Note that G_v induces a red complete subgraph for every $v \in V$. Thus, $g(v) \leq l$ for every $v \in V$. Let H_2 be the subgraph induced by $V_2 = V \setminus V_1$. One of the following three cases must occur.

Case I: All edges in H_2 are red. Consequently $p + 2s - l = |V_2| \leq l$, and this yields $l \geq (p+1)/2 + s$. But then, the p-subgraph induced by V_1 and $p - l$ vertices from V_2 has at most $((p+1)/2 + s)((p-1)/2 - s) = k - 1$ green edges, a contradiction.

Case II: All green edges in H_2 are incident to a common vertex u. Then $H_2 - u$ is a red complete graph and contains at most l vertices. In case of $l \geq (p+1)/2 + s$ we get a contradiction similar as above. It remains that $l = (p-1)/2 + s$. Consequently $|V_2| = (p+1)/2 + s \leq p - 1$. By the maximality of V_1, there exists a vertex $w \in G_u \cap V_1$. We obtain that $g(\{u, w\}, V_2 \setminus u) \leq (p-1)/2 + s$, as otherwise a green K_3 would occur. But then the p-subgraph induced by V_2, w and $(p-3)/2 - s$ other vertices from V_1 has at most $((p-3)/2 - s)l + (p-1)/2 + s + 1 \leq k - 1$ green edges.

Case III: There are at least two independent green edges $\{u_1, w_1\}$ and $\{u_2, w_2\}$ in H_2. Then $g(\{u_i, w_i\}, V_1) \leq l$ and we can assume that $g(u_1, V_1) \leq l/2$. Moreover, there are at most four green edges between u_1, u_2, w_1 and w_2, and at most two between any three of them. Note that $p \geq 7$ for $l \leq p - 3$ since $l \geq 3$. If $l \leq p - 4$, take a p-subgraph induced by V_1, u_1, u_2, w_1, w_2 and $p - l - 4$ other vertices from V_2. It would contain at most $(p - l - 4)l + 2l + 4 = (p - 2 - l)l + 4 \leq ((p-1)/2)(p-3)/2) + 4 \leq k - 1$ green edges. It remains that $l = p - 3$ or $l = p - 2$. But then we get a contradiction from the p-subgraphs induced by V_1, u_1, u_2 and w_2 or by V_1, u_1 and w_1.

Thus, each of the cases I to III leads to a contradiction and the proof of (i) is complete. The proof of (ii) can be completed similarly. □

Table 1 shows that for small p the bounds for s where equality in (2) or (3) holds can be improved. It seems that with more careful methods this might be possible in general. The following theorems improve the bounds for $r(K_3, \langle p, q\rangle)$ given in Theorem 2 for special q close to $\binom{p}{2}$.

Theorem 3. *For p even, $p \geq 4$,*

$$r(K_3, \langle p, \binom{p}{2} - p + 3\rangle) \geq \frac{5}{2}p - 4. \qquad (4)$$

Proof: Take five disjoint $K_{p/2-1}$ denoted by $H_1, \ldots, H_5$. Join, for $i = 1, \ldots, 5$, the vertices from H_i by green edges to the vertices from $H_{i+1 \pmod 5}$ and any two other vertices by a red edge. The resultant coloring of $K_{5p/2-5}$ contains no green K_3 and every subgraph with p vertices has at most $\binom{p}{2} - p + 2$ red edges. Thus, inequality (4) is established. □

Theorem 4. *Let s be a positive integer and $p \geq s+6$. Then*

$$r(K_3, \langle p, \binom{p}{2} - s\rangle) \geq 4p - 4s - 3. \tag{5}$$

Proof: In [7] it was shown that (5) holds for $s = 1$. Note that a $(p, \binom{p}{2} - s)$-graph contains a $(p-s+1, \binom{p-s+1}{2} - 1)$-subgraph. Consequently $r(K_3, \langle p, \binom{p}{2} - s\rangle) \geq r(K_3, \langle p - s + 1, \binom{p-s+1}{2} - 1\rangle) \geq 4(p - s + 1) - 7$ if $p - s + 1 \geq 7$ and inequality (5) is proved. □

3 The values of $r(K_3, \langle 8, q\rangle)$

The values of $r(K_3, \langle 8, q\rangle)$ given in Table 1 can be deduced from Theorems 1 and 2 for $q \leq 16$. For $27 \leq q \leq 28$ they were determined in [6] and [7] by computer methods. Furthermore, the value for $q = 27$ was obtained in [5] without using a computer. For the remaining case $17 \leq q \leq 26$ it can be seen from Theorems 2 to 4 and from the coloring in Figure 1 that the values in Table 1 are lower bounds for $r(K_3, \langle 8, q\rangle)$. To prove equality it suffices to establish them as upper bounds for $q = 21$, 22, 24, 25 and 26. This will be carried out in the following lemmas.

