Covers Of Finite Sets With Distinct Block Sizes

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ABSTRACT. A cover of a finite set N is a collection of subsets of N whose union is N. We determine the number of such covers whose blocks all have distinct sizes. The cases of unordered and ordered blocks are each considered.

1 Introduction

Any representation of a finite set N as $N = N_1 \cup N_2 \cdots \cup N_r$, with $N_j \neq \phi$ and $N_i \neq N_j$ for $i \neq j$ is called a cover of N. The sets N_j are called the blocks of the cover. If $V(n)$ denotes the number of covers of N where $|N| = n$, and $V(0) = 1$ then it is well known (see Comtet [C, p165] or [HW]) that $V(n)$ satisfies

$$\sum_{k=0}^{n} \binom{n}{k} V(k) = 2^{2^n - 1}.$$

Thus by binomial inversion,

$$\begin{aligned} V(n) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^{2^k - 1} \\ &= 2^{2^n - 1}(1 + O(2^{n - 2^{n-1}})) \text{ as } n \to \infty. \end{aligned} \tag{1.1}$$

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Suppose now that the order of the subsets in a cover of N is to be taken into account. If $X(n)$ denotes the number of such ordered covers with $X(0) = 1$, then

$$\sum_{k=1}^{n} \binom{n}{k} X(k) = \sum_{m=1}^{2^n-1} \binom{2^n-1}{m} m!.$$

This holds since both sides of the equation give the number of ordered unions $N_1 \cup N_2 \cdots \cup N_m \subset N$, $1 \le m \le 2n-1$.

The right hand side counts these according to the number m of blocks in the union while the left hand side counts these according to subsets $M \subset N$ with $N_1 \cup \cdots \cup N_m = M$ and $|M| = k$. Denote the sum $\Sigma_{m=0}^{2^n-1} \binom{2^n-1}{m} m!$ by $x(n)$ so that

$$\sum_{k=0}^{n} \binom{n}{k} X(k) = x(n),$$

and again by binomial inversion,

$$X(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x(k).$$

Now

$$\begin{aligned} x(n) = (2^n-1)! \sum_{m=0}^{2^n-1} \frac{1}{(2^n-1-m)!} &= (2^n-1)! \sum_{j=o}^{2^n-1} \frac{1}{j!} \\ &\sim e(2^n-1)!. \end{aligned} \tag{1.2}$$

Also since

$$\left| \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} x(k) \right| \le 2^n x(n-1)$$

we deduce that

$$X(n) = x(n)(1 + O(2^{n-n2^{n-1}})) \text{ as } n \to \infty, \tag{1.3}$$

where we have used Stirling's formula to estimate $x(n-1)/x(n)$.

The main aim of this paper is to determine the number of covers of N corresponding to each of the cases above under the additional restriction that the sizes of the blocks be distinct. This problem is the direct analog of the problem considered in [KORSW] of determining the number of partitions of an n element set with distinct block sizes. In a partition of a set we require that blocks be pairwise disjoint. Similar problems within the content of cycles of permutations were solved by Greene and Knuth [GK] and concerning irreducible factors of polynomials over a finite field by the present authors in [KW].

2 Unordered Coverings with distinct block sizes

Let $W(n)$ denote the numbers of covers of N with the added restriction that the sizes of the blocks be distinct and set $W(0) = 1$. Then we can write

$$W(n) = \widetilde{W}(n) + \widehat{W}(n)$$

where $\widetilde{W}(n)$ denotes the number of unordered covers of N with distinct block sizes where N occurs as one of the blocks and $\widehat{W}(n)$ denotes the remaining cases where N does not occur as one of the blocks. We introduce $\widetilde{W}(n)$ since as shown below there is a formula for calculating this. Firstly since the set N can be adjoined to any cover counted in $\widehat{W}(n)$ it follows that $\widehat{W}(n) \leq \widetilde{W}(n)$ and hence

