Log Concavity Involving the Number of Paths from the Origin to Points Along the Line $(a, b, c, d) + t(1, -1, 1, -1)$

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Abstract

This paper examines the numbers of lattice paths of length n from the origin to integer points along the line $(a, b, c, d) + t(1, -1, 1, -1)$. These numbers form a sequence which this paper shows is log concave, and for sufficiently large values of n, the location of the maximum of this sequence is shown. This paper also shows unimodality of such sequences for other lines provided that n is sufficiently large.

Introduction

Consider the d-dimensional integer lattice. One can form a walk on it by going at each step one unit in either direction parallel to an axis. Simple combinatorial expressions are known for the number of walks of length n from the origin to a given point if the dimension is 1 or 2. In 4 dimensions, a simple combinatorial expression exists for the number of walks of length n from the origin to the line $(a, b, c, d) + t(1, -1, 1, -1)$, but the known expressions for the number of walks of length n from the origin to a given point in the 4-dimensional lattice are not as simple. See [2] for details. We nonetheless wish to get some idea of the properties this number has.

A sequence x_0, x_1, ..., x_n is said to be unimodal if $x_0 \leq x_1 \leq ... \leq x_i \geq x_{i+1} \geq ... \geq x_n$ for some $i \in \{0, ..., n\}$. This sequence is said to be log concave if $x_i^2 \geq x_{i+1}x_{i-1}$ for all $i \in \{1, ..., n-1\}$. Note that all log concave sequences with no internal zeros and no negative numbers are unimodal. For a survey of some log concave and unimodal sequences, see [1] or [4]. Along a line in the integer lattice, one can define a sequence by letting the elements of the sequence be the number of walks of length n

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to integer points $(x_1, ..., x_m)$ such that $\sum_{i=1}^{m} |x_i| \le n$ and $\sum_{i=1}^{m} |x_i| \equiv n \pmod 2$. The questions we consider in this paper involve the log concavity and unimodality of such sequences.

It can be readily shown that such sequences are log concave in dimensions 1 and 2. In this paper, we show log concavity for such sequences formed by the line $(a, b, c, d) + t(1, -1, 1, -1)$. We also consider where the maximum is along the line via a method which can be generalized to show the unimodality for large enough n of such sequences along other lines.

Proof of Log Concavity for Specific 4-Dimensional Line

Let $P_n^d(p)$ be the number of walks of length n from the origin to the point p in the d-dimensional lattice. In this section, we shall show

Theorem 1: *Suppose a, b, c, and d are integers such that* $a+b+c+d \equiv n \pmod 2$. *Let* $g(t) = P_n^4((a, b, c, d) + t(1, -1, 1, -1))$. *Then the sequence* $..., g(-2), g(-1), g(0), g(1), g(2), ...$ *is log concave.*

Note that this theorem shows the unimodality of the sequence consisting of the number of paths of length n from the origin to points on the line $(a, b, c, d) + t(1, -1, 1, -1)$ since this sequence has no internal zeros. Also note that it suffices to prove that $g(0)^2 \ge g(1)g(-1)$.

The proof of this theorem will involve several lemmas. The first lemma provides an expression for $P_n^4((a, b, c, d))$.

Lemma 1:

$$P_n^4((a,b,c,d)) = \sum_{n_1+n_2=n} \binom{n}{n_1} \binom{n_1}{(n_1+a+b)/2} \binom{n_1}{(n_1+a-b)/2} \binom{n_2}{(n_2+c+d)/2} \binom{n_2}{(n_2+c-d)/2}$$

where the binomial coefficient $\binom{n}{k}$ *is defined to be* 0 *if* $k \notin \{0, ..., n\}$.

Proof: There are $P_{n_1}^2((a, b))P_{n_2}^2((c, d))$ lattice paths of length n from the origin to the point (a, b, c, d) such that n_1 given steps are parallel to one of the first 2 axes and the other steps are parallel to one of the other axes. There are, of course, $\binom{n}{n_1}$ ways to choose the n_1 given steps.

