An Extended Class Of Resolvable, Incomplete Lattice Designs

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ABSTRACT. A method for synthesizing combinatorial structures which are members of an extended class of resolvable incomplete lattice designs is presented. Square and rectangular lattices both are realizable, yet designs in the extended class are not limited in number of treatments by the classically severe restriction $v = s^2$ or $v = s(s-1)$. Rather, the current restriction is the condition that there exist a finite closable set of k-permutations on the objects of some group or finite field, which is then used as the generating array for a L(0,1) lattice design. A connection to Hadamard matrices H(p,p) is considered.

1 Introduction

The first of the systematically studied and applied resolvable designs were the square lattices, introduced by Yates (1936). If $v = s^2$ is the number of treatments, construction of a design possessing $R > 3$ replicates depends upon the existence of a set of $R-2$ mutually orthogonal latin squares (MOLS) of side s. Arranging the treatments in an array A of side s, the rows and columns of A form respectively the blocks of the first two replicates. Succesive replicates are formed by superimposing on A one of the latin squares, and taking as blocks those variables adjacent to the same symbol. Orthogonality of the latin squares prevents any two treatments from appearing in more than one block. Thus, a scarce design or else a balanced incomplete block design emerges, depending upon whether there exists a complete set of MOLS of side s.

Harshbarger (1949) extended the lattice principle to simple and triple rectangular lattices where $v = ks$, with $k = s - 1$. A more general approach via mutually orthogonal latin squares is presented in John (1987).

Treatments are initially set out in an $(s)x(s)$ array with cells belonging to one of the principle diagonals left blank. If there exists a complete set of mutually orthogonal latin squares of side s, then a set of $s-2$ squares can be found where the symbols each appear on the diagonal corresponding to the blank diagonal of the initial array. Again, rows and columns of the initial array respectively form blocks for the first two replicates. Successive replicates are found by superimposing each of the orthogonal squares on the initial array, with blocks formed from all treatments which are paired with the same symbol. Orthogonality forces a scarce design where any two treatments concur in at most one block.

Kempthorne (1952) has suggested constructing kx(k-l) lattice designs by superimposing a latin square of side k in which l entries have been omitted from each row, each column, and the set of occurrences of each symbol. The idea is also mentioned by Street and Street (1987). However, few details are given; and particularly, the utilization of orthogonality is not considered.

The purpose of the present research is to present a scheme for constructing, when they exist, members of an extended broad class of equi-block lattice designs. A representative of the extended class of L(0,1) lattice designs is obtained somewhat as in the method of Yates. The treatments are initially set up in an array, L, of dimension (s)x(k), where $k \leq s$. Block size is k, and blocks of the first replicate consist of the rows of L. The construction of additional replicates depends upon the existence of a set of r mutually orthogonal latin rectangles (MOLR) of the same dimension as L. Superimposing each latin rectangle on L and taking as blocks those variables adjacent to the same symbol leads to r further replicates.

Orthogonality of the rectangles forces a design for which two arbitrary treatments concur in at most one block. Whenever $k = s$ and $r = s - 1$ are permissible parameter choices, the design can be extended to become a balanced incomplete block design, by taking columns of the initial array as the blocks of an additional parallel class.

Thus, a method for synthesizing combinatorial structures which are members of an extended class of resolvable, incomplete, lattice designs emerges. Both square and rectangular lattices may be realizable, yet designs in the extended class are not limited in number of treatments by the classical restriction $v = s^2$ or $v = s(s-1)$. In fact, due to the non-existence of orthogonal latin squares of side six, the case $v = 30$ does not permit a classical lattice design, yet possesses a realizable L(0,1) design (see Section 3). However, the condition for obtaining an L(0,1) design on $v = ks$ treatments is that there exist an s-closable set of k-permutations taken from the s objects of some algebraic module S.

As there is a one-to-one correspondence between the set of Alpha(0,1) designs and the set of L(0,1) extended lattice designs, it might be expected that statistical efficiency properties of the two classes are somewhat equivalent. However, a number of application areas exist where the design itself is of more importance than is its statistical properties; viz, tournament scheduling in sports (Cooke, (1996)). An application of the L(0,1) design method to the scheduling of certain types of tennis and golf tournaments is presented in Section 4. Connections to Alpha(0,1) designs are investigated in Section 5, and some techniques for constructing s-closable latin rectangles are given in Section 6.