$$\widetilde{W}(n) \leq W(n) \leq 2\widetilde{W}(n). \tag{2.1}$$

We now derive a formula for $\widetilde{W}(n)$: Let us denote the sizes of the blocks (excluding N itself) in a cover belonging to $\widetilde{W}(n)$ by $m_1 < m_2 \cdots < m_k$, where $1 \leq k \leq n-1$. There are $\binom{n}{m_i}$ possible choices of elements for the block of size m_i, $1 \leq i \leq k$ and so $\binom{n}{m_1}\binom{n}{m_2}\ldots\binom{n}{m_k}$ covers with these block sizes. Summing over all possibilities for the sizes of the blocks leads to the formula

$$\widetilde{W}(n) = 1 + \sum_{k=1}^{n-1} \sum_{1 \leq m_1 < m_2 \cdots < m_k < n} \binom{n}{m_1}\binom{n}{m_2}\cdots\binom{n}{m_k} \tag{2.2}$$

where the term 1 counts the cover by the set N alone. Next we divide up the covers contributing to $\widetilde{W}(n)$ into the following classes:

(i) the set N itself,

(ii) the set of covers contributing to $\widehat{W}(n)$ with N adjoined to each such cover,

(iii) these cover with distinct block sizes whose union is a strict subset of N, if the block N itself is disregarded. If this union consists of m elements from N then we have $\binom{n}{m}$ choices for the elements and $W(m)$ covers corresponding to each such choice of elements. It follows from the above that

$$\begin{aligned}\widetilde{W}(n) &= 1 + \widehat{W}(n) + \sum_{m=1}^{n-1} \binom{n}{m} W(m) \\ &= \widehat{W}(n) + \sum_{m=0}^{n-1} \binom{n}{m} W(m) \text{ (using } W(0) = 1).\end{aligned}$$

Adding $\widetilde{W}(n)$ to both sides gives for $n \geq 1$,

$$2\widetilde{W}(n) = \sum_{m=0}^{n} \binom{n}{m} W(m) \tag{2.3}$$

and by binomial inversion we obtain the formula

$$W(n) = 2\left(\sum_{m=1}^{n} \binom{n}{m} (-1)^{n-m} \widetilde{W}(m)\right) + (-1)^n. \tag{2.4}$$

At this stage we obtain an asymptotic estimate for $\widetilde{W}(n)$ in terms of the largest term in the sum of equation (2.2). Let $P(n)$ denote the number of covers of N with n blocks of distinct size. Then

$$P(n) = \prod_{j=1}^{n} \binom{n}{j} = \frac{(n!)^{n+1}}{(\prod_{j=1}^{n} j!)^2}.$$

By (2.2) for $n > 1$,

$$\frac{\widetilde{W}(n)}{P(n)} = 1 + \sum_{r=1}^{n-1} \sum_{1 \leq j_1 < \cdots < j_r \leq n-1} \frac{1}{\binom{n}{j_1} \cdots \binom{n}{j_r}}.$$

Let us denote the inner sum above by $S_r(n)$. Then for $r \geq 3$,

$$S_r(n) \leq \frac{\binom{n}{r}}{M_r(n)}$$

where

$$\begin{aligned} M_r(n) : &= \min_{1 \leq j_1 < \cdots < j_r \leq n-1} \binom{n}{j_1} \cdots \binom{n}{j_r} \\ &= \left(\prod_{j=1}^{[\frac{r}{2}]} \binom{n}{j}\right)^2 a_r(n) \end{aligned}$$

where

$$a_r(n) : = \begin{cases} 1 & \text{if } r \text{ even} \\ \binom{n}{\frac{r+1}{2}} & \text{if } r \text{ odd,} \end{cases}$$

(by the unimodality of the binomial coefficient).

