Locally Invariant Positions of (0,1) Matrices

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ABSTRACT. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. Denote by $A(R,S)$ the class of $(0,1)$ matrices with row sum vector R and column sum vector S. We study a generalization of invariant positions called locally invariant positions of a class $A(R,S)$. For a normalized class, locally invariant positions have in common with invariant positions the property that they lie above and to the left of some simple rook path through the set of positions.

Introduction

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. Let $A(R,S)$ denote the class of $(0,1)$ matrices with row sum vector R and column sum vector S. The study of properties of these classes was begun in the late 1950's by Herbert J. Ryser and others. Basic information on $A(R,S)$ classes can be found in [1] and [2]. Ryser observed that replacing one of the following submatrices by the other leaves row and column sums unaltered:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ryser called such a replacement an *interchange* and proved that between any two matrices in an $A(R,S)$ class there exists a sequence of intermediate matrices in the class each obtained from the previous one by an interchange [1, page 68]. A position ij with $1 \le i \le m$, $1 \le j \le n$ is called an *invariant position* for the class $A(R,S)$ if all matrices in the class have the same entry in the ij position; that is, either condition a) or condition b) holds:

a) $A_{ij} = 1$ for all $A \in A(R,S)$

b) $A_{ij} = 0$ for all $A \in A(R, S)$.

If a) holds (b) holds) then ij is called an invariant 1-position (0-position) of $A(R, S)$. Invariant positions and invariant sets of positions have been studied in [3].

We now generalize the concept of invariant position. A position, ij, is a *locally invariant 1-position* of the class $A(R, S)$ if there exists a matrix $A \in A(R, S)$ such that

i) $A_{ij} = 1$

ii) no single interchange applied to A makes the ij entry equal to 0.

We will henceforth refer to these as properties i) and ii). Locally invariant 0-positions of $A(R, S)$ may be defined analogously.

To illustrate these ideas consider the class $A(R, S)$ with $R = S = (3, 2, 1, 1)$. It can be shown that the positions 11, 12, 21, 22 are locally invariant 1-positions of $A(R, S)$ and 23, 24, 32, 33, 34, 42, 43, 44 are locally invariant 0-positions. The matrix $A \in A(R, S)$ given below shows that 22 is a locally invariant 1-position of $A(R, S)$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix B shows that 11 is a locally invariant 1-position. B also shows that 34, 43, 44 are locally invariant 0-positions. Other matrices in $A(R, S)$ can be found to establish the local invariance of the positions listed. It is important to note that local invariance is a property of the class $A(R, S)$ and not of any particular matrix in the class. $B_{22} = 0$ and yet 22 is a locally invariant 1-position of the class $A(R, S)$. An invariant 1-position (0-position) is clearly a locally invariant 1-position (locally invariant 0-position). It is possible, however, for a position to be locally invariant yet not invariant for a class $A(R, S)$ as is demonstrated by the case $R = S = (3, 2, 1, 1)$ given above. There are no invariant positions in this class.

Our goal in this paper is to say as much as possible about the set of locally invariant positions of a class $A(R, S)$. Our task is simplified by considering only normalized classes: those in which $r_1 \geq r_2 \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$. We are able to show, in Theorem 2, that for a normalized nonempty class $A(R, S)$ the locally invariant 1-positions are to the left and above some simple rook path through the set of positions. This is a property shared with invariant 1-positions.

Results: We begin with the following very simple lemmas.

Lemma 1. *Let $A(R,S)$ be nonempty with $s_1 \geq s_2 \geq \cdots \geq s_n$. Let i be an integer with $1 \leq i \leq m$. Then there exists $A \in A(R,S)$ such that the ith row of A consists of r_i 1's followed by $n - r_i$ 0's.*

Proof: Let $B \in A(R,S)$. Suppose $B_{ij} = 0$ and $B_{ik} = 1$ for some $1 \leq j < k \leq n$. Monotonicity of the column sum vector S assures the existence of an interchange resulting in a new matrix $B' \in A(R,S)$ with $B'_{ij} = 1$ and $B'_{ik} = 0$ and other entries in row i unchanged. In this way 1's in row i may be moved to the left so long as they are preceded by a 0. The outcome of this recursive procedure is a matrix satisfying the conclusion of this lemma. □

