The Polytope Of Generalized Tournament Matrices With A Common Integral Score Vector*

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ABSTRACT. We consider the polytope $\mathcal{P}(\mathbf{s})$ of generalized tournament matrices with score vector s. For the case that s has integer entries, we find the extreme points of $\mathcal{P}(\mathbf{s})$ and discuss the graph-theoretic structure of its 1-skeleton.

1 Introduction

A *tournament* of order n is a loop free directed graph on the vertices $\{1, 2, \ldots, n\}$ with the property that for any distinct vertices i and j, either $i \to j$ or $j \to i$, but not both. A *tournament matrix* is the adjacency matrix of a tournament, and it follows that $\mathbf{T}$ is a tournament matrix if and only if it is a $(0,1)$ matrix satisfying $\mathbf{T} + \mathbf{T}^t = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the all ones matrix of the appropriate order. There is a wealth of literature on tournaments (see [2], [16] and their lists of references), while tournament matrices have been the subject of a number of recent papers (for example, [7], [8], [9], [10], [11], [12], [13], [18]).

In [17], Moon and Pullman introduced the notion of a *generalized tournament matrix*, that is an entrywise nonnegative matrix $\mathbf{M}$ satisfying $\mathbf{M} + \mathbf{M}^t = \mathbf{J} - \mathbf{I}$. Evidently the set of all $n \times n$ generalized tournament matrices

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forms a convex polytope, and it was shown in [17] that the extreme points of that polytope are the tournament matrices of order n. Thus $\mathbf{M}$ is a generalized tournament matrix if and only if it is a convex combination of tournament matrices.

Given a generalized tournament matrix $\mathbf{M}$, its *score vector* $\mathbf{s}$ is just the vector of row sums of $\mathbf{M}$ - i.e. $\mathbf{s} = \mathbf{M}\mathbf{1}$, where $\mathbf{1}$ is the all ones vector. In some recent work, the score vector has been of some use in discussing spectral properties of generalized tournament matrices, particularly in bounding their spectral radii (see [12], [8] and [13]). If $\mathbf{s}$ is the score vector of some generalized tournament matrix, it is clear that $\mathcal{P}(\mathbf{s})$, the set of all generalized tournament matrices with score vector $\mathbf{s}$, forms a convex polytope. It is natural to wonder about the structure of such a polytope, and in this paper we investigate that structure in the case that $\mathbf{s}$ is a score vector with (necessarily nonnegative) integer entries. Specifically, we focus on the 1-skeleton of $\mathcal{P}(\mathbf{s})$, which we denote by $\mathcal{S}(\mathbf{s})$. It is the (undirected) graph whose vertices are the extreme points of $\mathcal{P}(\mathbf{s})$, with two vertices adjacent in $\mathcal{S}(\mathbf{s})$ whenever they are on the same 1-face of $\mathcal{P}(\mathbf{s})$.

In Section 3 we show that for an integral score vector $\mathbf{s}$, the extreme points of $\mathcal{P}(\mathbf{s})$ are exactly the tournament matrices with score vector $\mathbf{s}$, and that two tournament matrices are adjacent in $\mathcal{S}(\mathbf{s})$ if and only if their associated tournaments differ only in the orientation of a single (directed) cycle. We show further that $\mathcal{S}(\mathbf{s})$ has two adjacent vertices with the same degree, and in Section 4, we examine some of the special properties of $\mathcal{S}(\hat{\mathbf{s}}_n)$, where $\hat{\mathbf{s}}_n = (1, 1, 2, 3, 4, \ldots, n-3, n-2, n-2)^t$.

We remark that Brualdi and Li [4] have also looked at a certain graph whose vertices are the tournament matrices with a common score vector and its relationship with $\mathcal{S}(\mathbf{s})$ is discussed in Section 3. In a different direction, another polytope consisting of generalized tournament matrices - the polytope of generalized transitive tournament matrices - has been studied by Brualdi and Hwang [3] and Cruse [6] in connection with a problem posed by Mirsky [14].

2 Preliminaries

Given an $n \times n$ nonnegative matrix $\mathbf{M}$, we say that it is *reducible* if $n > 1$ and there is a permutation matrix $\mathbf{P}$ such that $\mathbf{P}\mathbf{M}\mathbf{P}^t = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{X} & \mathbf{B} \end{array}\right]$ where $\mathbf{A}$ and $\mathbf{B}$ are (nonvacuous) square matrices and $\mathbf{0}$ is the zero matrix of the appropriate size. If $n = 1$, or no such permutation matrix $\mathbf{P}$ exists, we say that $\mathbf{M}$ is *irreducible*, and it is well-known that $\mathbf{M}$ is irreducible if and only if the directed graph associated with $\mathbf{M}$ is strongly connected. In the case that $\mathbf{M}$ is a generalized tournament matrix with score vector $\mathbf{s}$, Moon and Pullman [17] have shown that $\mathbf{M}$ is reducible if and only if there is a permutation matrix $\mathbf{P}$ such that $\mathbf{PMP^t} = \left[\begin{array}{c|c} \mathbf{M}_1 & \mathbf{0} \\ \hline \mathbf{J} & \mathbf{M}_2 \end{array}\right]$ where $\mathbf{M}_1$ and $\mathbf{M}_2$ are generalized tournament matrices of smaller order, and $\mathbf{J}$ is the all-ones matrix of the appropriate size. Moreover, $\mathbf{P}$ can be taken to be a permutation matrix such that the entries of $\mathbf{Ps}$ are in nondecreasing order. It follows that if $\overline{\mathbf{M}}$ is any other matrix in $\mathcal{P}(\mathbf{s})$, then $\mathbf{P}\overline{\mathbf{M}}\mathbf{P^t} = \left[\begin{array}{c|c} \overline{\mathbf{M}}_1 & \mathbf{0} \\ \hline \mathbf{J} & \overline{\mathbf{M}}_2 \end{array}\right]$ where $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ are generalized tournament matrices of the same orders as $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Thus we see that if one matrix in $\mathcal{P}(\mathbf{s})$ is irreducible, then they all are, while if one matrix in $\mathcal{P}(\mathbf{s})$is reducible, then all of the matrices in $\mathcal{P}(\mathbf{s})$ can be simultaneously permuted into a common block triangular form, using a common permutation matrix.

Now suppose that $\mathbf{M}$ is a reducible generalized tournament matrix with score vector $\mathbf{s}$. From our observations above, we may assume without loss of generality that $\mathbf{M}$ has the form $\left[\begin{array}{c|c} \mathbf{M}_1 & \mathbf{0} \\ \hline \mathbf{J} & \mathbf{M}_2 \end{array}\right]$ and hence that any $\overline{\mathbf{M}}$ in $\mathcal{P}(\mathbf{s})$ can be partitioned (using the same partitioning as for $\mathbf{M}$) as $\left[\begin{array}{c|c} \overline{\mathbf{M}}_1 & \mathbf{0} \\ \hline \mathbf{J} & \overline{\mathbf{M}}_2 \end{array}\right]$. Recall that $\mathbf{T}$ is an *extreme point* of $\mathcal{P}(\mathbf{s})$ if and only if, whenever we have $\mathbf{T} = \Sigma_{i=1}^k \alpha_i \mathbf{T}_i$ for matrices $\mathbf{T}_1, \ldots, \mathbf{T}_k$ in $\mathcal{P}(\mathbf{s})$ and nonnegative constants $\alpha_1, \ldots, \alpha_k$ such that $\Sigma_{i=1}^k \alpha_i = 1$, then we must have $\mathbf{T}_1 = \mathbf{T}$ for any i such that $\alpha_i > 0$. It now follows that if $\mathbf{M}$ is as above, then $\mathbf{M}$ is an extreme point of $\mathcal{P}(\mathbf{s})$ if and only if $\mathbf{M}_1$ is an extreme point of $\mathcal{P}(\mathbf{s}_1)$ and $\mathbf{M}_2$ is an extreme point of $\mathcal{P}(\mathbf{s}_2)$, where $\mathbf{s}_1$ and $\mathbf{s}_2$ are the score vectors of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively.