Put $s : = [\frac{r}{2}]$. Then since $s \geq 1$ for $r \geq 3$,

$$\prod_{j=1}^{s} \binom{n}{j} \geq n\binom{n}{2}^{s-1} = n^s \binom{n-1}{2}^{s-1}.$$

Since $\binom{n}{r} \le n^r$ we get

$$S_r(n) \le \frac{n^r}{(n^s(\frac{n-1}{2})^{s-1})^2 a_r(n)} = \frac{n^{r-2s}}{(\frac{n-1}{2})^{2s-2} a_r(n)}.$$

One has

$$s = \begin{cases} r/2 & \text{if } r \text{ even} \\ \frac{r-1}{2} & \text{if } r \text{ odd,} \end{cases}$$

hence for r even, $r > 2$,

$$S_r(n) \le \frac{1}{(\frac{n-1}{2})^{r-2}},$$

while for r odd, $r \ge 3$,

$$S_r(n) \le \frac{n}{(\frac{n-1}{2})^{r-3}\binom{n}{2}} = \frac{1}{(\frac{n-1}{2})^{r-2}}.$$

Now

$$\sum_{r=1}^{n-1} S_r(n) = S_1(n) + S_2(n) + \sum_{r=3}^{n-1} S_r(n).$$

We assume that $n \ge 5$, Then

$$\sum_{r=3}^{n-1} S_r(n) \le \sum_{r=3}^{\infty} \frac{1}{(\frac{n-1}{2})^{r-2}} \le \frac{4}{n-1}.$$

Also,

$$S_1(n) = \sum_{j=1}^{n-1} \frac{1}{\binom{n}{j}} \le \frac{2}{n} + \frac{n}{\binom{n}{2}} = O\left(\frac{1}{n}\right),$$

and

$$\begin{aligned} S_2(n) &= \sum_{1 \le j < k \le n-1} \frac{1}{\binom{n}{j}\binom{n}{k}} \\ &= \sum_{\substack{1 \le j < k \le n-1 \\ j=1 \text{ or } k=n-1}} \frac{1}{\binom{n}{j}\binom{n}{k}} + \sum_{2 \le j < k \le n-2} \frac{1}{\binom{n}{j}\binom{n}{k}} \\ &\le \frac{n}{n^2} + \frac{n}{n^2} + \frac{n^2}{\binom{n}{2}^2} = O\left(\frac{1}{n}\right). \end{aligned}$$

Thus

$$\widetilde{W}(n) = P(n)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{2.5}$$

We now estimate $\log P(n)$ using the following sharper version of Stirling's formula, (see e.g. Wilf [W, p121])

$$\log n! = n \log n - n + \frac{1}{2}\log n + c_1 + \frac{1}{12n} + R(n), n \to \infty \tag{2.6}$$

where $c_1 = \frac{1}{2}\log 2\pi$ and $R(n) = O(\frac{1}{n^3})$.
Put

$$c_2 : = \sum_{m=1}^{\infty} R(m).$$

Then

$$\sum_{m=1}^{n} R(m) = c_2 + O\left(\frac{1}{n^2}\right).$$

Also, let

$$A(n) : = \sum_{j=1}^{n} \log j!, \quad B(n) : = \sum j \log j.$$

Firstly

$$B(n) = \sum_{j=1}^{n} \log j \sum_{i=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{n} \log j = \sum_{i=1}^{n}(\log n! - \log(i-1)!)$$
$$= n \log n! - \sum_{j=1}^{n-1} \log j! = (n+1)\log n! - A(n).$$

Then

$$\begin{aligned}\log P(n) &= (n+1)\log n! - 2A(n)\\ &= B(n) - A(n).\end{aligned} \tag{2.7}$$

By (2.6) we have

$$\begin{aligned}A(n) &= \sum_{j=1}^{n}\left(j \log j - j + \frac{1}{2}\log j + c_1 + \frac{1}{12j} + R(j)\right)\\
&= B(n) - \frac{1}{2}n(n+1) + \frac{1}{2}\log n! + c_1 n\\
&\quad + \frac{1}{12}\left(\log n + \gamma + O\left(\frac{1}{n}\right)\right) + c_2 + O\left(\frac{1}{n^2}\right)\\
&= B(n) - \frac{1}{2}n(n+1)\\
&\quad + \frac{1}{2}\left(n \log n - n + \frac{1}{2}\log n + c_1\right)\\
&\quad + c_1 n + \frac{1}{12}(\log n + \gamma) + c_2 + O\left(\frac{1}{n}\right).\end{aligned}$$