2 Construction of Mutually Orthogonal Latin Rectangles

In combinatorial practice, two latin squares of side s with elements in the module $S = \{x_0, x_1, ...x_n\}$ are called orthogonal iff when superimposed every ordered pair (a,b) from SxS appears exactly once. However, the concept permits a useful generalization whereby orthogonality can be defined for rectanglular arrays. Consequently, the method of Bose (1938) for generating mutually orthogonal latin squares (MOLS) naturally extends to the problem of generating mutually orthogonal latin rectangles (MOLR).

Let A be a rectangular array of dimension (r)x(k), with $r, k \leq n$, whose elements are in the set $S = \{x_0, x_1, x_2, ...x_n\}$, where $|S| = s$. It is assumed that the elements of S can be combined with some binary differencing operation (-). Ryser (1963) defines array A a latin rectangle provided each row of A is a k-permutation chosen from S, with each column being an r-permutation. On the other hand, Hall (1987) stipulates the further requirement $k = s$ in order that A be a latin rectangle. For present purposes, the Ryser definition will be adopted. Thus, every sub-rectangle of a latin square is a latin rectangle, yet not every latin rectangle can be extended to a latin square.

Two latin rectangles of the same dimension will be called mutually orthogonal iff when the arrays are superimposed any ordered pair (a,b) from SxS appears at most once (some pairs may not appear). A set $\{L_j : j = 1, 2, ...r\}$ of latin rectangles having the same dimension will be called mutually orthogonal (MOLR) iff each pair is mutually orthogonal.

Closable Sets of Permutations

Let $s = n + 1$ be a positive integer, and $1 < k < s$. A finite set of k-permutations of the objects from S is called s-closable provided the vector difference of two arbitrary members is also a k-permutation. Usually, the differencing refers to arithmetic modulo s, but in all cases there is assumed a group or finite field structure on the elements of S within which the (-) operation for combining two scalar elements is well defined.

Latin rectangle A whose row vectors constitute an s-closable set of r-permutations shall be called s-closable. It shall be called s-closed iff every row difference (of two arbitrary rows) is also a row vector of A. As an example, if S is a finite field, the linear permutation functions $\{f_a(x) =$

$ax : 0 \neq a \epsilon S\}$ induce a closed set of permutations $\{f_a(S) = aS, a \neq 0\}$ from which a closed latin rectangle can be formed. Omission of the zero element of the field from each permutation will result in a closed latin square of side $s-1$. Moreover, any sub-rectangle is an s-closable latin rectangle, which may not be s-closed.

Bose's Method Applied To MOLR

If Bose's (1938) method for generating MOLS is applied to the row vectors of an s-closed (r)x(k) latin rectangle, there results an r-set of MOLR whose dimension is (s)x(k). The procedure here is simple: develope cyclicly each row vector R_i by adding successively each non-zero element from module S, to obtain the latin rectangles $\{L_i : i = 1, 2, ...r\}$ whose rows are $\{R_i + x_k : k = 0, 1, ...s-1\}$. The property that each row difference associated with the s-closable rectangle is a permutation forces mutual orthogonality of the $\{L_i\}$.

3 Constructing an $L(0,1)$ Lattice Design

As a first illustration of the construction process for an L(0,1) design, consider the case $S = \{0, 1, 2, 3, 4, 5, mod6\}$. As previously recalled, the non-existence of MOLR of side $s = 6$ prevents construction of a classical lattice design characterized by $v = 30$ treatments which possesses more than two replicates. However, as there exists an s-closable latin rectangle of dimension 4x5, there can be constructed an extended L(0,1) design having five replicates whose parallel classes are of dimension 6xk, with blocksize k, for $2 \leq k \leq 5$. The corresponding s-closable rectangle of maximal dimension 4x5 is given by the

Generating Rectangle, G:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 2 & 5 & 1 & 4 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{bmatrix}$$

By cyclic development of each row of the array G, there results a set of four MOLR of dimension $6x5$. For purposes of illustration, one such rectangle is the

First Latin Rectangle:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 0 & 2 \\ 4 & 5 & 0 & 1 & 3 \\ 5 & 0 & 1 & 2 & 4 \end{bmatrix}$$

Setting out the treatments in an array whose rows are blocks of the design, there results the