Furthermore, [2] shows that

$$P_n^2((a,b)) = \binom{n}{(n+a+b)/2} \binom{n}{(n+a-b)/2}.$$

Summing over all possible values of n_1 completes the lemma. ∎

Let x and y be non-negative integers less than or equal to $(n-(a+b+c+d))/2$. Let

$$(x,y,+) =$$
$$\binom{n}{a+b+2x}\binom{a+b+2x}{a+b+x}\binom{a+b+2x}{a+x}\binom{c+d+2\tilde{x}}{c+d+\tilde{x}}\binom{c+d+2\tilde{x}}{c+\tilde{x}}$$
$$\binom{n}{a+b+2y}\binom{a+b+2y}{a+b+y}\binom{a+b+2y}{a+y}\binom{c+d+2\tilde{y}}{c+d+\tilde{y}}\binom{c+d+2\tilde{y}}{c+\tilde{y}}$$

where $a+b+c+d+2x+2\tilde{x}=n$ and $a+b+c+d+2y+2\tilde{y}=n$. Let

$$(x,y,-) =$$
$$\binom{n}{a+b+2x}\binom{a+b+2x}{a+b+x}\binom{a+b+2x}{a+x+1}\binom{c+d+2\tilde{x}}{c+d+\tilde{x}}\binom{c+d+2\tilde{x}}{c+\tilde{x}+1}$$
$$\binom{n}{a+b+2y}\binom{a+b+2y}{a+b+y}\binom{a+b+2y}{a+y-1}\binom{c+d+2\tilde{y}}{c+d+\tilde{y}}\binom{c+d+2\tilde{y}}{c+\tilde{y}-1},$$

and let $(x,y)=(x,y,+)-(x,y,-)$.

Observe that

$$\begin{aligned}&(P_n^4((a,b,c,d))^2\\ &-P_n^4((a+1,b-1,c+1,d-1))P_n^4((a-1,b+1,c-1,d+1)))\\ &=\sum_{x=0}^{(n-a-b-c-d)/2}\sum_{y=0}^{(n-a-b-c-d)/2}(x,y).\end{aligned}$$

Thus we would like to show $\sum(x,y)\geq 0$. Unfortunately, (x,y) is sometimes less than 0. In addition, computer number-crunching has demonstrated that $(x,y)+(y,x)$ may also be negative.

The strategy we employ does involve matching each negative (x,y) term with a unique positive term. Consider the diagram in Figure 1. All terms represented on the diagonal and the lines just off the diagonal can be shown to be nonnegative. Negative terms below these lines are matched with unique positive terms in the upper triangle in Figure 2 while negative terms above the lines are matched with unique positive terms in the lower triangle in Figure 3. The 2 triangles intersect along the diagonal, but we shall show that at most one term is matched to a given term along the diagonal. Furthermore we shall show that each negative term has been matched with a positive term at least as large in absolute value as the negative term. Figure 4 contains output from a Maple run which gives the values of (x,y) for a specific example.

The following 3 lemmas provide this alternate method and together prove Theorem 1.

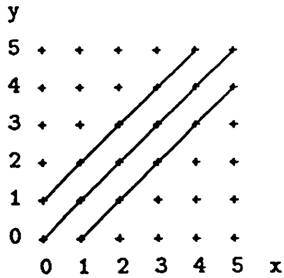

Figure 1

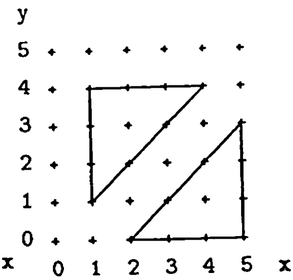

Figure 2

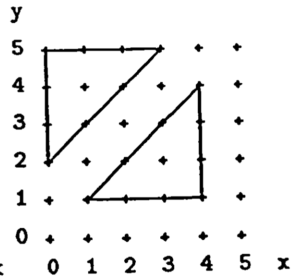

Figure 3

Lemma 2: *If* $|x - y| \leq 1$, *then* $(x, y) \geq 0$.
Lemma 3*: If* $(x, y) < 0$, *then* $(y, x) \geq 0$.
Lemma 4*: If* $(x, y) < 0$, *then* $(x, y) + (y + 1, x - 1) \geq 0$ *if* $x > y$ *and* $(x, y) + (y - 1, x + 1) \geq 0$ *if* $x < y$.

Theorem 1 follows from these lemmas by noting that each negative term in the sum can be matched up with a unique positive term such that the sum of the two terms is positive. Note that Lemma 3 is required to ensure that positive terms with $x = y$ are not matched up with 2 separate negative terms. For example, in Figure 4, $(3, 3)$ is matched up with a negative term $(2, 4)$ but the term $(4, 2)$ is not negative (and so does not have to be matched with $(3, 3)$ as well).