Therefore

$$A(n) = B(n) - \frac{1}{2}n^2 + \frac{1}{2}n\log n + (c_1 - 1)n + \frac{1}{3}\log n - c_3 + O\left(\frac{1}{n}\right) \qquad (2.8)$$

where γ is Euler's constant and

$$c_3 :\, = -\left(\frac{1}{2}c_1 + \frac{\gamma}{12} + c_2\right).$$

From (2.7) and (2.8) we deduce that

$$P(n) = E(n)\left(1 + O\left(\frac{1}{n}\right)\right) \qquad (2.9)$$

where

$$E(n) :\, = \exp\left(\frac{1}{2}n^2 - \frac{1}{2}n\log n + (1 - c_1)n - \frac{1}{3}\log n + c_3\right). \qquad (2.10)$$

We now compute c_3:

By using the Euler-Maclaurin summation formula, Wilf [W, p126] found the constant term in the asymptotic expansion of $B(n)$ to be

$$c_4 = \frac{\gamma}{12} + \frac{\log 2\pi}{12} + \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{\log k}{k^2}.$$

Since the constant term on the right hand side of

$$B(n) = (n+1)\log n! - A(n)$$

is $\frac{1}{2}\log 2\pi + \frac{1}{12} + c_3 - c_4$ by (2.8) we deduce that

$$\begin{aligned} c_3 &= 2c_4 - c_1 - \frac{1}{12} \\ &= \frac{\gamma}{6} - \frac{\log 2\pi}{3} - \frac{1}{12} + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\log k}{k^2} \\ &= \frac{\gamma}{6} - \frac{\log 2\pi}{3} - \frac{1}{12} - \frac{\zeta'(2)}{6\zeta(2)}, \end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta function.

Next by (2.4),

$$W(n) = 2\widetilde{W}(n)(1 + \rho(n))$$

where

$$|\rho(n)| \le 2^n\frac{\widetilde{W}(n-1)}{\widetilde{W}(n)}.$$

From (2.5) we see

$$\frac{\widetilde{W}(n-1)}{\widetilde{W}(n)} = \frac{P(n-1)}{P(n)}\left(1+O\left(\frac{1}{n}\right)\right)$$
$$= \frac{E(n-1)}{E(n)}\left(1+O\left(\frac{1}{n}\right)\right).$$

Now by (2.10),

$$\frac{E(n-1)}{E(n)} = \exp(-n+O(\log n)).$$

Thus

$$|\rho(n)| \le 2^n \exp(-n+O(\log n)) \ll \frac{1}{n}.$$

We conclude that

$$W(n) = 2E(n)\left(1+O\left(\frac{1}{n}\right)\right) \tag{2.11}$$

where $E(n)$ is given by (2.10).

Finally we note that from (2.2), for $n \ge 1$,

$$2\widetilde{W}(n) = \prod_{m=1}^{n}\left(1+\binom{n}{m}\right) \tag{2.12}$$

and so by (2.4),

$$W(n) = \sum_{m=1}^{n}\binom{n}{m}(-1)^{n-m}\prod_{j=1}^{m}\left(1+\binom{m}{j}\right)+(-1)^n.$$

By using (2.12) in place of (2.2) we obtain an alternative and shorter way to derive (2.5): By (2 12),

$$\widetilde{W}(n)/P(n) = \prod_{m=1}^{n-1}\left(1+\binom{n}{m}^{-1}\right) \ge 1$$

and

$$\widetilde{W}(n)/P(n) \le \left(1+\frac{1}{n}\right)^2\left(1+\binom{n}{2}^{-1}\right)^n$$
$$\le \left(1+\frac{1}{n}\right)^2 \exp\left(n/\binom{n}{2}\right)$$
$$= 1+O\left(\frac{1}{n}\right).$$

However, the reason for including the original method is that this same approach will be applied to estimate a sum like (2.2) without a simple closed form, that arises in section 3 on ordered covers.