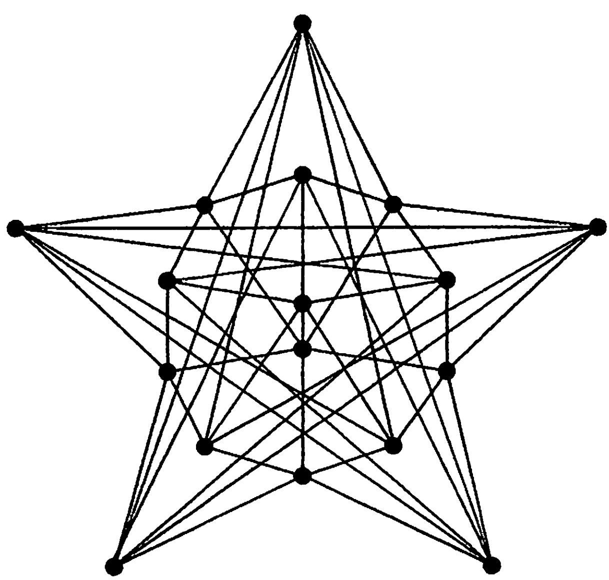

Figure 1. The green subgraph of a $(K_3, \langle 8, 25\rangle)$-coloring of K_{17}

Lemma 1. *Let $p \geq 3$ and $q \geq \lfloor p/2 \rfloor$. Let χ be a 2-coloring of K_n containing a red $(p-1, q - \lfloor p/2 \rfloor)$-graph but no green K_3. Then χ contains* a *red* (p,q)*-graph if*

$$(i)\quad 2p-3 \le n \le 2p-2, q \le \binom{p-1}{2} \text{ or}$$

$$(ii)\quad n \ge 2p-1, q \le \binom{p-1}{2} + \left\lfloor \frac{p}{2} \right\rfloor.$$

Proof: Let V_1 be the vertex set of a red $(p-1, q-\lfloor p/2 \rfloor)$-graph in χ. If $V_2 = V_1 \setminus V$ contains a vertex u with $g(u, V_1) \le \lfloor (p-1)/2 \rfloor$, a red (p,q)-graph occurs for $q \le \binom{p-1}{2} + \lfloor p/2 \rfloor$. Otherwise, any two vertices in V_2 have a common green neighbor in V_1. This leads to a red K_{n-p+1} induced by V_2. If then $n \ge 2p-1$ we have a red K_p containing a red (p,q)-graph for $q \le \binom{p}{2}$. For $n = 2p-2$, a red K_{p-1} is induced by V_2. This yields a red (p,q)-graph for $q \le \binom{p-1}{2}$. It remains that $n = 2p-3$. If then a red K_{p-1} is induced by V_1, we obtain a red (p,q)-graph for $q \le \binom{p-1}{2}$ as before. Otherwise there is at least one green edge between vertices u_1 and u_2 in V_1. But then u_1 and u_2 together with V_2 yield a red (p,q)-graph for $q \le \binom{p-1}{2}$. □

Lemma 2. $r(K_3, \langle 8, 21 \rangle) \le 13$, $r(K_3, \langle 8, 22 \rangle) \le 15$.

Proof: Using that $r(K3, \langle 7, 17 \rangle) = 13$ and $r(K_3, \langle 7, 18 \rangle) = 14$ (see Table 1) Lemma 2 follows from Lemma 1. □

Lemma 3. *A* $(K_3, \langle 8, q \rangle)$*-coloring* χ *of* K_n *has the following properties.*

(P_1) *For* $n \ge 15$, *there is no red* $K_7 - e$ *in* χ *if* $q = 24$ *and no red* K_7 *if* $24 \le q \le 25$.

(P_2) *If* W *is the vertex set of a red* K_l *in* χ *with* $2 \le l \le 7$, $w \in W$ *and* $U \subseteq G_w$ *such that* $|U| = 9 - l$, *then* $g(U, W \setminus \{w\} \ge 29 - q$.

Proof: (P_1) is an immediate consequence of Lemma 1. Taking into account that G_w induces a red complete graph the proof of (P_2) is straightforward. □

Some further notation will be used in the following lemmas. If χ is a given 2-coloring of K_n, the green subgraph is denoted by G. We define Δ to be the maximum degree in G and always use u to denote a vertex of degree Δ in G. The green and red neighbors of u are, respectively, denoted by $1, \ldots, \Delta$ and $\Delta + 1, \ldots, n-1$.

Lemma 4. $r(K_3, \langle 8, 24 \rangle) \le 16$.

Proof: Suppose that we have a $(K_3, \langle 8, 24 \rangle)$-coloring χ of K_{16}. Then χ must have properties (P_1) and (P_2) from Lemma 3. (P_1) implies the further property

(P_3) If W is the vertex set of a red K_6 in χ, then $g(v,W) \geq 2$ for every $v \in V$. Equality holds for at most two vertices.

Moreover, (P_1) yields $\Delta \leq 6$. We distinguish the following three cases depending on the value of Δ.