Proof of Lemma 2: Let x and y be given. Note that $(x, y) \geq 0$ if

$$\binom{a+b+2x}{a+x}\binom{a+b+2y}{a+y} - \binom{a+b+2x}{a+x+1}\binom{a+b+2y}{a+y-1} \geq 0 \qquad (*)$$

and

$$\binom{c+d+2\tilde{x}}{c+\tilde{x}}\binom{c+d+2\tilde{y}}{c+\tilde{y}} - \binom{c+d+2\tilde{x}}{c+\tilde{x}+1}\binom{c+d+2\tilde{y}}{c+\tilde{y}-1} \geq 0. \qquad (**)$$

If any of $b + x - 1$, $a + x + 1$, $b + y + 1$, and $a + y - 1$ is negative, then (*) clearly holds. Otherwise, since

$$\binom{a+b+2x}{a+x}\binom{a+b+2y}{a+y} - \binom{a+b+2x}{a+x+1}\binom{a+b+2y}{a+y-1}$$
$$= \binom{a+b+2x}{a+x}\binom{a+b+2y}{a+y}\left(1 - \frac{(b+x)}{(a+x+1)}\frac{(a+y)}{(b+y+1)}\right),$$

(*) follows from the fact that $0 < b+x \leq b+y+1$ and $0 < a+y \leq a+x+1$. (**) can be shown likewise. ∎

```
> print(P);

proc(n,a,b,c,d)
    p := 1/2*n-1/2*a-1/2*b-1/2*c-1/2*d;
    M := array(0 .. p,0 .. p);
    N := array(0 .. 2*p);
    for i from 0 to 2*p do  N[i] := 0 od;
    for i from 0 to p do
        for j from 0 to p do
            M[i,j] := binomial(n,a+b+2*i)*binomial(a+b+2*i,a+b+i)*
                binomial(a+b+2*i,a+i)*binomial(c+d+2*p-2*i,c+d+p-i)*
                binomial(c+d+2*p-2*i,c+p-i)*binomial(n,a+b+2*j)*
                binomial(a+b+2*j,a+b+j)*binomial(a+b+2*j,a+j)*
                binomial(c+d+2*p-2*j,c+d+p-j)*binomial(c+d+2*p-2*j,c+p-j)-
                binomial(n,a+b+2*i)*binomial(a+b+2*i,a+b+i)*
                binomial(a+b+2*i,a+i+1)*binomial(c+d+2*p-2*i,c+d+p-i)*
                binomial(c+d+2*p-2*i,c+p-i+1)*binomial(n,a+b+2*j)*
                binomial(a+b+2*j,a+b+j)*binomial(a+b+2*j,a+j-1)*
                binomial(c+d+2*p-2*j,c+d+p-j)*binomial(c+d+2*p-2*j,c+p-j-1);
            e := N[i+j];
            N[i+j] := e+M[i,j]
        od
    od;
    print(M);
    print(N)
end

> P(20,4,6,3,-1);
                    array(0 .. 4,0 .. 4,, [
                        (0, 0) = 5257744318681126272000000
                        (0, 1) = 8030009504894811033600000
                        (0, 2) = 2330136686688226416000000
                        (0, 3) = -3156163977736433664000000
                        (0, 4) = -814068018982842624000000
                        (1, 0) = 12236204959839712051200000
                        (1, 1) = 20556824332530716246016000
                        (1, 2) = 7953533223895812833280000
                        (1, 3) = 155380380442409041920000
                        (1, 4) = -1157785626997820620800000
                        (2, 0) = 7545204509276161728000000
                        (2, 1) = 13634628383821393428480000
                        (2, 2) = 6202601942184374126400000
                        (2, 3) = 552328696103875891200000
                        (2, 4) = -335964579262760448000000
                        (3, 0) = 1213909222206320640000000
                        (3, 1) = 2330705706636135628800000
                        (3, 2) = 1183561491651162624000000
                        (3, 3) = 154147202819850240000000
                        (3, 4) = 0
                        (4, 0) = 0
                        (4, 1) = 0
                        (4, 2) = 0
                        (4, 3) = 0
                        (4, 4) = 0
                    ])

...output continues...
```

Figure 4 - Maple output when (a,b,c,d)=(4,6,3,-1) and n=20. The array contains the values of (x,y).