Corollary 1. *Let $A(R,S)$ be nonempty with $r_1 \geq r_2 \geq \ldots r_m$. Let j be an integer with $1 \leq j \leq n$. Then there exists $A \in A(R,S)$ such that the jth column of A consists of s_j 1's above $m - s_j$ 0's.*

Proof: Transpose the matrix and interchange the terms row and column. □

Lemma 2. *Let $A(R,S)$ be a nonempty class of $2 \times n$ $(0,1)$ matrices with $r_1 \geq r_2$. Let $A \in A(R,S)$. Then there exists $A' \in A(R,S)$ such that*

a) *if $A_{2j} = 1$ then $A'_{1j} = 1$, $1 \leq j \leq n$*

b) *if $A_{1j} = 0$ then $A'_{2j} = 0$, $1 \leq j \leq n$*

c) *if $A_{2j} = A_{1j} = 1$ then $A'_{2j} = A'_{1j} = 1$, $1 \leq j \leq n$.*

Proof: For a matrix $B \in A(R,S)$ we define

$$\begin{aligned} B(1) &= \{j \mid B_{1j} = B_{2j} = 1\} \\ B(2) &= \{j \mid B_{1j} = 1, B_{2j} = 0\} \\ B(3) &= \{j \mid B_{1j} = 0, B_{2j} = 1\} \\ B(4) &= \{j \mid B_{1j} = 0, B_{2j} = 0\} \end{aligned} \tag{1}$$

We clearly have $B(1) \cup B(2) \cup B(3) \cup B(4) = \{1, 2, \ldots, n\}$. The left side of this equality suggest a partition of $\{1, 2, \ldots, n\}$ which determines the matrix B. Note that up to three of the $B(i)$ may be empty.

Given $A \in A(R,S)$ we form $A' \in A(R,S)$ by defining

$$\begin{aligned} A'(1) &= A(1) \\ A'(2) &= A(3) \cup A(2) - C) \\ A'(3) &= C \\ A'(4) &= A(4) \end{aligned} \tag{2}$$

where C is any subset of $A(2)$ with size $|C| = |A(3)|$. The condition $r_1 \geq r_2$ implies $|A(2)| \geq |A(3)|$ which makes selection of C possible. From (2) it is easy to verify that $A' \in A(R,S)$. Note that $A_{2j} = 1$ implies $j \in (A(1) \cup A(3)) \subseteq (A'(1) \cup A'(2))$. Thus $A'_{1j} = 1$ establishing condition a). Also if $A_{1j} = 0$ then $j \in (A(3) \cup A(4)) \subseteq (A'(2) \cup A'(4))$. Therefore $A_{2j} = 0$ establishing condition b). Condition c) is obvious. □

Theorem 1. *Let $A(R,S)$ be normalized and nonempty. The position ij is a locally invariant 1-position of $A(R,S)$ if and only if there exists a matrix $A \in A(R,S)$ of the form*

$$A = \begin{array}{c} \\ i \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & j & & \end{array} \\ \left[\begin{array}{c|c|c|c} * & U & * & * \\ \hline L & 1 & M & W \\ \hline * & V & * & * \\ \hline * & X & * & Y \end{array}\right] \end{array} \tag{3}$$

where L and M are row matrices with all entries equal to 1; U and V are column matrices of 1's; W, X and Y are zero matrices.

We allow the possibility that one or more of the indicated sub-matrices may be trivial.

Proof: If there is a matrix $A \in A(R,S)$ of the form (3) then ij is a locally invariant 1-position of $A(R,S)$. Let ij be a locally invariant 1-position of $A(R,S)$ and let $A \in A(R,S)$ satisfy properties i) and ii). Then if $A_{iw} = 0$ for some $1 \leq w < j$ then monotonicity of the column sum vector S assures the existence of an interchange which places a 0 in the ij position.