Recall further that if $\mathbf{T}$ and $\overline{\mathbf{T}}$ are extreme points of $\mathcal{P}(\mathbf{s})$, then they are *adjacent as vertices of* $\mathcal{S}(\mathbf{s})$ if and only if for any $0 < t < 1$, we have that if both $t\mathbf{T} + (1-t)\overline{\mathbf{T}} + \mathbf{X}$ and $t\mathbf{T} + (1-t)\overline{\mathbf{T}} - \mathbf{X}$ are in $\mathcal{P}(\mathbf{s})$, then $\mathbf{X}$ must be a scalar multiple of $\mathbf{T} - \overline{\mathbf{T}}$. Consequently, if $\mathbf{M}$ and $\overline{\mathbf{M}}$ above are extreme points of $\mathcal{P}(\mathbf{s})$, then they are adjacent in $\mathcal{S}(\mathbf{s})$ if and only if either $\mathbf{M}_1$ and $\overline{\mathbf{M}}_1$ are adjacent in $\mathcal{S}(\mathbf{s}_1)$ and $\mathbf{M}_2 = \overline{\mathbf{M}}_2$, or $\mathbf{M}_2$ and $\overline{\mathbf{M}}_2$ are adjacent in $\mathcal{S}(\mathbf{s}_2)$ and $\mathbf{M}_1 = \overline{\mathbf{M}}_1$. Thus $\mathcal{S}(\mathbf{s})$ is the Cartesian product of the graphs $\mathcal{S}(\mathbf{s}_1)$ and $\mathcal{S}(\mathbf{s}_2)$, so that the problem of determining the structure of $\mathcal{S}(\mathbf{s})$ is reduced to that of studying $\mathcal{S}(\mathbf{s}_1)$ and $\mathcal{S}(\mathbf{s}_2)$, both of which involve matrices of lower order. In other words, the reducible case can always be discussed in terms of the irreducible case. For this reason we will assume henceforth without loss of generality that our polytopes arise from irreducible generalized tournament matrices of order 2 or more (the 1×1 zero matrix is the only generalized tournament matrix of order 1, and is not especially interesting).

We note that for any nonnegative n-vector $\mathbf{s}$, the following straightforward test determines whether or not $\mathbf{s}$ is the score vector of an irreducible generalized tournament matrix:

Let $\sigma_1, \ldots, \sigma_n$ be the entries of sordered so that $\sigma_1 \leq \cdots \leq \sigma_n$. Then there is an irreducible generalized tournament matrix

with score vector s if and only if $\Sigma_{i=1}^{k} \sigma_i > \binom{k}{2}$ for $\qquad (*)$

$1 \leq k \leq n-1$ and $\Sigma_{i=1}^{n} \sigma_i = \binom{n}{2}$.

(This follows from a result of Moon [15].)

Throughout the sequel, we will deal only with score vectors consisting of integer entries. We will refer to a nonnegative integral n-vector s which satisfies $(*)$ as a *strong tournament score vector.*

3 General Properties of $\mathcal{S}(\mathbf{s})$

We begin by determining the extreme points of $\mathcal{P}(\mathbf{s})$ when s is a strong tournament score vector.

Theorem 1. *Let* s *be a strong tournament score vector. A matrix* **T** *is an extreme point of* $\mathcal{P}(\mathbf{s})$ *if and only if* **T** *is a tournament matrix with score vector* s.

Proof: Suppose that **T** is a tournament matrix with score vector s, and that $\mathbf{T} = \Sigma_{i=1}^{k} \alpha_i \mathbf{M}_i$ for some collection of matrices $\mathbf{M}_1, \ldots, \mathbf{M}_k$ in $\mathcal{P}(\mathbf{s})$ and nonnegative constants $\alpha_1, \ldots, \alpha_k$ such that $\Sigma_{i=1}^{k} \alpha_i = 1$. For each pair of indices i and j such that the (i,j) entry of **T** is 0, we must have that for any m such that $\alpha_m > 0$, the (i,j) entry of $\mathbf{M}_m$ is also 0. Further, from the fact that $\mathbf{M}_m + \mathbf{M}_m^t = J - I$, it follows that if the (i,j) entry of **T** is 1, then so is the (i,j) entry of $\mathbf{M}_m$. Hence $\mathbf{M}_m$ must equal **T** whenever $\alpha_m > 0$, so that **T** is an extreme point.

Now suppose that $\mathbf{M} \in \mathcal{P}(\mathbf{s})$, but that **M** is not a tournament matrix. Then some entry of **M**, say m_{pq}, is strictly between 0 and 1. Since the row sums of **M** are integers, and since necessarily $0 < m_{qp} < 1$, there is an index r such that $0 < m_{qr} < 1$. Similarly, there is an index s such that $0 < m_{rs} < 1$. Continuing in this way, we find that there are indices $i_1, i_2, \ldots, i_k$ such that $0 < m_{i_j i_{j+i}} < 1$ for $1 \leq j \leq k-1$, and $0 < m_{i_k i_1} < 1$. From the fact that **M** is a generalized tournament matrix, we see that also, $0 < m_{i_{j+1} i_j} < 1$ for $1 \leq j \leq k-1$, and $0 < m_{i_1 i_k} < 1$. Now let **C** be the $(0,1)$ matrix with $c_{i_j i_{j+1}} = 1$ for $1 \leq j \leq k-1$, $c_{i_k i_1} = 1$, and zeros elsewhere. Then for a sufficiently small $\varepsilon > 0$, we see that both $\mathbf{M} + \varepsilon(\mathbf{C} - \mathbf{C}^t)$ and $\mathbf{M} - \varepsilon(\mathbf{C} - \mathbf{C}^t)$ are in $\mathcal{P}(\mathbf{s})$, and that **M** can be written as $\mathbf{M} = (1/2)\{\mathbf{M} + \varepsilon(\mathbf{C} - \mathbf{C}^t)\} + (1/2)\{\mathbf{M} - \varepsilon(\mathbf{C} - \mathbf{C}^t)\}$. Consequently, **M** is not an extreme point of $\mathcal{P}(\mathbf{s})$. □

As we remarked above, Moon and Pullman [17] showed that any generalized tournament matrix can be written as a convex combination of tournament matrices. Theorem 1 has a similar flavor, showing in particular that if $\mathbf{s}$ is a strong tournament score vector, then any generalized tournament matrix with score vector $\mathbf{s}$ can be written as a combination of tournament matrices with that score vector. Our next result determines adjacency in $\mathcal{S}(\mathbf{s})$.

Theorem 2. *Let $\mathbf{s}$ be a strong tournament score vector and let $\mathbf{T}_1$ and $\mathbf{T}_2$ be two extreme points of $\mathcal{P}(\mathbf{s})$. Then $\mathbf{T}_1$ and $\mathbf{T}_2$ are adjacent in $\mathcal{S}(\mathbf{s})$ if and only if their associated tournaments differ only in the orientation of a single simple directed cycle.*

Proof: It suffices to show that $\mathbf{T}_1$ and $\mathbf{T}_2$ are adjacent in $\mathcal{S}(\mathbf{s})$ if and only if there are distinct indices $i_1, i_2, \ldots, i_k$ such that $\mathbf{T}_1 - \mathbf{T}_2 = \mathbf{P} - \mathbf{P}^t$, where $\mathbf{P}$ is the (0,1) matrix with $p_{i_j i_{j+1}} = 1$ for $1 \le j \le k-1$, $p_{i_k i_1} = 1$, and all other entries 0.