Proof of Lemma 3: Observe that

$$(x,y)=(x,y,+)\left(1-\frac{(b+x)}{(a+x+1)}\frac{(a+y)}{(b+y+1)}\frac{(d+\tilde{x})}{(c+\tilde{x}+1)}\frac{(c+\tilde{y})}{(d+\tilde{y}+1)}\right)$$

while

$$(y,x)=(x,y,+)\left(1-\frac{(b+y)}{(a+y+1)}\frac{(a+x)}{(b+x+1)}\frac{(d+\tilde{y})}{(c+\tilde{y}+1)}\frac{(c+\tilde{x})}{(d+\tilde{x}+1)}\right)$$

Note that since $(x,y)<0$ and hence $(x,y,-)\neq 0$, all of the following are non-negative: $b+x-1$, $a+y-1$, $d+\tilde{x}-1$, $c+\tilde{y}-1$, $a+x+1$, $b+y+1$, $c+\tilde{x}+1$, and $d+\tilde{y}+1$. If any one of $b+y+1$, $a+x+1$, $d+\tilde{y}+1$, and $c+\tilde{x}+1$ is zero, then $(y,x,-)=0$ and the lemma holds. Otherwise, since

$$\frac{(b+x)}{(a+x+1)}\frac{(a+y)}{(b+y+1)}\frac{(d+\tilde{x})}{(c+\tilde{x}+1)}\frac{(c+\tilde{y})}{(d+\tilde{y}+1)}>1,$$

then

$$\frac{(a+x+1)}{(b+x)}\frac{(b+y+1)}{(a+y)}\frac{(c+\tilde{x}+1)}{(d+\tilde{x})}\frac{(d+\tilde{y}+1)}{(c+\tilde{y})}<1.$$

Since

$$\frac{(b+y)}{(a+y+1)}\frac{(a+x)}{(b+x+1)}\frac{(d+\tilde{y})}{(c+\tilde{y}+1)}\frac{(c+\tilde{x})}{(d+\tilde{x}+1)}$$
$$<\frac{(a+x+1)}{(b+x)}\frac{(b+y+1)}{(a+y)}\frac{(c+\tilde{x}+1)}{(d+\tilde{x})}\frac{(d+\tilde{y}+1)}{(c+\tilde{y})},$$

the previous inequality implies $(y,x)\geq 0$ and the lemma follows. ∎

Proof of Lemma 4: First let's consider the case where $x>y$. Provided that no divisors are 0, we can write

$$(x,y)+(y+1,x-1)=$$
$$(x,y,-)\left(\left(\frac{(x,y,+)}{(x,y,-)}-1\right)+\frac{(y+1,x-1,+)}{(x,y,-)}\left(1-\frac{(y+1,x-1,-)}{(y+1,x-1,+)}\right)\right).$$

Note that $(x,y,-)\neq 0$ since $(x,y)<0$. To conclude this expression is well-defined, we need to also show the following proposition.

Proposition 1: $(y+1,x-1,+)>0$ *in this case.*

Since $(x,y,-)>0$, the following are all non-negative: $a+b+2x$, $c+d+2\tilde{x}$, $a+b+2y$, $c+d+2\tilde{y}$, $a+b+x$, x, $a+x+1$, $b+x-1$, $c+d+\tilde{x}$, $\tilde{x}$, $c+\tilde{x}+1$, $d+\tilde{x}-1$, $a+b+2y$, $c+d+2\tilde{y}$, $a+b+y$, y, $a+y-1$, $b+y+1$, $c+d+\tilde{y}$, $\tilde{y}$, $c+\tilde{y}-1$, and $d+\tilde{y}+1$.