Case I: $\Delta \leq 4$. Suppose first that there is a red (7,19)-graph in χ with vertex set W_1. Let W_2 be a set of eight vertices in $V \setminus W_1$. Then either W_2 yields a red (8,24)-graph or there exists $v \in W_2$ with $g(v, W_2 \setminus \{v\}) \geq 2$. But then $g(v, W_1) \leq 2$ because of $\Delta \leq 4$, and W_1 together with v yields a red (8,24)-graph. Now suppose that a red K_6 with vertex set W would occur. Then (P_3) implies that $g(W, V \setminus W) \geq 28$ in contradiction with $\Delta \leq 4$. Thus, in case of $\Delta \leq 4$ we obtain the additional property

(P_4) There is neither a red (7,19)-graph nor a red K_6 in χ.

Now consider a vertex u incident to Δ green edges. Then $r(K_3, K_4) = 9$ guarantees a red K_4 in R_u yielding a red K_5 (with vertex set U) together with u. (P_4) implies that $g(v,U) \geq 1$ for every $v \in V \setminus U$ with equality for at most two vertices. A vertex v with $g(v,U) \geq 3$ implies $g(U, V \setminus U) > 20$ in contradiction with $\Delta \leq 4$. The case of $g(v,U) \leq 2$ for every $v \in V \setminus U$ remains. This gives $g(U, V \setminus U) = 20$ and $g(w) = 4$ for every $w \in U$. But then (P_2) is violated.

Case II: $\Delta = 5$. Let V_i be the set of vertices $v \in R_u$ with $g(v, G_u) = i$. Then

$$\sum_{i \geq 1} i|V_i| = g(G_u, R_u) \leq 20. \tag{6}$$

First suppose that $V_0 \neq \emptyset$. Then $G_u \cup \{v\}$ where $v \in V_0$ induces a red K_6. (P_3) forces $g(R_u \setminus \{v\}, G_u \cup \{v\}) \geq 25$. This yields $g(v) = 5$ and $G_u \cup \{u\}$ induces a second red K_6. Because of (P_3), there exist vertices $w_1, w_2 \in R_u \cap R_v$ with $g(w_i, G_v) \geq 3$. But then the conditions $K_3 \notin G$ and $\Delta = 5$ imply a red (8,24)-graph in $G_u \cup \{v, w_1, w_2\}$. Thus, it remains that $V_0 = \emptyset$ and

$$\sum_{i \geq 1} |V_i| = |R_u| = 10. \tag{7}$$

Moreover, $|V_2| \geq 3$, as otherwise (6) and (7) would yield that (P_2) is violated for a vertex in G_u.

First let $|V_1| = 3$. Then we obtain from (6) and (7) that $|V_2| \geq 4$ and $g(V_1 \cup V_2, G_u) \geq 11$. We can assume that $g(1, V_1 \cup V_2) \geq 3$. Let $V_1 = \{6,7,8\}$. Without loss of generality, the edges $\{6,8\}$ and $\{7,8\}$ are green and $\{6,7\}$ red. (P_1) and $K_3 \not\subset G$ imply that $g(1, V_1) \leq 1$. Suppose that $g(1, V_2) \geq 3$. Then, since $K_3 \not\subset G$, either $G_u \setminus \{1\}$ together with 8 and three vertices in $G_1 \cap V_2$ leads to a red $(8,24)$-graph or G_u together with 6, 7 and one vertex in $G_1 \cap V_2$. It remains that $g(1,V_1) = 1$ and $g(1,V_2) = 2$. If $g(1,R_u) = 3$ we get $g(G_u,R_u) < 20$. Otherwise, $g(1,R_u) = 4$, and then (P_2) forces that $V_i \neq \emptyset$ for some $i \geq 4$. In both cases (6) and (7) imply $|V_2| \geq 5$. Then $g(V_1 \cup V_2, G_u) \geq 13$, and there must be a vertex $v \neq 1$ in G_u with $g(v, V_1 \cup V_2) \geq 3$. As above we get $g(v,V_1) = 1$, and we can assume that $\{1,6\}$ is green. But then $K_3 \not\subset G$ leads to a red $(8,24)$-graph among the vertices $2,\dots,8$ and the two vertices in $G_1 \cap V_2$.

It remains that $|V_1| \geq 4$. Let $6,7,8,9 \in V_1$. Without loss of generality, the edges $\{6,7\}$ and $\{8,9\}$ are red and the other edges between these four vertices must be green. Since $\Delta = 5$ implies $g(G_u, \{10,\dots 15\} \leq 16$ there is a vertex v in $\{10,\dots 15\}$ with $g(v,G_u) \leq 2$. But then the subgraph induced be v and $\{1,\dots,9\}$ contains a red $(8,24)$-graph if a green K_3 is avoided.