Suppose that $\mathbf{T}_1 - \mathbf{T}_2 = \mathbf{P} - \mathbf{P}^t$ for such a $\mathbf{P}$, and notice that the entries of $\mathbf{T}_1$ and $\mathbf{T}_2$ agree except in the positions where either $\mathbf{P}$ or $\mathbf{P}^t$ has a nonzero entry: denote by L the set of positions where $\mathbf{T}_1$ and $\mathbf{T}_2$ do not agree. We want to show that the set $\mathcal{E} = \{\mathbf{T}_1 + t(\mathbf{P}^t - \mathbf{P}) | 0 \le t \le 1\}$ is a 1-face of $\mathcal{P}(\mathbf{s})$. Fix a $0 < t < 1$, and suppose that both $\mathbf{T}_1 + t(\mathbf{P}^t - \mathbf{P}) + \mathbf{M}$ and $\mathbf{T}_1 + t(\mathbf{P}^t - \mathbf{P}) - \mathbf{M}$ are in $\mathcal{P}(\mathbf{s})$. Note that $\mathbf{M}$ must be a skew-symmetric matrix with all row and column sums equal to 0. Fix a position (i,j) which is not in the set L. Then $p_{ij} = p_{ji} = 0$, and either the (i,j) or the (j,i) entry of $\mathbf{T}_1$ is 0. Since both $\mathbf{T}_1 + t(\mathbf{P}^t - \mathbf{P}) + \mathbf{M}$ and $\mathbf{T}_1 + t(\mathbf{P}^t - \mathbf{P}) - \mathbf{M}$ are nonnegative, we see that one of m_{ij} and m_{ji} must be zero, and since $\mathbf{M}$ is skew-symmetric, in fact both must be zero. Consequently, $\mathbf{M}$ can have nonzero entries only in positions listed in L.

Suppose that $m_{i_1 i_2} = \alpha$. From the facts that $\mathbf{M1} = \mathbf{0}$ and that $\mathbf{M}$ can have nonzero entries only in positions listed in L, it follows that $m_{i_1 i_k} = -\alpha$. Further, since $\mathbf{M}$ is skew-symmetric, $m_{i_2 i_1} = -\alpha$, which in turn yields that $m_{i_2 i_3} = \alpha$. Continuing in this way, we see that $\mathbf{M} = \alpha(\mathbf{P} - \mathbf{P}^t)$, and hence that $\mathcal{E}$ is a 1-face of $\mathcal{P}(\mathbf{s})$.

Now suppose that $\mathbf{T}_1$ and $\mathbf{T}_2$ are adjacent vertices of $\mathcal{S}(\mathbf{s})$, so that $\mathcal{E} = \{t\mathbf{T}_1 + (1-t)\mathbf{T}_2 | 0 \le t \le 1\}$ is a 1-face of $\mathcal{P}(\mathbf{s})$. Since $\mathbf{T}_1 - \mathbf{T}_2$ is a $(0, 1, -1)$ skew-symmetric matrix with all row and column sums equal to zero, we see that any row or column of $\mathbf{T}_1 - \mathbf{T}_2$ which contains a 1 must also contain a -1. Let $\mathbf{A} = \mathbf{T}_1 - \mathbf{T}_2$ and suppose that $a_{pq} = 1$ for some indices p and q. Then $a_{qp} = -1$, so there is an index r such that $a_{qr} = 1$, Similarly, there is an index s such that $a_{rs} = 1$. Continuing in this way, we find that there are distinct indices $i_1, i_2, \ldots, i_k$ such that $a_{i_j i_{j+1}} = 1$ for $1 \le j \le k-1$, and $a_{i_k i_1} = 1$. Let $\mathbf{P}$ be the (0,1) matrix having ones in positions (i_j, i_{j+1}), $1 \le j \le k-1$ and in position (i_k, i_1), and zeros elsewhere. It follows that each nonzero entry of $\mathbf{P} - \mathbf{P}^t$ is equal to the corresponding entry in A. In particular, $\mathbf{T}_1 + \mathbf{T}_2 \geq \mathbf{P} - \mathbf{P}^t$ and $\mathbf{T}_1 + \mathbf{T}_2 \geq \mathbf{P}^t - \mathbf{P}$, so that for sufficiently small $\varepsilon > 0$, both $(1/2)\mathbf{T}_1 + (1/2)\mathbf{T}_2 + \varepsilon(\mathbf{P} - \mathbf{P}^t)$ and $(1/2)\mathbf{T}_1 + (1/2)\mathbf{T}_2 - \varepsilon(\mathbf{P} - \mathbf{P}^t)$ are nonnegative, both have row sums given by $\mathbf{s}$, and hence both are in $\mathcal{P}(\mathbf{s})$. Since $\mathcal{E}$ is a 1-face of $\mathcal{P}(\mathbf{s})$, $\varepsilon(\mathbf{P} - \mathbf{P}^t)$ must be a scalar multiple of $\mathbf{T}_1 - \mathbf{T}_2$, and hence $\mathbf{P} - \mathbf{P}^t$ is a scalar multiple of $\mathbf{T}_1 - \mathbf{T}_2$. Since both are $(0, 1, -1)$ matrices, we see that either $\mathbf{T}_1 - \mathbf{T}_2 = \mathbf{P} - \mathbf{P}^t$ or $\mathbf{T}_1 - \mathbf{T}_2 = \mathbf{P}^t - \mathbf{P}$. In either case, the result now follows. □

We remark that taken together, Theorems 1 and 2 provide a viewpoint on the tournament matrices and their associated tournaments: each tournament matrix $\mathbf{T}$ can be thought of as a vertex of $\mathcal{S}(\mathbf{T1})$, and the cycles in the tournament associated with $\mathbf{T}$ are in $1-1$ correspondence with the vertices of $\mathcal{S}(\mathbf{T1})$ which are adjacent to $\mathbf{T}$. In particular, the following corollary is immediate.

Corollary 2.1. *Let $\mathbf{s}$ be a strong tournament score vector, and let $\mathbf{T}$ be a tournament matrix with score vector $\mathbf{s}$. The degree of $\mathbf{T}$ as a vertex of $\mathcal{S}(\mathbf{s})$ is equal to the number of cycles in the tournament associated with $\mathbf{T}$.*

It has been shown (see [16]) that any strongly connected tournament on n vertices contains at least $(n-1)(n-2)/2$ cycles. This observation yields the following.

Corollary 2.2. *Let $\mathbf{s}$ be a strong tournament score vector with n entries. The degree of any vertex of $\mathcal{S}(\mathbf{s})$ is at least $(n-1)(n-2)/2$.*

In [4], Brualdi and Li consider a strong tournament score vector and define what they call the interchange graph, $\mathcal{G}(\mathbf{s})$, as follows: its vertices are the tournament matrices with score vector $\mathbf{s}$, and two vertices are adjacent if and only if one can be obtained from the other by the reversal of the orientation of the arcs on a single 3-cycle in the associated tournament. While the interchange graph was formulated without reference to the polytope $\mathcal{P}(\mathbf{s})$, it is clear that the interchange graph $\mathcal{G}(\mathbf{s})$ is a spanning subgraph of $\mathcal{S}(\mathbf{s})$. In particular, Brualdi and Li's result [4, Theorem 2.6] that $\mathcal{G}(\mathbf{s})$ is 2-connected leads immediately to the following:

Corollary 2.3. *If $\mathbf{s}$ is a strong tournament score vector, then $\mathcal{S}(\mathbf{s})$ is 2-connected.*

Our last result of this section gives a little of the structure of $\mathcal{P}(\mathbf{s})$. To prove it, we need the following fact.