Observe that

$$(y+1,x-1,+) = \binom{n}{a+b+2y+2}\binom{a+b+2y+2}{a+b+y+1}\binom{a+b+2y+2}{a+y+1}$$
$$\binom{c+d+2\tilde{y}-2}{c+d+\tilde{y}-1}\binom{c+d+2\tilde{y}-2}{c+\tilde{y}-1}$$
$$\binom{n}{a+b+2x-2}\binom{a+b+2x-2}{a+b+x-1}\binom{a+b+2x-2}{a+x-1}$$
$$\binom{c+d+2\tilde{x}+2}{c+d+\tilde{x}+1}\binom{c+d+2\tilde{x}+2}{c+\tilde{x}+1}.$$

Since $x > y$ and hence $x \geq y+1$ and $\tilde{y} \geq \tilde{x}+1$, we may conclude that the following are also non-negative: $a+b+2x-2$, $a+b+x-1$, $x-1$, $a+x-1$, $c+d+\tilde{y}-1$, $d+\tilde{y}-1$, $\tilde{y}-1$, and $c+d+2\tilde{y}-2$. Thus all the binomial coefficients in $(y+1,x-1,+)$ are non-zero and hence $(y+1,x-1,+) > 0$. ■

To continue the proof of Lemma 4, we shall show

Proposition 2:

$$\left|\frac{(x,y,+)}{(x,y,-)} - 1\right| \leq 1 - \frac{(y+1,x-1,-)}{(y+1,x-1,+)}.$$

Observe that implicit in this proposition is the fact that $(y+1,x-1,-) < (y+1,x-1,+)$ and hence $(y+1,x-1) > 0$.

Proof of Proposition 2: This proof is quite straightforward. Observe that

$$\frac{(x,y,+)}{(x,y,-)} = \frac{(a+x+1)}{(b+x)}\frac{(b+y+1)}{(a+y)}\frac{(c+\tilde{x}+1)}{(d+\tilde{x})}\frac{(d+\tilde{y}+1)}{(c+\tilde{y})}$$

and

$$\frac{(y+1,x-1,-)}{(y+1,x-1,+)} = \frac{(a+x-1)}{(b+x)}\frac{(b+y+1)}{(a+y+2)}\frac{(c+\tilde{x}+1)}{(d+\tilde{x}+2)}\frac{(d+\tilde{y}-1)}{(c+\tilde{y})}.$$

Note that all terms are non-negative and that inequality follows by a term by term comparison. ■

The following proposition completes the proof in the case that $x > y$.

Proposition 3:

$$(y+1,x-1,+) \geq (x,y,-).$$

Proof: Observe that

$$\frac{(y+1,x-1,+)}{(x,y,-)} =$$
$$\frac{(a+b+x)x(a+x)(a+x+1)(a+b+2y+1)(a+b+2y+2)}{(a+b+2x-1)(a+b+2x)(a+y)(a+y+1)(a+b+y+1)(y+1)}$$
$$\frac{(c+d+\tilde{y})\tilde{y}(d+\tilde{y})(d+\tilde{y}+1)(c+d+2\tilde{x}+1)(c+d+2\tilde{x}+2)}{(c+d+2\tilde{y}-1)(c+d+2\tilde{y})(d+\tilde{x})(d+\tilde{x}+1)(c+d+\tilde{x}+1)(\tilde{x}+1)}.$$

Observe that

$$\frac{(a+b+x)x(a+x)(a+x+1)(a+b+2y+1)(a+b+2y+2)}{(a+b+2x-1)(a+b+2x)(a+y)(a+y+1)(a+b+y+1)(y+1)}$$
$$=\frac{(a+b+x)x(a+b+2y+2)(a+b+2y+2)}{(a+b+2x-1)(a+b+2x)(a+b+y+1)(y+1)}$$
$$\frac{(a+x)(a+b+2y+1)}{(a+y)(a+b+2y+2)}\frac{(a+x+1)}{(a+y+1)}.$$

Since $x > y$ and the terms are positive, it is clear that $(a+x+1)/(a+y+1) > 1$. Furthermore,

$$\begin{aligned}\frac{(a+x)(a+b+2y+1)}{(a+y)(a+b+2y+2)} &= \frac{(a+x)/(a+y)}{(a+b+2y+2)/(a+b+2y+1)}\\ &= \frac{1+((x-y)/(a+y))}{1+(1/(a+b+2y+1))}\\ &\geq \frac{1+(1/(a+y))}{1+(1/(a+b+2y+1))}.\end{aligned}$$

Because $b+y+1 \geq 0$, we may conclude that $a+y \leq a+b+2y+1$. Since $a+y > 0$, we may conclude the above fraction is at least 1.