Case III: $\Delta = 6$. Then (P_3) implies $g(G_u,R_u) \geq 25$ and we can assume that $g(1) = 6$. Let $G_1 = \{u,11,\dots,15\}$. By (P_3), we can assume that $g(7, G_1 \setminus \{u\}) \geq 3$ and $g(8, G_1 \setminus \{u\}) \geq 3$. Then the edge $\{7,8\}$ must be red. The interdiction of a red $(8,24)$–graph implies that $g(7,G_u) \geq 3$ or $g(8,G_u) \geq 3$, say $g(7,G_u) \geq 3$. We can assume that $G_7 = \{4,5,6,11,12,13\}$. The K_7 induced by $\{u,7,\dots,10,14,15\}$ must contain a green edge $\{v,w\}$. Then $g(\{v,w\},\{4,5,6\}) \leq 3$. We can assume that the edges $\{v,4\}$ and $\{v,5\}$ are red, and, because of (P_1), that $\{v,11\}$ and $\{v,3\}$ are green. Using $K_3 \not\subset G$ and (P_1) the edge $\{3,11\}$ has to be red, $\{3,12\}$ and $\{3,13\}$ must be green, $\{v,12\}$ and $\{v,13\}$ red, $\{v,6\}$ green, $\{v,14\}$ or $\{v,15\}$ green, say $\{v,14\}$ green, $\{14,6\}$ and $\{14,3\}$ red. But then $G_7 \cup \{3,14\}$ yields a red $(8,24)$-graph. □

Lemma 5. $r(K_3, \langle 8,25 \rangle) \leq 18$.

Proof: Suppose that we have a $(K_3, \langle 8,25\rangle)$-coloring χ of K_{18}. Then χ must have properties (P_1) and (P_2) from Lemma 3. (P_1) and (P_2) imply

(P_5) If W is the vertex set of a red K_6, then $g(v,W) \geq 1$ for every $v \in V \setminus W$ with equality for at most one vertex, and $g(v,W) \leq 2$ for at most six $v \in V \setminus W$.

Using the fact that a subgraph induced by six vertices contains a red K_3 in χ (since $r(K_3,K_3) = 6$) we obtain

(P_6) If U is the vertex set of a red K_5, then $g(v,U) \leq 1$ for at most five $v \in V \setminus U$.

Moreover, (P_1) implies $\Delta \leq 6$. We distinguish three cases depending on Δ.
Case I: $\Delta \leq 4$. Since $r(K_3,K_6) = 18$ there must be a red K_6 in χ. Let W be its vertex set. Then (P_5) implies that $g(V \setminus W, W) \geq 29$ contradicting $\Delta \leq 4$.

Case II: $\Delta = 5$. Then $g(v, G_u) \geq 1$ for every $v \in R_u$ as otherwise (P_5) would contradict $\Delta = 5$. By (P_6) we obtain $19 \leq g(G_u, R_u) \leq \Delta(\Delta - 1) = 20$. But then (P_2) must be violated for at least one vertex in G_u.
Case III: $\Delta = 6$. Then (P_5) can be used to prove the χ has the additional property

(P_7) If $g(v) = 6$ and $U \subset R_v$ such that $|U| = 6$, then $g(U, G_v) \geq 12$.

This together with (P_5) implies that $g(G_v, R_v) \geq 27$ if $g(v) = 6$. Thus, we obtain

(P_8) If $g(v) = 6$ then there are at least three vertices $w \in G_v$ with $g(w) = 6$.

Now consider a vertex u with $g(u) = 6$. By (P_8), we can assume that $g(1) = g(2) = 6$. Let $W_1 = G_1 \cap G_2$, $W_2 = R_1 \cap R_2$, $W_3 = G_1 \cup G_2$ and $k = |W_1|$. Then $1 \leq k \leq 6$ and $g(W_1, W_2) \leq 4k$. Without loss of generality $G_1 \setminus \{u\} = \{13, \ldots, 17\}$ and $G_2 \setminus \{u\} = \{17, 16, \ldots 19 - k\} \cup \{12, 11, \ldots, 7 + k\}$. We distinguish six subbases depending on the value of k.
III.1: $k = 1$. Then (P_5) implies that $g(7, G_u) \geq 1$. Moreover, there must be at least three green edges from 7 to each of the two red subgraphs $K_7 - e$ induced by $G_1 \cup \{2\}$ and $G_2 \cup \{1\}$. This contradicts $g(7) \leq 6$.
III.2: $k = 4$. Then $g(\{11, 12\}, \{13, 14\}) = 4$ and all other edges in the subgraph induced by W_3 must be red. (P_5) and $K_3 \not\subset G$ imply that $g(v, W_1) \geq 1$ for every $v \in W_2$. Since $g(W_1, W_2) \leq 4k$, one of the following two cases must occur.