Proposition 1. *Suppose that $\mathbf{T}$ is a tournament and that two vertices of $\mathbf{T}$ have the same outdegree. Then those two vertices are on a 3-cycle.*

Proof: Without loss of generality, we will suppose that the two vertices with common outdegree are 1 and 2, and that $1 \to 2$. If it were that case that for every i such that $2 \to i$ we also had $1 \to i$, then the outdegree of 1 would exceed the outdegree of 2, contrary to the hypothesis. Hence there is some vertex i_1 such that $2 \to i_1$ and $i_1 \to 1$, so that $1 \to 2 \to i_1 \to 1$ is a 3-cycle in T. □

Theorem 3. *Suppose that* $\mathbf{s}$ *is a strong tournament score vector of order* n*. Then there are two adjacent vertices in* $S(\mathbf{s})$ *which have the same degree.*

Proof: We begin by noting that since $\mathbf{s}$ is a strong tournament score vector, there are at least two entries of $\mathbf{s}$ which are the same (this follows from the fact that if an integral score vector were to have all distinct entries, those entries would necessarily be $0, 1, \ldots, n-1$). So without loss of generality, we may assume that the first two entries of $\mathbf{s}$ are equal. So if $\mathbf{T}$ is a tournament matrix with score vector $\mathbf{s}$, by Proposition 1, the vertices 1 and 2 in the tournament corresponding to $\mathbf{T}$ are on a 3-cycle: again without loss of generality, we will suppose that that 3-cycle is $1 \to 2 \to 3 \to 1$. It now follows that by appropriate reordering of the rows and columns numbered 4 and above, $\mathbf{T}$ can be taken to have the following form:

$$\mathbf{T} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} & \mathbf{H} \\ \hline
\begin{array}{c} \mathbf{A}' \\ \hline \mathbf{B}' \\ \hline \mathbf{C}' \\ \hline \mathbf{D}' \\ \hline \mathbf{E}' \\ \hline \mathbf{F}' \\ \hline \mathbf{G}' \\ \hline \mathbf{H}' \end{array} & \multicolumn{8}{c}{\mathbf{X}}
\end{array}\right]$$

where $\mathbf{X}$ is a tournament matrix of lower order, and where the other blocks, some of which may be empty, are defined as

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{1}_a^t \\ \hline \mathbf{0}_a^t \\ \hline \mathbf{0}_a^t \end{array}\right]_{3\times a}, \mathbf{B} = \left[\begin{array}{c} \mathbf{1}_b^t \\ \hline \mathbf{0}_b^t \\ \hline \mathbf{1}_b^t \end{array}\right]_{3\times b}, \mathbf{C} = \left[\begin{array}{c} \mathbf{0}_c^t \\ \hline \mathbf{1}_c^t \\ \hline \mathbf{0}_c^t \end{array}\right]_{3\times c}, \mathbf{D} = \left[\begin{array}{c} \mathbf{0}_d^t \\ \hline \mathbf{1}_d^t \\ \hline \mathbf{1}_d^t \end{array}\right]_{3\times d},$$

$$\mathbf{E} = \left[\begin{array}{c} \mathbf{1}_e^t \\ \hline \mathbf{1}_e^t \\ \hline \mathbf{1}_e^t \end{array}\right]_{3\times e}, \mathbf{F} = \left[\begin{array}{c} \mathbf{1}_f^t \\ \hline \mathbf{1}_f^t \\ \hline \mathbf{0}_f^t \end{array}\right]_{3\times f}, \mathbf{G} = \left[\begin{array}{c} \mathbf{0}_g^t \\ \hline \mathbf{0}_g^t \\ \hline \mathbf{1}_g^t \end{array}\right]_{3\times g}, \mathbf{H} = \left[\begin{array}{c} \mathbf{0}_h^t \\ \hline \mathbf{0}_h^t \\ \hline \mathbf{0}_h^t \end{array}\right]_{3\times h},$$

and $\mathbf{Z}' = \mathbf{J} - \mathbf{Z}^t$ when $\mathbf{Z}$ is one of $\mathbf{A}, \ldots, \mathbf{H}$. (Here $\mathbf{1}_k$ and $\mathbf{0}_k$ denote the all ones and all zeros k-vectors, respectively.) Note that the submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ correspond to places in $\mathbf{T}$ (away from the leading 3×3 principal submatrix) where rows 1 and 2 disagree.

We claim that there is a tournament matrix $\overline{\mathbf{T}}$ in $\mathcal{P}(\mathrm{s})$ which has the form

$$\overline{\mathbf{T}} = \left[\begin{array}{ccc|c|c|c|c} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & \overline{\mathbf{E}} & \overline{\mathbf{F}} & \overline{\mathbf{G}} & \overline{\mathbf{H}} \\ 1 & 0 & 0 & & & & \\ \hline \multicolumn{3}{c|}{\overline{\mathbf{E}}'} & & & & \\ \cline{1-3} \multicolumn{3}{c|}{\overline{\mathbf{F}}'} & \multicolumn{4}{c}{\mathbf{X}} \\ \cline{1-3} \multicolumn{3}{c|}{\overline{\mathbf{G}}'} & & & & \\ \cline{1-3} \multicolumn{3}{c|}{\overline{\mathbf{H}}'} & & & & \end{array}\right],$$

where

$$\overline{\mathbf{E}} = \left[\begin{array}{c} \mathbf{1}_p^t \\ \hline \mathbf{1}_p^t \\ \hline \mathbf{1}_p^t \end{array}\right], \overline{\mathbf{F}} = \left[\begin{array}{c} \mathbf{1}_q^t \\ \hline \mathbf{1}_q^t \\ \hline \mathbf{0}_q^t \end{array}\right], \overline{\mathbf{G}} = \left[\begin{array}{c} \mathbf{0}_r^t \\ \hline \mathbf{0}_r^t \\ \hline \mathbf{1}_r^t \end{array}\right], \overline{\mathbf{H}} = \left[\begin{array}{c} \mathbf{0}_s^t \\ \hline \mathbf{0}_s^t \\ \hline \mathbf{0}_s^t \end{array}\right],$$

for some p, q, r and s (note that in $\overline{\mathbf{T}}$, rows 1 and 2 agree everywhere except on the leading 3×3 principal submatrix). In order to prove the claim, we will show that if one of a, b, c and d is nonzero, then there is a tournament matrix $\tilde{\mathbf{T}} \in \mathcal{P}(\mathrm{s})$ having the same 3×3 leading principal submatrix as $\mathbf{T}$ does, the same trailing principal submatrix of order $n-3$ as $\mathbf{T}$ does, but where rows 1 and 2 of $\tilde{\mathbf{T}}$ agree in more positions than rows 1 and 2 of T do. There are several cases to consider.

Case 1: b and c both nonzero. If $b \le c$, construct $\tilde{\mathbf{T}}$ from $\mathbf{T}$ by replacing the submatrix

$$[\mathbf{B}|\mathbf{C}] = \left[\begin{array}{c|c} \mathbf{1}_b^t & \mathbf{0}_c^t \\ \hline \mathbf{0}_b^t & \mathbf{1}_c^t \\ \hline \mathbf{1}_b^t & \mathbf{0}_c^t \end{array}\right] \text{ by } \left[\tilde{\mathbf{B}}|\tilde{\mathbf{C}}\right] = \left[\begin{array}{c|c|c} \mathbf{1}_b^t & \multicolumn{2}{c}{\mathbf{0}_c^t} \\ \hline \mathbf{1}_b^t & \mathbf{0}_b^t & \mathbf{1}_{c-b}^t \\ \hline \mathbf{0}_b^t & \mathbf{1}_b^t & \mathbf{0}_{c-b}^t \end{array}\right]$$

and $[\mathbf{B}'|\mathbf{C}']$ by $\mathbf{J} - \left[\tilde{\mathbf{B}}|\tilde{\mathbf{C}}\right]^t$. If $b > c$, replace $[\mathbf{B}|\mathbf{C}]$ by

$$\left[\tilde{\mathbf{B}}|\tilde{\mathbf{C}}\right] = \left[\begin{array}{c|c|c} \multicolumn{2}{c|}{\mathbf{1}_b^t} & \mathbf{0}_c^t \\ \hline \mathbf{0}_{b-c}^t & \mathbf{1}_c^t & \mathbf{0}_c^t \\ \hline \mathbf{1}_{b-c}^t & \mathbf{0}_c^t & \mathbf{1}_c^t \end{array}\right]$$

and $[\mathbf{B}'|\mathbf{C}']$ by $\mathbf{J} - \left[\tilde{\mathbf{B}}|\tilde{\mathbf{C}}\right]^t$.

