Directed Triple Systems with a Class of Automorphisms*

Zhike Jiang and Mary McLeish
Department of Computing and Information Science
University of Guelph
Guelph, Ontario
Canada N1G 2W1

ABSTRACT. A directed triple system of order v, denoted by $DTS(v)$, is called (f,k)-rotational if it has an automorphism consisting of f fixed points and k cycles each of length $(v-f)/k$. In this paper, we obtain a necessary and sufficient condition for the existence of (f,k)-rotational $DTS(v)$ for any arbitrary positive integer k.

1 Introduction

A *directed triple system* of order v, denoted by $DTS(v)$, is an ordered pair $(V,\mathcal{B})$, where V is a set of v elements and $\mathcal{B}$ is a set of transitive triples, called *blocks* briefly, such that each ordered pair of elements from V is contained in exactly one block in $\mathcal{B}$. We use $[a,b,c]$ to denote the block containing the three ordered pairs $(a,b),(b,c)$ and (a,c). Hung and Mendelsohn [6] introduced directed triple systems as a generalization of Steiner triple systems; they showed that a $DTS(v)$ exists if and only if $v \equiv 0,1 \pmod 3$. A *subsystem* of order w of a $DTS(v)$ $(V,\mathcal{B})$ is a $DTS(w)$ $(W,\mathcal{A})$ such that $W \subseteq V$ and $\mathcal{A} \subseteq \mathcal{B}$. It is known (c.f. [3, 10]) that a $DTS(v)$ with a subsystem $DTS(w)$ exists if and only if $v,w \equiv 0,1 \pmod 3$ and $v=w$ or $v \geq 2w+1$.

An *automorphism* of a $DTS(v)$ $(V,\mathcal{B})$ is a permutation on V which preserves $\mathcal{B}$. An automorphism π of a $DTS(v)$ $(V,\mathcal{B})$ induces a partition of $\mathcal{B}$ into equivalent classes, which are called *orbits* of π, such that two blocks T_1,T_2 are in the same orbit if and only if $\pi^{\alpha}(T_1)=T_2$ for some integer

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α. A representative block of an orbit is called a *base block*. A $DTS(v)$ is (f,k)-*rotational* if it admits an automorphism which consists of f fixed points and k cycles of the same length. It is clear that the set of fixed points of an (f,k)-rotational $DTS(v)$ forms a subsystem. Several problems on directed triple systems which have been explored in the literature turn out to be special cases of existence of (f,k)-rotational $DTS(v)$. A cyclic $DTS(v)$ is one which is $(0,1)$-rotational and it is shown in [4] that a cyclic $DTS(v)$ exists if and only if $v \equiv 1,4,7 \pmod{12}$. For a positive integer k, a $(1,k)$-rotational $DTS(v)$ is called a k-rotational $DTS(v)$. The existence of a k-rotational $DTS(v)$ is determined by Cho, Chae and Hwang [2] and this result is quoted in the following lemma.

Lemma 1.1 *A k-rotational $DTS(v)$ exists if and only if $kv \equiv 0 \pmod 3$ and $v \equiv 1 \pmod k$.*

Gardner [5] studies k-near-rotational $DTS(v)$, a $DTS(v)$ admitting a $(3,k)$-rotational automorphism, and proves

Lemma 1.2 *A k-near-rotational $DTS(v)$ exists if and only if $k(v+2) \equiv 0 \pmod 3$, $v \equiv 3 \pmod k$ and $v \equiv 0,1 \pmod 3$.*

Micale and Pennisi [7] fix k to be one and consider exhaustively all the possible values of f such that there is an $(f,1)$-rotational $DTS(v)$ (such a system is defined to be $(v-f)$-cyclic because the only nontrivial cycle of the automorphism has length $v-f$). In the following is the result they obtain:

Lemma 1.3 *A $(v-f)$-cyclic $DTS(v)$ exists if and only if $v \geq 2f+1$ and, further, $v \equiv 0 \pmod 3$ and $f \equiv 1 \pmod 3$ or $v \equiv 1 \pmod 3$ and $f \equiv 0 \pmod 3$.*

In this paper, we consider (f,k)-rotational directed triple systems for arbitrary values of $f \geq 1$ and $k \geq 1$, and prove a necessary and sufficient condition for the existence of such systems which generalizes Lemmas 1.1, 1.2 and 1.3.