(i) $g(w, W_1) = 1$ for some $w \in W_2$. Then $g(w, G_1) = 1$ or $g(w, G_2) = 1$, say $g(w, G_1) = 1$, and $g(w, \{11, 12\}) = 2$. Moreover, $g(v, W_1) = 1$ for at most one $v \in W_2 \setminus \{w\}$, and there must be $v_1, \ldots, v_5 \in W_2 \setminus \{w\}$ with $g(v_i, W_1) \leq 2$. Since $g(13, G_2) = g(14, G_2) = 2$, (P_5) implies that $g(v_i, G_2) \geq 3$ for some v_i. But then $g(v_i, \{11, 12\}) \geq 1$ for this v_i and $G_1 \cup \{v_i, w\}$ yields a red $(8, 25)$-graph if $K_3 \not\subset G$.

(ii) $g(v, W_1) = 2$ for every $v \in W_2$. Then $\{7, \ldots, 10\}$ induces a red K_4. Applying (P_2) to u (in G_1 and G_2) and (P_5) to G_1 and G_2 we obtain that (without loss of generality) $g(x, \{11, 12\}) \geq 1$ for $x = 3, 4, 7, 8$ and $g(y, \{13, 14\}) \geq 1$ for $y = 5, 6, 9, 10$. Moreover, $g(\{7, 8\}, \{11, 12\}) \geq 3$ and $g(\{9, 10\}, \{13, 14\}) \geq 3$. We can assume that $g(7, \{11, 12\}) = g(10, \{13, 14\}) = 2$. Then $g(7, \{3, 4\}) = g(10, \{5, 6\}) = 0$ since $K_3 \not\subset G$, and $g(7, \{5, 6\}) = g(10, \{3, 4\}) = 2$ as otherwise a red $(8, 25)$-graph would occur. Since $\Delta = 6$ we obtain that $g(\{13, 14\}, \{3, \ldots, 6\}) \geq 3$. Consequently $g(\{a, b\}, \{3, \ldots, 6\}) \geq 1$ for every two vertices $a, b \in \{15, 16, 17\}$. Thus, we can assume that $g(7, \{16, 17\}) = g(10, \{15, 16\}) = 2$. But then a red $(8, 25)$-graph is induced by $G_7 \cup \{10, u\}$.

III.3: $k = 5$. (P_1) implies that $\{12, 13\}$ is green, and W_3 induces a red $K_7 - e$. Then $g(w, W_3) \geq 3$ and $g(w, W_1) \geq 2$ for every $w \in W_2$. Since $g(W_1, W_2) \leq 20$ there are $w_1, \ldots, w_7 \in W_2$ with $g(w_i, W_1) = 2$. We can assume that $g(13, \{w_1, \ldots, w_4\}) = 4$. Then $G_{13} \cup \{2\}$ induces a second red $K_7 - e$. Since $g(12) \leq 6$, there is some $v \in W_2 \setminus \{w_1, \ldots, w_4\}$ such that $\{v, 12\}$ is red. The interdiction of a red $(8, 25)$-graph and a green K_3 implies that $g(v, \{w_1, \ldots, w_4\}), g(v, W_1) \geq 3$. But then $G_v \cup \{12\}$ induces a red K_7 contradicting (P_1).

III.4: $k = 6$. (P_5) and (P_7) applied to G_1 yield $g(v) = 6$ for every $v \in G_1$ and $g(w, G_1) \leq 3$ for every $w \in W_2$. Moreover, $g(\{3, \ldots, 6\}, G_1 \setminus \{u\}) \geq 6$ because of (P_2). This implies that $g(v, \{3, \ldots, 6\}) \geq 2$ for some $v \in G_1 \setminus \{u\}$, say for $v = 17$. If none of the preceding cases is to occur (with u and 17 instead of 1 and 2), then $G_u = G_{17}$ and, as above, $g(v, G_u) \leq 3$ for every $v \in R_u \cap R_{17}$. This implies $g(v, \{3, \ldots, 6\}) = 1$ for every $v \in G_1 \setminus \{u, 17\}$, since (P_2) forces $g(\{3, \ldots, 6\}, G_1 \setminus \{u, 17\}) \geq 4$. We obtain that $g(v, \{7, \ldots, 12\}) = 3$ for every $v \in G_1 \setminus \{u, 17\}$ and $g(G_1 \setminus \{u, 17\}, \{7, \ldots, 12\}) = 12$. But then (P_5) implies a contradiction to (P_2) for one vertex in $G_1 \setminus \{u, 17\}$.

III.5: $k = 2$. Since $g(2, G_1) = g(1, G_1) = 2$ we obtain that $g(v, G_1), g(v, G_2) \geq 2$ for every $v \in W_2$. We distinguish two cases:

(i) $g(v, \{13, \ldots, 16\}) \leq 1$ or $g(v, \{9, \ldots, 12\}) \leq 1$ for every $v \in W_2$. Then $g(17, \{7, 8\}) = 2$ and, if none of the preceding cases is to occur (for u and 17 instead of 1 and 2), $g(17, \{3, \ldots, 6\}) \leq 1$. But then (P_2) is violated for u and G_1 or G_2.