Case 2: $b = 0$ and $d > 0$. Note that $a = c + d$ in this case. Construct $\tilde{\mathbf{T}}$ from $\mathbf{T}$ by replacing

$$[\mathbf{A}|\mathbf{D}] = \left[\begin{array}{c|c} \mathbf{1}_a^t & \mathbf{0}_d^t \\ \hline \mathbf{0}_a^t & \mathbf{1}_d^t \\ \hline \mathbf{0}_a^t & \mathbf{1}_d^t \end{array}\right] \text{ by } \left[\tilde{\mathbf{A}}|\tilde{\mathbf{D}}\right] = \left[\begin{array}{c|c|c} \mathbf{1}_{a-d}^t & \mathbf{0}_d^t & \mathbf{1}_d^t \\ \hline \multicolumn{2}{c|}{\mathbf{0}_a^t} & \mathbf{1}_d^t \\ \hline \mathbf{1}_{a-d}^t & \mathbf{1}_d^t & \mathbf{0}_d^t \end{array}\right]$$

and $[\mathbf{A}'|\mathbf{D}']$ by $\mathbf{J} - \left[\tilde{\mathbf{A}}|\tilde{\mathbf{D}}\right]^t$.

Case 3: $b = d = 0$. Note that $a = c$ in this case, so in particular, a and c can be assumed to be positive. The submatrix of T on rows and columns 1, 2, 3, 4 and $a+4$ is either

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ a+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & a+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ a+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & a+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

To construct $\tilde{\mathbf{T}}$ from $\mathbf{T}$, replace the former by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ a+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & a+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \text{or the latter by} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ a+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & a+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Case 4: $c = 0$ and $a > 0$. Note that $d = a + b$ in this case. To construct $\tilde{\mathbf{T}}$ from $\mathbf{T}$, replace

$$[\mathbf{A}|\mathbf{D}] = \left[\begin{array}{c|c} \mathbf{1}_a^t & \mathbf{0}_d^t \\ \hline \mathbf{0}_a^t & \mathbf{1}_d^t \\ \hline \mathbf{0}_a^t & \mathbf{1}_d^t \end{array}\right] \text{ by } \left[\tilde{\mathbf{A}}|\tilde{\mathbf{D}}\right] = \left[\begin{array}{c|c|c} \mathbf{0}_a^t & \mathbf{1}_a^t & \mathbf{0}_{d-a}^t \\ \hline \mathbf{0}_a^t & \multicolumn{2}{c}{\mathbf{1}_d^t} \\ \hline \mathbf{1}_a^t & \mathbf{0}_a^t & \mathbf{1}_{d-a}^t \end{array}\right]$$

and $[\mathbf{A}'|\mathbf{D}']$ by $\mathbf{J} - \left[\tilde{\mathbf{A}}|\tilde{\mathbf{D}}\right]^t$.

Case 5: $c = a = 0$. Note that $b = d$ in this case, and both can be taken to be positive. The submatrix of $\mathbf{T}$ on lines 1, 2, 3, 4 and $b+4$ is either

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ b+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & b+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ b+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & b+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

To construct $\tilde{\mathbf{T}}$ from $\mathbf{T}$, replace the former by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ b+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & b+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \text{ or the latter by } \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ b+4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & b+4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

In all five cases, to construct $\tilde{\mathbf{T}}$ we replaced blocks of $\mathbf{T}$ with $(0,1)$ blocks having the same row and column sums as the corresponding blocks in $\mathbf{T}$. Thus the row and column sums of $\tilde{\mathbf{T}}$ agree with those of $\mathbf{T}$. Further, the replacements were made in such a way as to ensure that $\tilde{\mathbf{T}}$ is still a tournament matrix (necessarily in $\mathcal{P}(\mathbf{s})$), and that its first two rows agree in more places than the first two rows of $\mathbf{T}$.

It now follows that there is a tournament matrix $\overline{\mathbf{T}}$ in $\mathcal{P}(\mathbf{s})$ having the form

$$\overline{\mathbf{T}} = \left[\begin{array}{ccc|c|c|c|c} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & \overline{\mathbf{E}} & \overline{\mathbf{F}} & \overline{\mathbf{G}} & \overline{\mathbf{H}} \\ 1 & 0 & 0 & & & & \\ \hline \multicolumn{3}{c|}{\overline{\mathbf{E}}'} & & & & \\ \multicolumn{3}{c|}{\overline{\mathbf{F}}'} & \multicolumn{4}{c}{\mathbf{X}} \\ \multicolumn{3}{c|}{\overline{\mathbf{G}}'} & & & & \\ \multicolumn{3}{c|}{\overline{\mathbf{H}}'} & & & & \end{array}\right],$$

with

$$\overline{\mathbf{E}} = \left[\begin{array}{c} \mathbf{1}_p^t \\ \hline \mathbf{1}_p^t \\ \hline \mathbf{1}_p^t \end{array}\right], \overline{\mathbf{F}} = \left[\begin{array}{c} \mathbf{1}_q^t \\ \hline \mathbf{1}_q^t \\ \hline \mathbf{0}_q^t \end{array}\right], \overline{\mathbf{G}} = \left[\begin{array}{c} \mathbf{0}_r^t \\ \hline \mathbf{0}_r^t \\ \hline \mathbf{1}_r^t \end{array}\right], \text{ and } \overline{\mathbf{H}} = \left[\begin{array}{c} \mathbf{0}_s^t \\ \hline \mathbf{0}_s^t \\ \hline \mathbf{0}_s^t \end{array}\right].$$

Let $\mathbf{S}$ be the tournament matrix found from $\overline{\mathbf{T}}$ by replacing its leading 3×3 principal submatrix by $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $\mathbf{S}$ and $\overline{\mathbf{T}}$ are adjacent in $\mathcal{S}(\mathbf{s})$ since their associated tournaments differ only in the orientation of the arcs on the 3-cycle involving 1, 2 and 3. Further, $\mathbf{S}$ and $\overline{\mathbf{T}}$ are permutationally similar since exchanging rows and columns 1 and 2 of $\overline{\mathbf{T}}$ yields $\mathbf{S}$. Consequently, their associated tournaments contain the same number of cycles. Hence $\mathbf{S}$ and $\overline{\mathbf{T}}$ are adjacent vertices of $\mathcal{S}(\mathbf{s})$ having the same degree. □

We close this section by giving an explicit description of $\mathcal{S}(\mathbf{s})$ when $\mathbf{s} = [2\ 2\ 2\ 2\ 2]^t$.

Example 1. Let $\mathbf{s} = [2\ 2\ 2\ 2\ 2]^t$; according to the appendix in Moon [16], all tournament matrices in $\mathcal{P}(\mathbf{s})$ are permutationally similar, since up to relabeling of vertices, there is just one tournament with score sequence (i.e. outdegree sequence) s. Let $\mathbf{T}_{11} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ and note that it is a vertex of $\mathcal{S}(\mathbf{s})$. Moreover, it is a circulant matrix, so it can be written as a polynomial in the matrix $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. Now any vertex of $\mathcal{S}(\mathbf{s})$ can be written as $\mathbf{PT}_{11}\mathbf{P}^t$ for some permutation matrix $\mathbf{P}$, and it follows from the fact that $\mathbf{T}_{11}$ is a circulant that the permutation matrices satisfying $\mathbf{PT}_{11}\mathbf{P}^t = \mathbf{T}_{11}$ are $\mathbf{I}, \mathbf{C}, \ldots, \mathbf{C}^4$. Thus we find that of the 5! permutation matrices of order 5, exactly $5!/5 = 24$ of them yield distinct vertices of $\mathcal{S}(\mathbf{s})$ when we perform the similarity transformation $\mathbf{PT}_{11}\mathbf{P}^t$. Hence, $\mathcal{S}(\mathbf{s})$ has 24 vertices, and here they are:

$$\mathbf{T}_{11}=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{T}_{12}=\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \mathbf{T}_{13}=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{T}_{14}=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{T}_{21}=\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{T}_{22}=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{T}_{23}=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{T}_{24}=\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \mathbf{T}_{31}=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T}_{32}=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{T}_{33}=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \mathbf{T}_{34}=\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T}_{41}=\begin{bmatrix}0&1&0&1&0\\0&0&1&1&0\\1&0&0&0&1\\0&0&1&0&1\\1&1&0&0&0\end{bmatrix},\mathbf{T}_{42}=\begin{bmatrix}0&0&0&1&1\\1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\\0&1&1&0&0\end{bmatrix},\mathbf{T}_{43}=\begin{bmatrix}0&0&1&0&1\\1&0&0&0&1\\0&1&0&1&0\\1&1&0&0&0\\0&0&1&1&0\end{bmatrix}$$