First Replicate:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 \end{bmatrix}$$

Superimposing in turn on this array each of the MOLR and grouping into blocks those variables which are adjacent to a common symbol, with these symbols placed in natural order, there results four additional parallel classes:

Second Replicate:

$$\begin{bmatrix} 0 & 9 & 18 & 22 & 26 \\ 1 & 5 & 14 & 23 & 27 \\ 2 & 6 & 10 & 19 & 28 \\ 3 & 7 & 11 & 15 & 24 \\ 8 & 12 & 16 & 20 & 29 \\ 4 & 13 & 17 & 21 & 25 \end{bmatrix}$$

Third Replicate:

$$\begin{bmatrix} 0 & 7 & 14 & 21 & 28 \\ 3 & 5 & 12 & 19 & 26 \\ 1 & 8 & 10 & 17 & 24 \\ 6 & 13 & 15 & 22 & 29 \\ 4 & 11 & 18 & 20 & 27 \\ 2 & 9 & 16 & 23 & 25 \end{bmatrix}$$

Fourth Replicate:

$$\begin{bmatrix} 0 & 16 & 13 & 24 & 27 \\ 2 & 5 & 18 & 21 & 29 \\ 4 & 7 & 10 & 23 & 26 \\ 1 & 9 & 12 & 15 & 28 \\ 3 & 6 & 14 & 17 & 20 \\ 8 & 11 & 19 & 22 & 25 \end{bmatrix}$$

Fifth Replicate:

$$\begin{bmatrix} 0 & 11 & 17 & 23 & 29 \\ 4 & 5 & 16 & 22 & 28 \\ 3 & 9 & 10 & 21 & 27 \\ 2 & 8 & 14 & 15 & 26 \\ 1 & 7 & 13 & 19 & 20 \\ 6 & 12 & 18 & 24 & 25 \end{bmatrix}$$

4 Sports Application - An Unbiased Tournament Schedule

The problems associated with designing an unbiased, round-robin doubles tennis tournament schedule are discussed in Cooke (1996). Basicly, such a schedule is a resolvable, incomplete, equi-block combinatorial design possessing at least five parallel classes (rounds). To be unbiased as regards choices of partners and opponents, two blocks can intersect at most once. The block size is $k = 4$, and $v = ks$, where s is the number of tennis courts simultaneously available.

Schedules of 5, 5, and 7 rounds were found for the respective cases $v = 16$, 20, and 28 players, but difficulties were encountered when attempting a five round schedule for 24 players. Street and Street (1987) tabulate a three round schedule (an Alpha(0,1) design). The purpose of the present section is to record a five round L(0,1) design, thought to be the longest possible such schedule. Again, the maximality question for length of a resolvable scarce design, over all possible combinatorial design types, appears to be an open problem, due basicly to the broadness in scope of the question.

We start with a sub-rectangle of G as the

Generating Rectangle, GS:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 4 & 3 & 2 \end{bmatrix}$$

By cyclic development of each row of the array GS, there results a set of four MOLR of dimension $6x4$.

Setting out the treatments in a $6x4$ array whose rows are blocks of the design, there results the

First Round:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \end{bmatrix}$$

Superimposing in turn on this array each of the MOLR and grouping into blocks those variables which are adjacent to a common symbol, in the order that they appear in successive columns, there results four additional parallel classes:

Second Round:

$$\begin{bmatrix} 0 & 21 & 18 & 15 \\ 4 & 1 & 22 & 19 \\ 8 & 5 & 2 & 23 \\ 12 & 9 & 6 & 3 \\ 16 & 13 & 10 & 7 \\ 20 & 17 & 14 & 11 \end{bmatrix}$$

Third Round:

$$\begin{bmatrix} 0 & 17 & 6 & 23 \\ 4 & 21 & 10 & 3 \\ 8 & 1 & 14 & 7 \\ 12 & 5 & 18 & 11 \\ 16 & 9 & 22 & 15 \\ 20 & 13 & 2 & 19 \end{bmatrix}$$

Fourth Round:

$$\begin{bmatrix} 0 & 13 & 22 & 11 \\ 4 & 17 & 2 & 15 \\ 8 & 21 & 6 & 19 \\ 12 & 1 & 10 & 23 \\ 16 & 5 & 14 & 3 \\ 20 & 9 & 18 & 7 \end{bmatrix}$$