(ii) $g(w, \{13, \ldots, 16\}) \geq 2$ and $g(w, \{9, \ldots, 12\}) \geq 2$ for some $w \in W_2$. We may assume that $11, 12, 15, 16 \in G_w$ and that a red K_6 is induced by these four vertices together with u and 17. Then $g(\{11, 12\}, \{13.14\}) = g(\{9, 10\}, \{15, 16\}) = 4$. If $g(17) = 6$, a situation equivalent to $k \geq 4$ occurs. If $g(15) = g(16) = 6$ and a situation equivalent to $k \geq 4$ is avoided, $\{1, 3, 4, 5, 6, 9, 10\}$ must induce a red K_7 if $K_3 \not\subset G$. This contradicts (P_1). We obtain that $g(\{15, 16, 17, u\}, W_2) \leq 12$ and similarly $g(\{11, 12, 17, u\}, W_2) \leq 12$. It can be shown that $g(v, \{15, 16, 17, u\}) \leq 1$ or $g(v, \{11, 12, 17, u\}) \leq 1$ for some $v \in W_2$ is impossible. This implies that $g(v, \{15, 16, 17, u\}) = g(v, \{11, 12, 17, u\}) = 2$ for all $v \in W_2$. Then $g(17, \{3, \ldots, 6\}) \geq 3$ would yield a red K_7 induced by $\{11, 12, 15, 16\}$ and three green neighbors of 17 in $\{3, \ldots, 6\}$, contradicting (P_1). Thus, we can assume that $3, 4 \in R_{17}$. But this leads to a red $(8, 25)$-graph induced by $\{2, 3, 4, 13, \ldots, 17\}$.

III.6: $k = 3$. We can assume that a situation equivalent to one of the preceding cases with $k \neq 3$ does not occur. Then χ must have the property

(P_9) If $z \in V$ such that $g(z) = 6$ then $|G_x \cap G_y| = 3$ for any two vertices $x, y \in G_z$ with $g(x) = g(y) = 6$.

As for (P_8) there are two vertices in $G_1 \setminus \{u\}$ and two in $G_2 \setminus \{u\}$ each one incident to six green edges. This leads to one of the following three cases:

(i) $g(16) = g(17) = 6$. Then (P_9) is violated for at least two of the vertices u, 16 and 17.

(ii) $g(11) = g(12) = g(13) = g(14) = 6$. Then (P_1) implies that the K_7 induced by $\{11, 12, 13, 14, 16, 17, u\}$ must contain a green edge. We can assume that $\{12, 13\}$ is green. By (P_9) we obtain that $14, 15 \in G_{12}$ and $10, 11 \in G_{13}$. But then (P_9) is violated for 13 and 14 or for 2 and 14.

(iii) $g(13) = g(16) = 6$, $g(17) \leq 5$. Since $|G_{16} \cap G_u| = 3$, we may assume that $G_{16} = \{1, 2, 6, 7, 8, 9\}$. Moreover, $|G_{13} \cap G_u| = |G_{13} \cap G_{16}| = 3$. Then $6 \notin G_{13}$ would imply a red K_7 induced by 1, 2, 6 and the four vertices in $\{3, 4, 5, 7, 8, 9\} \cap G_{13}$ contradicting (P_1). Thus, we can assume that $G_{13} = \{1, 3, 6, 9, 10, 11\}$. As for (P_8) (applied to G_2), there must be a vertex in $\{10, 11, 12\}$ incident to six green edges, and, as for the vertex 13, it must be joined green to 6. This implies $g(10), g(11) \leq 5$, $g(12) = 6$, and $\{12, 6\}$ has to be green. By (P_9), $g(6) \leq 5$. Now (P_8) applied to G_{13} yields $g(3) = g(9) = 6$, and (P_9) implies that $G_3 = \{u, 7, 8, 12, 13, 17\}$, $G_9 = \{4, 5, 12, 13, 16, 17\}$ and $G_{12} = \{2, 3, 6, 9, 14, 15\}$. If a red $(8, 25)$-graph is avoided in the subgraphs induced by $\{1, \ldots, 9\}$ and $\{10, \ldots, 17, u\}$ then the edges $\{4, 8\}$, $\{5, 7\}$, $\{10, 15\}$ and $\{11, 14\}$ must be green. Moreover, $|R_6 \cap \{10, 11, 14, 15\}| \geq 3$. We can assume that $10, 11, 14 \in R_6$. But then one of the vertices 10, 11 and 14 yields a red $(8, 25)$-graph together with seven suitable vertices in $\{1, \ldots, 9\}$.

□

Lemma 6. $r(K_3, \langle 8, 26 \rangle) \leq 21$.

Proof: Suppose that we have a $(K_3, \langle 8, 26 \rangle)$-coloring χ of K_{21}. It is easy to see that χ must have two further properties besides (P_2).

(P_{10}) If W is the vertex set of a red K_7 then $g(v, W) \geq 3$ for every $v \in V \setminus W$.

(P_{11}) If U is the vertex set of a red K_6 then $g(v, U) \leq 1$ for at most two $u \in V \setminus W$.