$$\mathbf{T}_{44}=\begin{bmatrix}0&1&1&0&0\\0&0&0&1&1\\0&1&0&0&1\\1&0&1&0&0\\1&0&0&1&0\end{bmatrix},\mathbf{T}_{51}=\begin{bmatrix}0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&1\\1&1&0&0&0\\1&0&0&1&0\end{bmatrix},\mathbf{T}_{52}=\begin{bmatrix}0&0&1&1&0\\1&0&0&0&1\\0&1&0&1&0\\0&1&0&0&1\\1&0&1&0&0\end{bmatrix}$$

$$\mathbf{T}_{53}=\begin{bmatrix}0&0&0&1&1\\1&0&0&1&0\\1&1&0&0&0\\0&0&1&0&1\\0&1&1&0&0\end{bmatrix},\mathbf{T}_{54}=\begin{bmatrix}0&1&0&0&1\\0&0&1&1&0\\1&0&0&0&1\\1&0&1&0&0\\0&1&0&1&0\end{bmatrix},\mathbf{T}_{61}=\begin{bmatrix}0&1&0&0&1\\0&0&1&1&0\\1&0&0&1&0\\1&0&0&0&1\\0&1&1&0&0\end{bmatrix}$$

$$\mathbf{T}_{62}=\begin{bmatrix}0&1&0&1&0\\0&0&0&1&1\\1&1&0&0&0\\0&0&1&0&1\\1&0&1&0&0\end{bmatrix},\mathbf{T}_{63}=\begin{bmatrix}0&0&1&1&0\\1&0&0&0&1\\0&1&0&0&1\\0&1&1&0&0\\1&0&0&1&0\end{bmatrix},\mathbf{T}_{64}=\begin{bmatrix}0&0&1&0&1\\1&0&1&0&0\\0&0&0&1&1\\1&1&0&0&0\\0&1&0&1&0\end{bmatrix}$$

The tournament corresponding to $\mathbf{T}_{11}$ has 12 cycles, and since all of the vertices above are permutationally similar, we see that $\mathcal{S}(\mathrm{s})$ is a 12-regular graph. In order to describe the adjacency, we consider each "row", $\mathbf{T}_{k1}$, $\mathbf{T}_{k2}$, $\mathbf{T}_{k3}$, $\mathbf{T}_{k4}$ of vertices in the list above, and give the adjacencies in the subgraph induced by any single row, and by any pair of rows.

For any single row, row k say, we have the following induced subgraph:

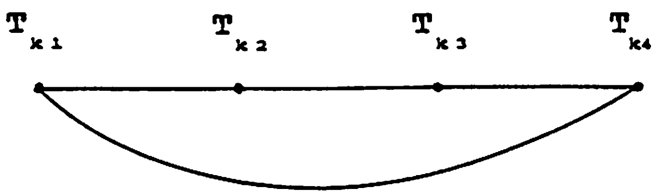

Figure 1

For rows 1 and k, $6 \geq k \geq 2$, $\mathbf{T}_{1j}$ is adjacent to both $\mathbf{T}_{kj}$ and $\mathbf{T}_{k5-j}$, for $1 \leq j \leq 4$.

For pairs of rows involving rows $2, 3, \ldots, 6$ we will use the following convention: (Row i, Row j) $\rightarrow G_1$ will mean that the subgraph of $\mathcal{S}(\mathrm{s})$ induced by rows i and j can be described by relabeling the vertices in the graph G_1 below by replacing v_{1p} by $\mathbf{T}_{ip}$, $1 \le p \le 4$, and v_{2p} by $\mathbf{T}_{jp}$, $1 \le p \le 4$. The notation (Row i, Row j) $\rightarrow G_2$ is defined analogously.

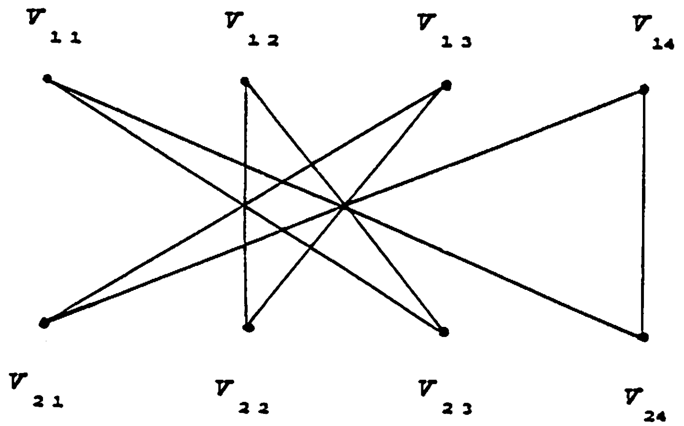

Figure 2

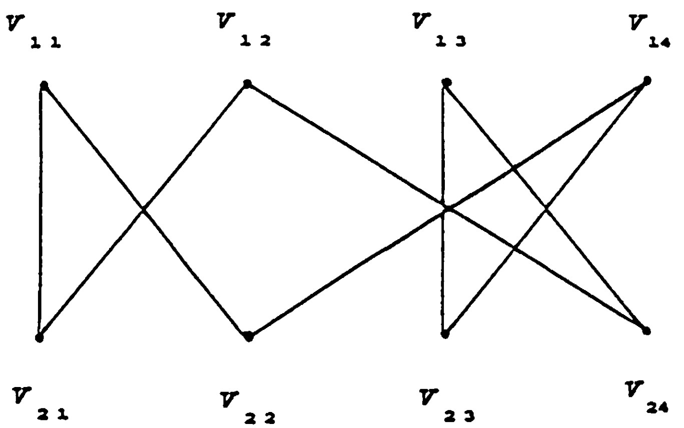

Figure 3

We have the following list of subgraphs of $\mathcal{S}(\mathbf{s})$ which are induced by pairs of rows: (Row i, Row $i+1$) $\rightarrow G_1$ for $2 \le i \le 5$, (Row i, Row $i+2$) $\rightarrow G_2$ for $2 \le i \le 4$, (Row i, Row $i+3$) $\rightarrow G_2$ for $2 \le i \le 3$, and (Row i, Row $i+4$) $\rightarrow G_1$ for $i = 2$. Finally, we remark that it is not difficult to see that G_1 is isomorphic to G_2.

4 A Polytope with Extremal Properties

As part of their discussion on the interchange graph of tournaments with a common score vector, Brualdi and Li consider the set of tournament matrices of order n whose score vector is $\hat{\mathbf{s}}_n = [1\ 1\ 2\ 3\ \dots\ n-3\ n-2\ n-2]^t$ (it is easy to see that for $n \geq 3$, $\hat{\mathbf{s}}_n$ is a strong tournament score vector, since it satisfies (*)). They then use the special structure of $\hat{\mathbf{s}}_n$ to produce some specific information about the corresponding interchange graph. In this section, we will present some properties of $\mathcal{S}(\hat{\mathbf{s}}_n)$ which distinguish it from the 1-skeletons associated with other score vectors. The following result will be useful.

Proposition 2. **(Brualdi and Li [4]).** *Given two tournament matrices* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*, their join,* $\mathbf{T}_1 * \mathbf{T}_2$ *is defined as* $\mathbf{T}_1 * \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{T}_1 & 0 \\ \hline \mathbf{J} & \mathbf{T}_2 \end{array}\right]$, *and define* $\mathbf{T}_1 * \mathbf{T}_2 * \cdots * \mathbf{T}_k$ *inductively by* $\mathbf{T}_1 * \mathbf{T}_2 * \cdots * \mathbf{T}_k = \mathbf{T}_1 * (\mathbf{T}_2 * \cdots * \mathbf{T}_k)$. *For* $k \geq 2$*, let* $\mathbf{U}_k$ *be the tournament matrix of order* k *given by*

$$\mathbf{U}_k = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 0 \\ 0 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}$$

If $\mathbf{T}$ *is a tournament matrix with score vector* $\hat{\mathbf{s}}_n$*, then there are integers* $k_1, \dots, k_u$ *with* $k_1 + \cdots + k_u - u + 1 = n$ *such that* $\mathbf{T} = \mathbf{U}_{k_1} * \mathbf{U}_{k_2} * \cdots * \mathbf{U}_{k_u}$.