Now let's look at the first fraction.

$$\frac{(a+b+x)x(a+b+2y+2)(a+b+2y+2)}{(a+b+2x-1)(a+b+2x)(a+b+y+1)(y+1)}$$
$$\geq \frac{(a+b+x)x}{(a+b+2x)(a+b+2x)}\frac{(a+b+2y+2)(a+b+2y+2)}{(a+b+y+1)(y+1)}$$
$$=\frac{f(x)}{f(y+1)}$$

where

$$f(w) := \frac{(w+(a+b))w}{(2w+(a+b))^2}.$$

The first fraction can be shown to be greater than 1 by Proposition 4; this proposition will complete the proof of Proposition 3.

Proposition 4: *If $x \geq y+1$ and $a+b+y$ and y are non-negative, then $f(x) \geq f(y+1)$.*

Proof: We can rewrite $f(x)$ as follows.

$$\begin{aligned}
&f(x)\\
&= \frac{((x+(1/2)(a+b))+(1/2)(a+b))((x+(1/2)(a+b))-(1/2)(a+b))}{2^2(x+(1/2)(a+b))^2}\\
&= \frac{1}{4}\left(1+\frac{(1/2)(a+b)}{(x+(1/2)(a+b))}\right)\left(1-\frac{(1/2)(a+b)}{(x+(1/2)(a+b))}\right)\\
&= \frac{1}{4}\left(1-\frac{((1/2)(a+b))^2}{(x+(1/2)(a+b))^2}\right).
\end{aligned}$$

Likewise

$$f(y+1)=\frac{1}{4}\left(1-\frac{((1/2)(a+b))^2}{((y+1)+(1/2)(a+b))^2}\right).$$

Since $0 \leq (y+1)+(1/2)(a+b) \leq x+(1/2)(a+b)$, it follows that $f(y+1) \leq f(x)$. ∎

The case where $x < y$ can be proved from the case $x > y$ by observing that (x,y) along the line $(a,b,c,d)+t(1,-1,1,-1)$ is the same as (y,x) along the line $(b,a,d,c)+t(1,-1,1,-1)$. ∎

Location of the Maximum

At what value of t is the maximum of $P_n^4((a,b,c,d)+t(1,-1,1,-1))$? Computer number-crunching shows that for certain lines this point may vary depending on n. As n increases, the point where the maximum occurs approaches a certain location which is described in the following theorem.

Theorem 2: *The maximum of $P_n^4((a,b,c,d)+t(1,-1,1,-1))$ occurs at an integer value of t such that $(a+t)-(b-t)+(c+t)-(d-t)$ is as close to 0 as possible provided that n is sufficiently large.*

Proof: Let us examine the expression

$$P_n^4((a,b,c,d)) = \sum_{n_1}\binom{n}{n_1}\binom{n_1}{\frac{1}{2}(n_1+a+b)}\binom{n_1}{\frac{1}{2}(n_1+a-b)}\binom{n_2}{\frac{1}{2}(n_2+c+d)}\binom{n_2}{\frac{1}{2}(n_2+c-d)}.$$

For what range of n_1 can we find the main portion of the sum? For large enough n, we shall find that the sum is concentrated around $n_1 = n/2$.

Let

$$t(n, n_1, a, b, c, d) :=$$
$$\binom{n}{n_1}\binom{n_1}{\frac{1}{2}(n_1+a+b)}\binom{n_1}{\frac{1}{2}(n_1+a-b)}\binom{n_2}{\frac{1}{2}(n_2+c+d)}\binom{n_2}{\frac{1}{2}(n_2+c-d)}$$