Since $\Delta \geq 8$ would imply a red K_8 containing a red $(8, 26)$-graph we obtain that $\Delta \leq 7$. Thus, one of the following four cases must occur.

Case I: $\Delta \le 4$. Since $r(K_3, K_6) = 18$, χ must contain a red K_6. Let U be its vertex set. But then (P_{11}) and $\Delta \le 4$ would imply $26 \le g(U, V \backslash U) \le 24$.
Case II: $\Delta = 5$. Then there is no red K_7 in χ because of (P_{10}) and we obtain that $g(v, G_u) \ge 1$ for every $v \in R_u$ by (P_{11}). Let V_i be the set of vertices v in R_u such that $g(v, G_u) = i$. If $|V_1| > 10$, we would obtain that $g(w, V_1) \ge 3$ for some $w \in G_u$ and this would lead to a red K_7. It remains that $|V_1| = 10$, $V_2 = R_u \setminus V_1$ and $g(w, R_u) = 4$ for every $w \in G_u$, since $\Delta = 5$ implies $g(G_u, R_u) \le 20$. But then (P_2) is violated for G_u.
Case III: $\Delta = 7$. Because of (P_{10}), $g(G_u, V \setminus G_u) \ge 46$. Thus, we can assume that $g(1) = g(2) = 7$. Let $k = |G_1 \cap G_2|$. We obtain that $k \le 6$ as otherwise (P_{10}) would imply $g(G_1, V \backslash G_1) \ge 50$, contradicting $\Delta = 7$. Then (P_{10}) applied to 2 and G_1 and to $v \in G_2 \backslash G_1$ and G_1 yields $3 \le k \le 4$. Since $g(G_1 \cap G_2, R_1 \cap R_2) \le 5k$, there exists $v \in R_1 \cap R_2$ so that $g(v, G_1 \cap G_2) \le 1$ if $k = 3$ and $g(v, G_1 \cap G_2) \le 2$ if $k = 4$. But then, by (P_{10}), v is joined green to a vertex $w \in G_2 \setminus G_1$, to two vertices from $G_1 \setminus G_2$ if $k = 3$ and to one vertex from $G_1 \setminus G_2$ if $k = 4$. Thus, $g(w, G_1) \le 2$ contradicting (P_{10}).
Case IV: $\Delta = 6$. As an immediate consequence of (P_{10}) we obtain the property

(P_{12}) If U is the vertex set of a red K_6 and $g(v, U) \ge 4$ for some $v \in V \setminus U$, then $g(v, U) \ge 1$ for every $v \in V \setminus U$.

Next we will prove

(P_{13}) If $x, y, z \in V$, $g(x) = 6$ and $g(y, G_x) = g(z, G_x) = 2$, then $|(G_y \cup G_z) \cap G_x| \ge 3$.

Assume the contrary. Then $(G_y \cup G_z) \cap G_x) = \{v_1, v_2\}$. Denote the four other vertices in G_x by $v_3, \dots, v_6$. Let $A = \{v_3, \dots, v_6\}$ and let $B = V \setminus (G_x \cup \{x, y, z\})$. Note that $\Delta = 6$ implies that $g(A, B) \le 20$ and $g(G_x, B) \le 26$. Consider the two red K_6 induced by G_x and $A \cup \{x, y\}$. (P_{12}) implies that $g(w, A) \ge 1$ for every $w \in B$ with equality for at most four vertices because of (P_{11}). Since $g(A, B) \le 20$ there must be four vertices $w_1, \dots, w_4$ in B such that $g(w_i, A) = 1$ and eight vertices $w \in B$ with $g(w, A) = 2$. Moreover, $g(v_i) = 6$ for $i = 3, \dots, 6$. We can assume that $g(w_1, G_x) = g(w_2, G_x) = 1$ and $g(w_3, A \cup \{y, z\}) = g(w_4, A \cup \{y, z\}) = 1$. Then the edge $\{w_1, w_2\}$ must be green and, without loss of generality, also $\{w_1, v_5\}$ and $\{w_2, v_6\}$. Moreover, we can assume that $\{w_1, y\}$ and $\{w_2, z\}$ are green because of (P_{11}). This implies that $\{w_1, z\}$ is red. Similarly it can be shown that $g(w_3, G_x) = g(w_4, G_x) = 2$. Since $g(B, G_x) \le 26$, there are at most four $w \in B$ such that $g(w, G_x) \ge 3$. This together with (P_2) and $g(v_5) = g(v_6) = 6$ implies that v_5 is joined green to three vertices w_5, w_6, w_7, where v_6 is joined green to two of them, say to w_5 and w_6. But then $\{v_3, v_4, y, z, w_1, w_5, w_6, w_7\}$ leads to a red $(8, 26)$-graph if $K_3 \not\subset G$. Thus, (P_{13}) is proved.