The tournament associated with $\mathbf{U}_{k_1} * \mathbf{U}_{k_2} * \cdots * \mathbf{U}_{k_u}$ is pictured in Figure 4. Any arc not shown in that and all subsequent figures is taken to be oriented from the higher numbered vertex to the lower numbered vertex. It is clear from Figure 4 that the vertices of $\mathcal{S}(\hat{\mathbf{s}}_n)$ can be placed in one-to-one correspondence with the subsets of $\{2, 3, \dots, n-1\}$, which yields the following result, also obtained by Brualdi and Li [4].

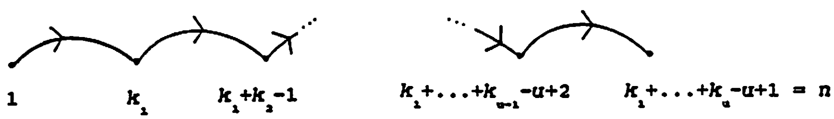

Figure 4

Proposition 3. *$\mathcal{S}(\hat{\mathbf{s}}_2)$ has 2^{n-2} vertices.*

The special structure of the tournament associated with $\mathbf{U}_{k_1} * \mathbf{U}_{k_2} * \cdots * \mathbf{U}_{k_u}$ allows us to enumerate its cycles. This leads us to the next result.

Theorem 4. *Suppose that $n \geq 3$ and that $\mathbf{U}_{k_1} * \mathbf{U}_{k_2} * \cdots * \mathbf{U}_{k_u}$ is a vertex of $\mathcal{S}(\hat{\mathbf{s}}_n)$. Then its degree is equal to*

$$\sum_{m=p+2}^{u} \sum_{p=0}^{u-2} 2^{k_{p+1}+\cdots+k_m-2m+2p} + \sum_{p=1}^{u} (2^{k_p-2} - 1).$$

Proof: For notational convenience, define k_0 to be 1. Referring to Figure 4, we see that there are two types of cycles in the tournament associated with $\mathbf{U}_{k_1} * \mathbf{U}_{k_2} * \cdots * \mathbf{U}_{k_u}$: those involving vertices numbered between $k_0 + \cdots + k_p - p$ and $k_0 + \cdots + k_m - m$ for some $p \geq 0$ and $m \geq p + 2$, and those involving only vertices numbered between $k_0 + \cdots + k_p - p$ and $k_0 + \cdots + k_{p+1} - p - 1$. The cycles of the former type are in one-to-one correspondence with the subsets (including the empty set) of $\cup_{j=p}^{m-1}\{k_0 + \cdots + k_j - j + 1, k_0 + \cdots + k_j - j + 2, \ldots, k_0 + \cdots + k_{j+1} - j - 2\}$, while the cycles of the latter type are in one-to-one correspondence with the nonempty subsets of $\{k_0 + \cdots + k_p - p + 1, k_0 + \cdots + k_p - p + 2, \ldots, k_0 + \cdots + k_{p+1} - p - 2\}$. The result now follows directly. □

Recall that Corollary 2.2 asserted that for a strong tournament score vector $\mathbf{s}$ with n entries, each vertex has degree at least $(n-1)(n-2)/2$. Our next result shows that $\hat{\mathbf{s}}_n$ is (up to reordering of its entries) the unique score vector yielding equality in that lower bound on the degree.

Theorem 5. *Suppose that $\mathbf{s}$ is a strong tournament score vector of order $n \geq 4$, and that its entries are in nondecreasing order. If there is a vertex $\mathbf{T}$ of $\mathcal{S}(\mathbf{s})$ with degree $(n-1)(n-2)/2$, then $\mathbf{s} = \hat{\mathbf{s}}_n$ and $\mathbf{T}$ is one of the following:* $\mathbf{U}_2 * \mathbf{U}_2 * \cdots * \mathbf{U}_2$, $\mathbf{U}_3 * \mathbf{U}_2 * \cdots * \mathbf{U}_2$, $\mathbf{U}_2 * \mathbf{U}_2 * \cdots * \mathbf{U}_2 * \mathbf{U}_3$, $\mathbf{U}_3 * \mathbf{U}_2 * \cdots * \mathbf{U}_2 * \mathbf{U}_3$.

Proof: First note that each of the matrices listed above is permutationally similar to the adjacency matrix of the tournament T_n in Figure 5. We will be done if we can show that the vertices of the tournament G corresponding to $\mathbf{T}$ can be relabeled to yield T_n.

Figure 5

We will proceed by induction on n; the result is easy to verify for $n = 4$. Suppose that $n \geq 5$. A result of Moon [16] asserts that each vertex of a strongly connected tournament on n vertices is on a cycle of length k

for $3 \leq k \leq n$. It follows that any such tournament contains a strongly connected subtournament on $n-1$ vertices. Without loss of generality, we'll suppose that the subtournament of G on vertices $2, \ldots, n$, G' say, is strongly connected. Now vertex 1 is on at least $n-2$ cycles, and by the comment preceding Corollary 2.2, G' contains at least $(n-2)(n-3)/2$ cycles. But since G is assumed to have exactly $(n-1)(n-2)/2 = n-2+(n-2)(n-3)/2$ cycles, we find that 1 is on exactly one cycle of length k for $3 \leq k \leq n$, and that G' has exactly $(n-2)(n-3)/2$ cycles. By the induction step, G' can be relabeled to give T_{n-1}, so without loss of generality we can suppose that G' is as pictured below.

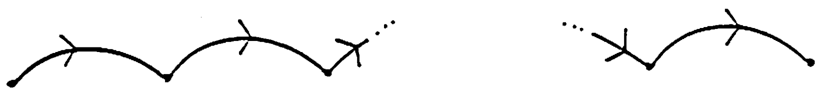

Figure 6

Since 1 is on a cycle of length n, it follows from the structure of G' that one of the following must hold:

i) $1 \to k$ and $k-1 \to 1$ for some $4 \leq k \leq n-1$,

ii) $1 \to n$ and $n-1 \to 1$,

iii) $1 \to 3$ and $2 \to 1$, or

iv) $1 \to 2$ and $n \to 1$.

If i) holds then both $1 \to k \to k+1 \to k-1 \to 1$ and $1 \to k \to k-2 \to k-1 \to 1$ are 4-cycles through 1, a contradiction. If ii) holds, then $1 \to n \to n-2 \to n-1 \to 1$ is the only 4-cycle through 1. Hence $1 \to n-2$, otherwise $1 \to n \to n-3 \to n-2 \to 1$ is a 4-cycle, and $n-3 \to 1$, otherwise $1 \to n-3 \to n-2 \to n-1 \to 1$ is a 4-cycle. But now both $1 \to n-2 \to n-1 \to 1$ and $1 \to n \to n-3 \to 1$ are 3-cycles, a contradiction. If iii) holds, then the only 4-cycle through 1 is $1 \to 3 \to 4 \to 2 \to 1$. Hence $1 \to 5$, otherwise $1 \to 3 \to 4 \to 5 \to 1$ is a 4-cycle. If $1 \to 4$, then both $1 \to 4 \to 2 \to 1$ and $1 \to 5 \to 2 \to 1$ are 3-cycles, a contradiction, while if $4 \to 1$, then both $1 \to 3 \to 4 \to 1$ and $1 \to 5 \to 2 \to 1$ are 3-cycles, another contradiction.

Thus we see that iv) must hold - i.e. $1 \to 2$ and $n \to 1$. If $1 \to n-1$, then $1 \to n-1 \to n \to 1$ is the only 3-cycle through 1, in which case we must have $1 \to k$ for $3 \leq k \leq n-3$, otherwise $1 \to n-1 \to k \to 1$ is a 3-cycle. Further $1 \to n-2$, otherwise $1 \to n-3 \to n-2 \to 1$ is a 3-cycle.