This contradicts the fact that A has property ii). Thus $A_{iw} = 1$ for $1 \leq w \leq j$. We may conclude by similar reasoning that $A_{vj} = 1$ for $1 \leq v \leq i$. We may now choose the 0's in row i to be located in the rightmost positions. This follows by applying Lemma 1 to the submatrix of A consisting of columns $j+1, j+2, \ldots, n$. Similarly the 0's in column j may be chosen to be in the lowest positions.

Now let an entry of matrix Y be 1. Then a single interchange makes the ij position 0, a contradiction. Thus Y is a zero matrix and A has form (3). □

Theorem 2. *Let $A(R,S)$ be a nonempty normalized class. If ij is a locally invariant 1-position of $A(R,S)$ then so is every position vw satisfying $1 \leq v \leq i$ and $1 \leq w \leq j$.*

Proof: Since ij is a locally invariant 1-position of $A(R,S)$ there exists, by Theorem 1, $A \in A(R,S)$ of form (3). Now fix a number $v, 1 \leq v < i$, and apply Lemma 2 to the $2 \times n$ submatrix of A consisting of rows v and i. We conclude that there exists $A' \in A(R,S)$ of the form (3) with the distinguished 1 in the vj position. The matrix A' shows that the vj position is a locally invariant 1-position of $A(R,S)$. Similarly, using the column version of Lemma 2, we may prove that the positions iw with $1 \leq w \leq j$ are locally invariant 1-positions of $A(R, S)$. The Theorem follows. $\square$

Corollary 2. *Let $A(R, S)$ be a nonempty normalized class. If ij is a locally invariant 0-position of $A(R, S)$ then so is every position vw satisfying $i \leq v \leq m$, $j \leq w \leq n$.*

Proof: Reflect matrices in $A(R, S)$ once about the horizontal axis, once about the vertical axis and interchange zeros and ones. Applying theorem 2 to the resulting class of matrices gives corollary 2. $\square$

Discussion: Theorem 2 shows that for a normalized, nonempty class $A(R, S)$ the set of locally invariant 1-positions occupy all positions above and to the left of some simple rook path throughout the m x n array of positions. For the example given earlier in which $R = S = (3, 2, 1, 1)$ the locally invariant positions are as indicated below.

$$\begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 2 & 1 & 1 \end{array} \\ \begin{bmatrix} * & * & \& & \& \\ * & * & \# & \# \\ \& & \# & \# & \# \\ \& & \# & \# & \# \end{bmatrix} \end{array}$$

* – a locally invariant 1-position
& – not a locally invariant position
\# – a locally invariant 0-position

Ryser has proved the following.

Proposition. (Theorem 5.2 in [2]) *Suppose ij is an invariant 1-position in the normalized, nonempty class $A(R, S)$, then there exists integers e, f with $i \leq e \leq m$ and $j \leq f \leq n$ such that every matrix $A \in A(R, S)$ has the form*

$$A = \begin{pmatrix} J_{ef} & A_1 \\ A_2 & 0 \end{pmatrix} \tag{4}$$

where J_{ef} is the $e \times f$ matrix all of whose entries equal 1, and 0 is the $(m - e) \times (n - f)$ zero matrix. $\square$

A similar result holds for invariant 0-positions. The variability in the class containing matrix A in (5) is found in the positions given by submatrices A_1 and A_2. In a sense the class containing A has split into two smaller classes. From this example we see that the study of local invariance may be restricted to classes without invariant positions.

The $(m+1) \times (n+1)$ matrix T defined by $T_{ij} = ij + (r_{i+1} + r_{i+2} + \ldots r_m) - (s_1 + s_2 + \ldots s_j)$ is called the structure matrix of the class $A(R, S)$. It is well known that matrices in $A(R, S)$ have the form (5) if and only if $T_{ef} = 0$ [2, page 186–7]. Thus the structure matrix prescribes the invariant positions of a class. Is there also a simple relationship between the structure matrix and locally invariant positions?

References

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