3 Ordered Coverings with Distinct Block Sizes

Let $Y(n)$ be the number of ordered covers of N with the restriction that the sizes of the blocks be distinct, $Y(0) = 1$. As before we write

$$Y(n) = \widetilde{Y}(n) + \widehat{Y}(n)$$

where $\widetilde{Y}(n)$ is the number of ordered covers of N with distinct block sizes where N occurs as one of the blocks and $\widehat{Y}(n)$ counts the remaining cases where N does not occur as a block. In addition the above variables with subscript r denote covers with precisely r blocks (excluding the block N itself in the case of $\widetilde{Y}_r(n)$). We begin as in the previous section by noting the inequalities,

$$\widetilde{Y}(n) \leq Y(n) \leq 2\widetilde{Y}(n). \tag{3.1}$$

Now let $1 \leq r \leq n$. Since there are $r+1$ positions in which the block N can be added to a cover with r blocks that excludes N we have

$$\widetilde{Y}_r(n) = (r+1)\widehat{Y}_r(n) + \sum_{m=r}^{n-1} \binom{n}{m} (r+1) Y_r(n)$$

From this

$$\begin{aligned}\sum_{r=1}^{n-1} \frac{\widetilde{Y}_r(N)}{r+1} &= \widehat{Y}(n) + \sum_{r=1}^{n-1}\sum_{m=r}^{n-1} \binom{n}{m} Y_r(m) \\ &= \widehat{Y}(n) + \sum_{m=1}^{n-1} \binom{n}{m} \sum_{r=1}^{m} Y_r(m) \\ &= \widehat{Y}(n) + \sum_{m=1}^{n-1} \binom{n}{m} Y(m).\end{aligned}$$

Adding $\widetilde{Y}(n)$ to each side leads to

$$\widetilde{Y}(n) + \sum_{r=0}^{n-1} \frac{\widetilde{Y}(n)}{r+1} = \sum_{m=0}^{n} \binom{n}{m} Y(m). \tag{3.2}$$

If we denote the left hand side of (3.2) by $Z(n)$ then binomial inversion gives the formula

$$Y(n) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} Z(m), \tag{3.3}$$

where we define $Z(0) = 1$.

Again we have an explicit formula for $\tilde{Y}(n) = \sum_{r=0}^{n-1} \tilde{Y}_r(n)$ since $\tilde{Y}_0(n) = 1$ and for $1 \leq r < n$,

$$\tilde{Y}(n) = \sum_{1 \leq m_1 < m_2 \cdots < m_r < n} \binom{n}{m_1}\binom{n}{m_2}\cdots\binom{n}{m_r}(r+1)!. \tag{3.4}$$

Let $Q(n)$ denote the number of ordered covers of N with n blocks of distinct size so that

$$Q(n) = n! \prod_{j=1}^{n} \binom{n}{j} = n!P(n). \tag{3.5}$$

There does not appear to be a compact expression like (2.12) for $Z(n)$, but we can estimate the sums (3.4) as we did with (2.2) to show that

$$Z(n) = Q(n)\left(1 + O\left(\frac{1}{n}\right)\right)$$

as well as

$$Y(n) = Z(n)\left(1 + O\left(\frac{1}{n}\right)\right).$$

From this and (2.9) we deduce that

$$\begin{aligned} Y(n) &= n!P(n)\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{2}n!W(n)\left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{3.6}$$

Remark From (3.6) and (2.7) we see that

$$\log W(n) \sim \log Y(n).$$

On the other hand the estimates of the introduction show that

$$\log X(n) \sim n \log V(n).$$

In either case, however, the proportion of covers of N with distinct block sizes is asymptotically zero. In particular we deduce for both the ordered and unordered cases: Almost all covers of a finite set with n elements have two distinct blocks of the same size.

In fact there is a simple direct proof of this. To every unordered cover of N with distinct block sizes associate those covers consisting of the same blocks and an additional two singleton blocks. There are either $\binom{n}{2}$ or $\binom{n-1}{2}$ such associated covers depending on whether or not the original cover had a singleton block. This implies that the proportion of cover of N with distinct block sizes is $O\left(\frac{1}{n^2}\right)$. In the ordered case, the ordering of the blocks only increases the number of associated covers with at least two singleton blocks.

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