Fifth Round:

$$\begin{bmatrix} 0 & 9 & 14 & 19 \\ 4 & 13 & 18 & 23 \\ 8 & 17 & 22 & 3 \\ 12 & 21 & 2 & 7 \\ 16 & 1 & 6 & 11 \\ 20 & 5 & 10 & 15 \end{bmatrix}$$

5 Connections to Alpha Designs

The Alpha method for combinatorial design provides a series of resolvable, incomplete block designs which can be generated readily. Williams (1975) provides a large table of such designs. As an alternative to the tables, Paterson and Patterson (1983) give a computer algorithm for choosing a suitable generating array, based on the aim of choosing block designs with high efficiency factors. It is established in John (1987) that the Alpha(0,1) designs, as opposed to Alpha(0,1,2) designs, etc, are characterized by the most optimal efficiency factors.

A spin-off of the present research is the provision of means to obtain, without computer search, initial arrays which always generate Alpha(0,1) designs. A sufficient condition for this to happen is that the initial array be a top zero-bordered s-closable Latin rectangle. If $s = n + 1$ is a prime or prime power, there is a module, S, whose elements constitute a finite Galois field, from which there can be chosen an s-closable Latin rectangle of maximal dimension (s-1)x(s). (See Section 6 below). Thus, by appending a top row of zeroes, there is obtained an (s)x(s) array which developed in the usual manner generates an Alpha(0,1) design whose s parallel classes are of maximal dimension (s)x(s).

Each L(0,1) design is not dual to some Alpha(0,1) design, as the dual is classically defined (see Raghavaro (1971)); yet the terminology would appear well-deserved. The condition for generating either an L(0,1) or an Alpha(0,1) design is the same. Each requires an s-closable set of permutations as a generating array, and the same initial array can be used for generating a design of either type.

If the s-closable set is a (k-1)x(r) array of r-permutations, by cyclicly developing the columns an Alpha(0,1) design of blocksize k, resolution classes (k)x(s), and replication number r emerges. Likewise, by cyclicly developing the rows of the same array and then employing the resulting MOLR to group treatments, there emerges an L(0,1) design of blocksize r, resolution classes of dimension sxr, and replication number k. At most two additional replicates are obtainable by grouping as blocks rows and columns of the initial array.

6 Construction of S-closable Latin Rectangles

Recall from Section 2, Latin rectangle A whose row vectors constitute an s-closable set of r-permutations is called s-closable. It is called s-closed iff every row difference (of two arbitrary rows) is also a row vector of A.

As an example, if S is a finite field, the linear permutation functions $\{f_a(x) = ax : 0 \neq a \epsilon S\}$ induce a closed set of permutations $\{f_a(S) = aS, a \neq 0\}$ from which a closed latin rectangle can be formed. Omission of the zero element of the field from each permutation will result in a closed latin square of side $s-1$. Moreover, any sub-rectangle is an s-closable latin rectangle, which may or may not be closed.

When S is not a finite field, s-closable (r)x(k) rectangles may sometimes be obtained, but whose k-permutations are usually characterized by small values of r and/or k. For example, Patterson, Williams, and Hunter (1978) give a set of basic generating arrays for Alpha designs, which (except for one case) by omission of a row of zeros provide $3xk$ s-closable arrays characterized by $v = ks \leq 100$. When s is not a prime power, for larger values $r, k \leq s$ the question of existence and methods for constructing (r)x(k) s-closable arrays appears to be an open problem. Indeed, the existence of such an array of dimension (s-1)x(s) would guarantee the existence of a complete set of MOLS of side s.

When S is the group Z_p of residue classes modulo $s = p$, a set of elements

$\{a_k : k \le n\}$ which satisfy $gcd(a_i, s) = 1$ and $gcd(a_i - a_j, s) = 1, i \ne j$, yields an s-closable set of permutation functions $f_k = a_k x$. For $s = 15$, three such distinct sets exist: $\{1,2\}, \{7,8\}, and \{13,14\}$. Thus, three sets of triple L(0,1) lattice designs are known to be possible. In fact, by considering subrectangles, triple L(0,1) lattice designs with parallel classes of dimension $15xk$, $k \le 15$ are possible, in triplicate sets.