It then follows that G can be relabeled to give T_n. On the other hand, if $n-1 \to 1$, then $n-1 \to 1 \to 2 \to 3 \cdots \to n-2 \to 1$ is the only $n-1$ cycle through 1. Hence $3 \to 1$, otherwise $1 \to 3 \to 4 \to \cdots \to n \to 1$ is another $n-1$ cycle. Thus $1 \to 2 \to 3 \to 1$ is the only 3-cycle through 1. But then $n-2 \to 1$, (otherwise $1 \to n-2 \to n-1 \to 1$ is a 3-cycle), which in turn yields that $n-3 \to 1$ (otherwise $1 \to n-3 \to n-2 \to 1$ is a 3-cycle), and so on. It now follows that $k \to 1$ for $3 \leq k \leq n-1$, and hence that G is T_n. □

Our next result establishes another property which is unique to $\mathcal{S}(\hat{\mathbf{s}}_n)$.

Theorem 6. *Suppose that* s *is a strong tournament score vector of order* $n \geq 4$ *and that its entries are in nondecreasing order. If there is a vertex* $\mathbf{T}$ *of* $\mathcal{S}(\mathbf{s})$ *which is adjacent to every vertex in* $\mathcal{S}(\mathbf{s})$, *then* $\mathbf{s} = \hat{\mathbf{s}}_n$ *and* $\mathbf{T}$ *is one of the following:* $\mathbf{U}_n, \mathbf{U}_2 * \mathbf{U}_{n-1}, \mathbf{U}_{n-1} * \mathbf{U}_2, \mathbf{U}_2 * \mathbf{U}_{n-2} * \mathbf{U}_2$.

Proof: First note that for each of the matrices listed above, the corresponding tournament can be relabeled to give the tournament U_n pictured in Figure 7. We will be done if we can show that the tournament G corresponding to $\mathbf{T}$ can be relabeled to yield U_n (equivalently, if there is a permutation matrix $\mathbf{Q}$ such that the tournament associated with $\mathbf{Q}^t\mathbf{T}\mathbf{Q}$ is $\mathbf{U}_n$).

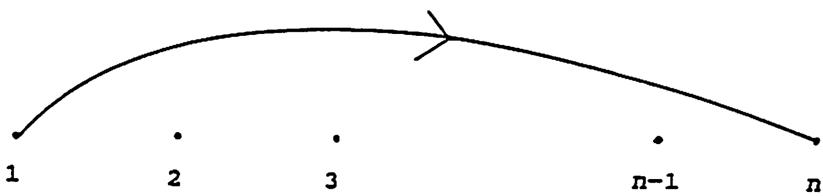

Figure 7

We will use induction on $n \geq 4$; since (1,1,2,2) is the only strong score vector of order 4 whose entries are nondecreasing, the case that $n = 4$ is easily verified. So suppose that $n \geq 5$ and that $\mathbf{T}$ is adjacent to every vertex of $\mathcal{S}(\mathbf{s})$. Again appealing to a result of Moon [16], we can suppose without loss of generality (by simultaneously permuting the rows and columns of $\mathbf{T}$ if necessary) that the subtournament of G on vertices $2, \ldots, n$, G' say, is strongly connected. Let $\overline{\mathbf{T}}$ be the trailing principal submatrix of $\mathbf{T}$ of order $n-1$, and let $\bar{\mathbf{s}}$ be its score vector. If there is a vertex $\mathbf{T}_1$ in $\mathcal{S}(\bar{\mathbf{s}})$ which is not adjacent to $\overline{\mathbf{T}}$ then the tournament associated with $\mathbf{T}_1$ can not be obtained from that associated with $\overline{\mathbf{T}}$ by the reversal of a single directed cycle. Let $\mathbf{T}_2$ be the tournament matrix of order n whose first row and column agree with $\mathbf{T}$ and whose trailing principal submatrix of order $n-1$

is $\mathbf{T}_1$. Since $\mathbf{T}$ and $\mathbf{T}_2$ have the same first row and column, it follows from Theorem 2 that $\mathbf{T}_2$ is adjacent to $\mathbf{T}$ in $\mathcal{S}(\mathbf{s})$ if and only if $\mathbf{T}_1$ is adjacent to $\overline{\mathbf{T}}$ in $\mathcal{S}(\mathbf{s})$. We thus obtain a contradiction if $\overline{\mathbf{T}}$ is not adjacent to every vertex of $\mathcal{S}(\overline{\mathbf{s}})$. Hence we see that $\overline{\mathbf{T}}$ is adjacent to any vertex in $\mathcal{S}(\overline{\mathbf{s}})$, and applying the induction step, we will assume without loss of generality that G' is as pictured below.

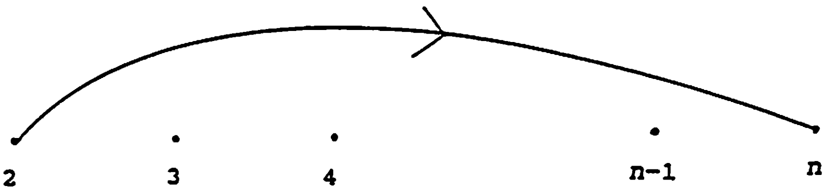

Figure 8

Note that G can not have two arc-disjoint directed cycles, otherwise the tournament obtained by reversing the orientations of both cycles would correspond to a vertex of $\mathcal{S}(\mathbf{s})$ which is not adjacent to $\mathbf{T}$. Now vertex 1 of G is on a cycle of length n, and it follows from the structure of G' that one of the following must hold:

i) $k \to 1$ and $1 \to k-1$ for some $4 \le k \le n-1$,

ii) $n \to 1$ and $1 \to n-1$,

iii) $3 \to 1$ and $1 \to 2$, or

iv) $2 \to 1$ and $1 \to n$.

Suppose that i) holds, and note that $n \to 3 \to 2 \to n$ is a cycle in G. Then $j \to 1$ for $k+1 \le j \le n$, otherwise $1 \to j \to k \to 1$ is a cycle in G disjoint from the one above. Similarly, $n \to n-1 \to 2 \to n$ is a cycle in G, which yields that $1 \to j$ for $2 \le j \le k-2$, and we find that G can be relabeled to give U_n.

If ii) holds then $n \to n-1 \to 2 \to n$ is a cycle in G. Then $1 \to j$ for $3 \le j \le n-2$, otherwise $1 \to n-1 \to j \to 1$ is a cycle disjoint in G from the one above. Further, $1 \to 2$ otherwise we have the cycle $2 \to 1 \to 3 \to 2$. It follows then that G can be relabeled to give U_n.

If iii) holds then again $n \to n-1 \to 2 \to n$ is a cycle in G. We must have $j \to 1$ for $4 \le j \le n$, otherwise $1 \to j \to 3 \to 1$ is a cycle in G disjoint from the one above, and so G can be relabeled to give U_n.

Finally, if iv) holds then $n \to 3 \to 2 \to n$ is a cycle in G. If $4 \to 1$, then $1 \to n \to 4 \to 1$ is a cycle in G, while if $1 \to 4$, then $1 \to 4 \to 2 \to 1$

is a cycle in G. In either case, we have two arc-disjoint cycles in G, a contradiction. Thus iv) cannot hold.

Consequently, we can relabel G to obtain U_n, as desired. □

Theorem 6 leads immediately to the following.

Corollary 6.1. *For $n \geq 4$, $S(\hat{\mathbf{s}}_n)$ is 4-connected.*

Proof: If we delete any three vertices (and their adjacent edges) from $S(\hat{\mathbf{s}}_n)$, then the resulting graph still has one of $\mathbf{U}_n$, $\mathbf{U}_2 * \mathbf{U}_{n-1}$, $\mathbf{U}_{n-1} * \mathbf{U}_2$, and $\mathbf{U}_2 * \mathbf{U}_{n-2} * \mathbf{U}_2$ as a vertex, and that vertex is adjacent to all of the remaining vertices. Hence the resulting graph remains connected, which yields the result. □

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