A Note on Paths in Edge-Coloured Tournaments

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Let T be a (finite) tournament whose edges are coloured with two colours. In [2] Sands, Sauer and Woodrow proved that there exists a vertex s of T such that there is a monochromatic path from any other vertex of T to s. Shorter proofs of this result were subsequently found by Reid [1] and Shen Minggang [3]. Still open is the following:

Problem 1: [2] *For every $n > 2$, is there a (least) positive integer $f(n)$ so that every tournament whose edges are coloured with n colours contains a set S of at most $f(n)$ vertices with the property that for every vertex v not in S there is a monochromatic path from v to a vertex of S?*

In this note we give an extension of the Sands-Sauer-Woodrow result, in which the edges of T are coloured with the elements of a partially ordered set P. In this case a directed path $v_1 v_2 \ldots v_n$ in T is called *monotone* if $\text{colour}(v_i v_{i+1}) \leq \text{colour}(v_{i+1} v_{i+2})$ in P for each i. Note that monochromatic paths are monotone, and they coincide if P is an antichain.

Let us define the *tournament colouring number*, $tc(P)$, of a poset P to be the smallest positive integer such that, for any edge-colouring of any tournament T by the elements of P, there is a set S of at most $tc(P)$ vertices of T with the property that there is a monotone path from any vertex of T not in S to a vertex of S. The result of [2] then says that $tc(P) = 1$ when P is a two-element antichain, and Problem 1 above asks whether $tc(P)$ exists for P a finite antichain of more than two elements.

Our main result is a characterization of those finite posets P with tournament colouring number 1.

Theorem 1: *The following are equivalent for a finite poset P:*

(*i*) $tc(P) = 1$;

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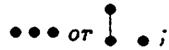
(*ii*) *P does not contain a subset isomorphic to*  *;*

(*iii*) *P is a linear sum of* 1- *and* 2-*element antichains.*

Here the *linear sum* of two disjoint posets P_1 and P_2 is the poset with elements $P_1 \cup P_2$ and order relations the union of the order relations of P_1 and P_2 together with $x_1 \leq x_2$ for all $x_1 \in P_1$, $x_2 \in P_2$. The linear sum of n posets is then defined inductively.

Actually, [2] contains the following stronger theorem which we will need for the proof of Theorem 1.

Theorem 2: (Sands, Sauer, Woodrow [2]) *Let D be a finite directed graph whose edges are coloured with two colours. Then there is a set S of vertices of D satisfying*

(a) *for every vertex v of $D-S$ there is a monochromatic directed path from v to a vertex of S;*

(b) *there is no monochromatic directed path in D between any two vertices of S.* □

If D is a tournament, it is obvious that S must consist of a single vertex.

Proof of Theorem 1.
$(i) \Rightarrow (ii)$. If P contains a copy of either poset of Figures 1a and 1b, we let T be the tournament of 3 vertices shown in Figure 1c, coloured as illustrated.

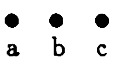


Figure 1a

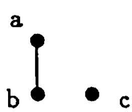


Figure 1b

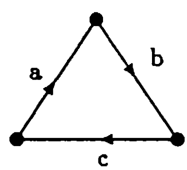
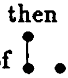


Figure 1c

Then T shows that $tc(P) \geq 2$.
$(ii) \Rightarrow (iii)$. Let M be the set of minimal elements of P. By (ii), $|M| \leq 2$. If $|M| = 1$, then P is the linear sum of M and $P-M$; since $P-M$ obviously satisfies (ii), (iii) follows for P by induction. If $|M| = 2$, then P is again the linear sum of M and $P - M$, else P contains a copy of  ; and now we use induction as before.
$(iii) \Rightarrow (i)$. This is by induction on $|P|$. Letting M again be the set of minimals of P, we have by (iii) that $|M| \leq 2$ and that P is the linear sum of M and $P - M$. Let T be a tournament and colour its edges with the elements of P. Let D be the digraph with all the vertices of T and with

edges only those edges of T coloured by elements of M. Since $|M| \leq 2$, by Theorem 2 we can find a set S of vertices of T such that

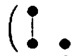
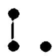
(a) for any vertex v of $D - S$ there is a monochromatic directed path (coloured by an element of M) from v to a vertex of S, and

(b) no two vertices of S are connected by a monochromatic directed path (coloured by an element of M) in D.

Now consider S as a subtournament of T. Its edges will all be coloured by elements of $P - M$, by (b); thus by induction there is a vertex s of S such that there is a monotone directed path from any other vertex of S to s. But now, given any vertex v of $T - \{s\}$, by (a) there is a path (coloured by an element of M) from v to a vertex $w \in S$, and combining this with a monotone path (coloured by elements of $P - M$) from w to s yields a monotone path from v to s. Thus the vertex s demonstrates that $tc(P) = 1$. □

The following 9-vertex tournament, given in [2], shows that $tc(P) \geq 3$, where P is the poset of Figure 1b:

vertices:	$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$	
directed edges:	$(a_1, a_2), (b_1, b_2), (c_1, c_2)$	coloured c
	$(a_2, a_3), (b_2, b_3), (c_2, c_3)$	coloured b
	$(a_3, a_1), (b_3, b_1), (c_3, c_1)$	coloured a
	(a_i, b_j)	coloured c for all i, j
	(b_i, c_j)	coloured b for all i, j
	(c_i, a_j)	coloured a for all i, j.

Problem 2: *Does tc* () *exist? Does tc* () $= 3$*?*
Incidentally, Theorem 1 and the above example show that there is no poset P for which $tc(P) = 2$.

We close with a further extension. One could replace the poset P by a directed graph D (with a loop at each vertex), colour the edges of a tournament T by the vertices of D, and instead of monotone paths consider "D-paths" in T, i.e. paths $v_1 v_2 \ldots v_n$ satisfying (colour$(v_i v_{i+1})$, colour $(v_{i+1} v_{i+2})$) is an edge or loop of D for all i. Here colour changes on the path are only permitted if the vertices of D corresponding to these colours are adjacent. The tournament colouring number $tc(D)$ of D could then be defined analogously as before. We do not know which digraphs D satisfy $tc(D) = 1$. We do not even know which *graphs* G satisfy $tc(G) = 1$, where the edges of G are taken to be directed both ways. Note, however, that (letting C_n be the n-vertex cycle) we have: $tc(C_3) = tc(C_4) = 1$, because

$tc($  $) = tc($  $) = 1$ by Theorem 1; and $tc(D) \geq 3$ whenever D contains a 3-vertex independent set, because $tc(\bullet\bullet\bullet) \geq 3$, thus $tc(C_n) \geq 3$ for $n \geq 6$.

Problem 3: *Does* $tc(C_5) = 1$ *?*

References

[1] K. B. Reid. Monochromatic reachability, complementary cycles, and single arc reversals in tournaments, *in* Graph Theory Singapore 1983, Lecture Notes in Mathematics 1073, Springer Verlag, (1984) 11-21.

[2] B. Sands, N. Sauer, and R. Woodrow. On monochromatic paths in edge coloured digraphs, *J. Combinatorial Theory Ser. B.* **33** (1982) 271-275.

[3] Shen Minggang. On monochromatic paths in m-coloured tournaments, *J. Combinatorial Theory Ser. B.* **45** (1988) 108-111.

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