where $n_2 = n - n_1$. Note that

$$\frac{t(n, n_1+2, a, b, c, d)}{t(n, n_1, a, b, c, d)}$$
$$= \frac{n_2(n_2-1)}{(n_1+2)(n_1+1)} \frac{(n_1+2)(n_1+1)}{(\frac{1}{2}(n_1+a+b)+1)(\frac{1}{2}(n_1-a-b)+1)}$$
$$\frac{(n_1+2)(n_1+1)}{(\frac{1}{2}(n_1+a-b)+1)(\frac{1}{2}(n_1-a+b)+1)}$$
$$\frac{\frac{1}{2}(n_2+c+d)\frac{1}{2}(n_2-c-d)}{n_2(n_2-1)} \frac{\frac{1}{2}(n_2+c-d)\frac{1}{2}(n_2-c+d)}{n_2(n_2-1)}$$
$$= \frac{n_2(n_2-1)}{(n_1+2)(n_1+1)} \frac{(1+\frac{2}{n_1})(1+\frac{1}{n_1})}{(1+\frac{a+b+2}{n_1})(1+\frac{-a-b+2}{n_1})} \frac{(1+\frac{2}{n_1})(1+\frac{1}{n_1})}{(1+\frac{a-b+2}{n_1})(1+\frac{-a+b+2}{n_1})}$$
$$\left(1+\frac{c+d}{n_2}\right)\left(1+\frac{-c-d+1}{n_2-1}\right)\left(1+\frac{c-d}{n_2}\right)\left(1+\frac{-c+d+1}{n_2-1}\right).$$

Suppose that $n_1 \geq \frac{1}{2}n + \sqrt{n}$. Then

$$\frac{t(n, n_1+2, a, b, c, d)}{t(n, n_1, a, b, c, d)} \leq 1 - \frac{2}{\sqrt{n}} + f_1(n)$$

where $f_1(n)/(1/n)$ is bounded as $n \to \infty$. Thus if $n_1 \geq \frac{1}{2}n + \sqrt{n} + n^{0.6}$, then for large enough n,

$$\frac{t(n, \frac{1}{2}n + \sqrt{n} + n^{0.6} + f_2(n), a, b, c, d)}{t(n, \frac{1}{2}n + \sqrt{n} + f_3(n), a, b, c, d)} \leq \exp(-2n^{0.1})$$

where $f_2(n)$ and $f_3(n)$ are added to make the expressions be integers of the correct parity; we let $f_2(n)$ and $f_3(n)$ be as close to 0 as possible; hence they have absolute value no more than 1. By similar reasoning, we may conclude that

$$\frac{t(n, \frac{1}{2}n - \sqrt{n} - n^{0.6} + f_4(n), a, b, c, d)}{t(n, \frac{1}{2}n - \sqrt{n} + f_5(n), a, b, c, d)} \leq \exp(-2n^{0.1})$$

for large enough n.

Now let us examine, for n_1 between $\frac{1}{2}n - \sqrt{n} - n^{0.6}$ and $\frac{1}{2}n + \sqrt{n} + n^{0.6}$, the following ratio:

$$\begin{aligned}
&\frac{t(n, n_1, a+1, b-1, c+1, d-1)}{t(n, n_1, a, b, c, d)} \\
&= \frac{\frac{1}{2}(n_1 - a + b)}{\frac{1}{2}(n_1 + a - b) + 1} \frac{\frac{1}{2}(n_2 - c + d)}{\frac{1}{2}(n_2 + c - d) + 1} \\
&= \frac{1 + \frac{-a+b}{n_1}}{1 + \frac{a-b+2}{n_1}} \frac{1 + \frac{-c+d}{n_2}}{1 + \frac{c-d+2}{n_2}} \\
&= 1 + \frac{2(-a+b-a+b-2-c+d-c+d-2)}{n} + f_6(n) \\
&= 1 + \frac{4(-a+b-c+d-2)}{n} + f_6(n)
\end{aligned}$$

where $\lim_{n\to\infty} f_6(n)/(1/n) = 0$. Since $\lim_{n\to\infty} n\exp(-2n^{0.1})/(1/n) = 0$, we may conclude

$$\frac{P_n^4((a+1, b-1, c+1, d-1))}{P_n^4((a, b, c, d))} = 1 + \frac{4(-a+b-c+d-2)}{n} + f_7(n)$$

where $\lim_{n\to\infty} f_7(n)/(1/n) = 0$.

For sufficiently large n, this ratio implies that the location of the maximum of $P_n^4((a, b, c, d) + t(1, -1, 1, -1))$ occurs at an integer t such that $(a+t) - (b-t) + (c+t) - (d-t)$ is nearest 0. ∎

This technique may be adapted to show the unimodality of the sequence of the number of paths of length n from the origin to integer points along other lines provided that n is sufficiently large.