Now consider a vertex u with $g(u) = 6$. Then $g(w, G_u) \geq 1$ for every $w \in R_u$ because of (P_{12}). Let m be the number of vertices $w \in R_u$ such that $g(w, G_u) = 1$. By (P_{11}), $0 \leq m \leq 2$. Using (P_2) and (P_{13}), it can be shown that in case of $m = 2$ the edges between G_u and R_u must be colored (up to isomorphism) as described by the matrix M_1 in Figure 2, where $\{i, j\}$ is green iff there is a "1" in the ith row and $(j-6)$th column. Similarly, in case of $m = 1$, the edges between G_u and R_u must be colored as described by the matrix M_2 in Figure 2.

$$M_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Figure 2.

We distinguish three cases depending on the value of m.

(i) $m = 2$. Consider the red K_6 induced by G_6. Then $g(9, G_6) \geq 1$ by (P_{12}), i.e. $g(9, \{13, 15\}) \geq 1$. First let $g(9, G_6) = 1$ and let $x \in \{13, 15\}$ such that $\{x, 9\}$ is green. Then $g(w, G_6) \geq 2$ for every $w \in R_6 \setminus \{9\}$ (otherwise, we would obtain the same situation for G_6 as for G_u, and x must be joined green to three of the four vertices $w \in R_6$ with $g(w, G_6) = 3$, contradicting $g(2, G_6) = g(5, G_6) = 3$ and $g(\{2, 5\}, \{13, 15\}) = 0$). This together with $g(G_6 \setminus \{x\}, R_6) \leq 25$ and $g(2, G_6 \setminus \{x\}) = g(5, G_6 \setminus \{x\}) = 3$ implies that there are two vertices $w \in R_6 \setminus \{9\}$ such that $g(w, G_6 \setminus \{x\}) = 1$ and the edges $\{w, x\}$ are green, contradicting (P_2). We still have to consider that $g(9, \{13, 15\}) = 2$. Similarly it can be proved that $g(9, G_2) = 2$, i.e. $g(9, \{8, 10\}) = 2$. But then $g(9, G_3) = 1$ since $g(9) \leq 6$ and we obtain a contradiction as above.

(ii) $m = 1$. First suppose that $g(7, \{9, \ldots, 12\}) \leq 2$. Then we can assume that the edges $\{7, 9\}$ and $\{7, 10\}$ are red. But this leads to a red $(8, 26)$-graph in $\{3, \ldots, 10\}$ if a green K_3 is avoided. Now suppose that $g(7, \{9, \ldots, 12\}) = 4$. Then one of the edges $\{7, 13\}$ and $\{7, 18\}$, say $\{7, 13\}$, must be red since $g(7) \leq 6$. This yields a red K_6 induced by

$\{2,5,6,7,13,19\}$. Moreover, $g(7) \leq 6$ implies that $g(7,\{14,16\}) = 0$ or $g(7,\{15,17\}) = 0$. Again we obtain a red $(8,26)$-graph if $K_3 \not\subset G$. It remains that $g(7,\{9,\ldots,12\}) = 3$. We can assume that the edge $\{7,9\}$ is red. Then $G_7 = \{1,10,11,12,13,18\}$ and $g(u,G_7) = g(9,G_7) = 1$. But this yields a situation equivalent to the preceding case (i).

(iii) $m = 0$. It is easy to see that in this case we can assume the property

(P_{14}) $g(w,G_x) \geq 2$ for every $x \in V$ with $g(x) = 6$ and every $w \in R_x$.

Thus, $g(G_u,R_u) \geq 28$ and, without loss of generality, $g(i) = 6$ for $i = 1,\ldots,4$. Let $k = |G_1 \cap G_2|$. By (P_{14}), $2 \leq k \leq 6$. We can assume that $G_1 = \{16,\ldots,20,u\}$ and $G_2 = \{u,20,19,\ldots,22-k\} \cup \{15,14,\ldots,10+k\}$. If $k \geq 5$ we obtain that $g(G_1,R_1) \geq 31$ by (P_{14}), contradicting $\Delta = 6$. If $k = 4$, then $K_3 \not\subset G$ forces that $g(\{14,15\},\{18,19,20,u\}) = 0$. Moreover, (P_{14}) implies that $g(\{14,15\},\{16,17\}) = 4$, contradicting (P_{13}). In case of $k = 2$ we obtain that $g(w,\{20,u\}) = 0$ for some $w \in \{3,\ldots,12\}$ and, by (P_{14}), that $g(w,\{16,\ldots,19\}) \geq 2$ and $g(w,\{12,\ldots,15\}) \geq 2$. We can assume that $12,13,16,17 \in G_w$. But then $K_3 \not\subset G$ and (P_{14}) yield that $G_{12} \cap G_1 = G_{13} \cap G_1 = \{18,19\}$, contradicting (P_{13}). The remaining case is $k = 3$. By symmetry, $|G_3 \cap G_1| = |G_4 \cap G_1| = 3$. But then (P_{14}) implies that $g(R_1,G_1) \geq 31$, contradicting $\Delta = 6$, and the proof of Lemma 6 is complete.

□

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