As another example, when $s = 35$, four such sets are possible: $\{\{1,2,3,4\} + 10i, i = 0,1,2,3\}$,

7 Connections to Hadmard Matrices

J. A. Butson (1962-1963) investigates the properties of generalized Hadamard matrices and relations to relative difference sets. S. S. Shrikhande (1964) studies the connections between generalized Hadamard matrices and orthogonal arrays of strength two. The reviewer of the present paper has questioned whether there is a relation between these concepts and the s-closeable generating rectangles of Section 6. For Hadamard matrices the question is affirmatively answered in this section. Here, it is demonstrated that for primes $p > 2$ the problem of constructing a generalized Hadamard matrix H(p,p) is combinatorially equivalent to constructing an affine resolvable balanced incomplete block design characterized by parameters

$$AR(p) : v = p^2, b = p^2 + p, r = p + 1, k = p, \lambda = 1. \qquad (1)$$

For clarity, the present demonstration considers a specific case $p = 5$; however, it is made clear that the procedure is valid for general prime p. Thus, the present result is a counterpart of Todd's (1933) discovery that construction of a classical Hadamard matrix H(2,4t) having integer $t > 1$ is combinatorially equivalent to the problem of constructing an unresolvable, symmetric, balanced incomplete block design whose parameters are ($v = b = 4t - 1, r = k = 2t - 1, \lambda = t - 1$).

Actually, the work of Shrikhande (1964) establishes that existence of a certain orthogonal array OA implies the existence of a Hadamard matrix $H(p, p^2[(p-1)t + 1])$ which for vanishing t becomes $H(p, p^2)$. He also establishes a combinatorial equivalence between a series of orthogonal arrays and a series of affine resolvable BIB designs which produces $AR(p)$ when a parameter is made to vanish. Thus, if Shrikhandes proof can be shown reversible, this implies $H(p, p^2)$ and $H(p, p)$ themselves are combinatorially equivalent.

For $p = 5$ and $x = exp(2\pi/p)$, consider the Hadamard H(p,p) matrix

$$H(5,5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x^1 & x^2 & x^3 & x^4 \\ 1 & x^2 & x^4 & x^1 & x^3 \\ 1 & x^3 & x^1 & x^4 & x^2 \\ 1 & x^4 & x^3 & x^2 & x^1 \end{bmatrix} \quad (2)$$

whose conjugate-transpose has the property $H * H^{CT} = 5I$.
The notation

$$H = x^E \quad (3)$$

means $h_{ij} = x^{e_{ij}}; i,j = 0,1,...p-1$, where

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{bmatrix} \quad (4)$$

It is observed that the core of E is a Latin square with an algebraic property: the mod p differences are such that the row vectors are an s-closed set (with $s = p = 5$). This, plus the fact that E is in the standard form required of the generating matrix for the α method (see J. A. John (1987)), assures that the $\alpha(0,1)$ design generated by E will be a group divisible (GDD), resolvable, incomplete block design characterized by m groups of n treatments whose parameters are

$$GDD(p) : v = mn; m = n = p = 5; k = r = p; b = p^2; \lambda_1 = 0; \lambda_2 = 1 \quad (5)$$

Moreover, as John (1987) observes, given design GDD(p) generated by the alpha method, its generating matrix E is readily inferred. Thus, the process can be reversed, after making sure that the generating matrix is in standard form, to obtain an $H(p,p)$ matrix. For, if the exponent core were not p-closable, the design could not be $\alpha(0,1)$, and if the nonzero core has side $p-1$, for prime p the core must be p-closed.

Finally, to complete the demonstration it is shown that AR(p) and GDD(p) can each be obtained from the other. To obtain AR(p) from GDD(p), as resolution class $p+1$ simply take the transpose of the first resolution class, whose columns are the p groups of 1st associates which have never been together in blocks of the design. The extended design is the affine resolvable BIB having parameters

$$AR(p) : v = p^2; k = p; r = p+1; b = p^2 + p; \lambda = 1 \quad (6)$$

Clearly, to obtain GDD(p) from AR(p), that resolution class is omitted whose rows qualify as the groups of first associates.

However, it is pointed out that the properties of Hadamard matrices indicated by Butson (1962-1963) make it clear the general H(p,p) matrix, for $p > 2$ a prime, has an exponent core which is p-closed. Thus, our demonstration holds for general prime p.

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