Theorem 3: *Let $a_1, ..., a_m, b_1, ..., b_m$ be integers such that $\sum_{i=1}^m b_i$ is even and $\sum_{i=1}^m a_i \equiv n \pmod 2$. Let $g(t) = P_n^m((a_1, ..., a_m) + t(b_1, ..., b_m))$. Then the sequence $..., g(-2), g(-1), g(0), g(1), g(2), ...$ is unimodal for sufficiently large n, and the maximum of this sequence corresponds to an integer t such that $(a_1, ..., a_m) + t(b_1, ..., b_m)$ is as close to the origin as possible.*

Sketch of proof: First note that we are only interested in a certain portion of the line; the portion of the line where all coordinates are decreasing in absolute value or all coordinates are increasing in absolute value will not contain a maximum of $g(t)$ except at the end of such a portion. Thus we may assume that we are looking at a portion whose length does not increase with n.

Without loss of generality, assume that for some integer d with $0 \le d \le m/2$, $b_1, ..., b_{2d}$ are all odd while $b_{2d+1}, ..., b_m$ are all even. Then

$$\begin{aligned}
&P_n^m(a_1, ..., a_m) \\
&= \sum_{\substack{m_1, ..., m_d, n_{2d+1}, ..., n_m \ge 0 \\ m_1 + ... + m_d + n_{2d+1} + ... + n_m = n}} t(n, m_1, .., m_d, n_{2d+1}, ..., n_m, a_1, ..., a_m)
\end{aligned}$$

where

$$t(n, m_1, \ldots, m_d, n_{2d+1}, \ldots, n_m, a_1, \ldots, a_m) := \binom{n}{m_1, \ldots, m_d, n_{2d+1}, \ldots, n_m}$$
$$\left(\prod_{i=1}^{d} \binom{m_i}{\frac{1}{2}(m_i + a_{2i-1} + a_{2i})} \binom{m_i}{\frac{1}{2}(m_i + a_{2i-1} - a_{2i})}\right)$$
$$\prod_{i=2d+1}^{m} \binom{n_i}{\frac{1}{2}(n_i + a_i)}.$$

By techniques similar to the proof of Theorem 2, we may conclude that if $(2/m)n - \sqrt{n} - n^{0.6} \le m_i \le (2/m)n + \sqrt{n} + n^{0.6}, i = 1, \ldots, d$ and $(1/m)n - \sqrt{n} - n^{0.6} \le n_i \le (1/m)n + \sqrt{n} + n^{0.6}, i = 2d+1, \ldots, m$, then

$$\frac{t(n, m_1, \ldots, m_d, n_{2d+1}, \ldots, n_m, a_1 + b_1, \ldots, a_m + b_m)}{t(n, m_1, \ldots, m_d, n_{2d+1}, \ldots, n_m, a_1, \ldots, a_m)}$$
$$= 1 - \frac{m}{n}(\sum_{i=1}^{m} a_i b_i + \frac{1}{2}\sum_{i=1}^{m} b_i^2) + o(1/n).$$

Furthermore, the terms where any of $m_1, \ldots, m_d, n_2d + 1, \ldots, n_m$ are outside this range are so small (for sufficiently large n) that we may conclude

$$\frac{P_n^m((a_1 + b_1, \ldots, a_m + b_m))}{P_n^m((a_1, \ldots, a_m))} = 1 - \frac{m}{n}(\sum_{i=1}^{m} a_i b_i + \frac{1}{2}\sum_{i=1}^{m} b_i^2) + o(1/n).$$

For sufficiently large n, this ratio implies that the maximum of $g(t)$ occurs at a value of t such that $(a_1, \ldots, a_m) + t(b_1, \ldots, b_m)$ is as close to the origin as possible. ∎

Problems for Further Study

A number of questions remain open in this field. For the line we studied, one wonders if there is a more elegant approach to showing log concavity of the sequence studied. Also, one may ask if such sequences are or are not Pólya frequency sequences of various orders; such sequences are log concave.

Showing (or disproving) log concavity along most other lines in 4 dimensions or along any line in dimensions other than 1, 2, or 4 remains open. It is not obvious how to generalize the techniques of Theorem 1 to these cases. It is not even clear what an appropriate matching would be. Is there a proof by induction on the number of dimensions? See [3] for some induction proofs on some problems involving log concavity and unimodality.

If direct proofs remain unavailable, can results on log concavity for these lines be shown for large n?

The location of the maximum, except for large n or direct computation in specific cases, remains unknown for the most part. How fast does this maximum move as n is increased?

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