2 $(f,3)$-rotational $DTS(v)$

In this section, we restrict the value of k to be 3 and give constructions for the systems in this special case. As Lemmas 1.1 and 1.2 have already dealt with the cases $f=1$ and $f=3$, we also restrict $f \geq 4$.

We use difference methods. Let $V = \{\infty_1, \ldots, \infty_f\} \bigcup (Z_n \times \{1,2,3\})$, $\pi = (\infty_1)\ldots(\infty_f)(0_1 \ldots (n-1)_1)(0_2 \ldots (n-1)_2)(0_3 \ldots (n-1)_3)$, where $v = 3n+f$. Then the existence of an $(f,3)$-rotational $DTS(v)$ on V with π

as an automorphism is equivalent to that of a set of base blocks which cover each of the pure and mixed differences in $Z_n \times \{1,2,3\}$ exactly once. (For the definitions of pure and mixed differences, and the standard representation of designs using difference methods, the reader is referred to [1].) In all the relevant cases of n and f, we prove the existence of an $(f,3)$-rotational $DTS(3n+f)$ on set V with π as an automorphism by constructing its base blocks. The construction in each case starts with exhibiting the base blocks of a $(4,3)$-rotational $DTS(v-(f-4))$ on a subset V_1 of V with $\pi|_{V_1}$ as an automorphism, where $V_1 = V \setminus \{\infty_5, \dots, \infty_f\}$. Then the base blocks of this $(4,3)$-rotational $DTS(v-(f-4))$ are modified, with $f-4$ new fixed points being introduced, to yield those for an $(f,3)$-rotational $DTS(v)$ on V with π as an automorphism. A $(4,3)$-rotational $DTS(v-(f-4))$ is *extendible* (to an $(f,3)$-rotational $DTS(v)$) if its base blocks can be modified for this purpose.

The base blocks of any $(4,3)$-rotational $DTS(3n+4)$ are of the following types:

1. $[\infty_i, \infty_j, \infty_m]$, where i,j,m are distinct and $i,j,m \in \{1,2,3,4\}$;
2. $[x_i, \infty_m, y_i]$, where $m \in \{1,2,3,4\}$, $x_i, y_i \in Z_n \times \{1,2,3\}$;
3. $[\infty_m, x_i, y_j]$, $[x_i, \infty_m, y_j]$, $[x_i, y_j, \infty_m]$, where $m \in \{1,2,3,4\}$, $i \neq j$, and $x_i, y_j \in Z_n \times \{1,2,3\}$;
4. $[x_i, y_j, z_m]$, where $x_i, y_j, z_m \in Z_n \times \{1,2,3\}$.

Each base block of the second type covers one pure difference, each base block of the third type covers one mixed difference, and each base block of the fourth type covers three differences, either pure or mixed.

Suppose that we have the base blocks of a $(4,3)$-rotational $DTS(v-(f-4))$. Then the blocks of the first type (which form a subsystem) clearly can be replaced by blocks of a $DTS(f)$ on $\{\infty_1, \dots, \infty_f\}$, whenever $f \equiv 0, 1 \pmod 3$, to deal with the appearance of pairs of the fixed points. When a new fixed point is introduced, its appearance with each non-fixed point in the system has to be guaranteed. This can be done by modifying some of the base blocks of the fourth type in the following ways. If we have a triplet of base blocks,

$$[x_1, y_1, z_1], \quad [a_2, b_2, c_2], \quad [p_3, q_3, r_3], \tag{1}$$

in the $DTS(v-(f-4))$, then we can replace them by nine new blocks of the second type, say,

$$\begin{array}{lll} [x_1, \infty_i, y_1], & [a_2, \infty_i, b_2], & [p_3, \infty_i, q_3], \\ [x_1, \infty_j, z_1], & [a_2, \infty_j, c_2], & [p_3, \infty_j, r_3], \\ [y_1, \infty_m, z_1], & [b_2, \infty_m, c_2], & [q_3, \infty_m, r_3], \end{array}$$

where $\infty_i, \infty_j, \infty_m \in \{\infty_5, \ldots, \infty_f\}$. The result of this procedure is that the differences covered by the original three base blocks are still covered by the new blocks while three new fixed points are introduced to the system (and their appearance with each non-fixed point is guaranteed by these base blocks). Similarly, if we have a base block

$$[x_i, y_j, z_m], \tag{2}$$

where (i, j, k) is a permutation of $(1, 2, 3)$, we can replace it by three new blocks

$$[\infty_\ell, x_i, y_j], \quad [y_j, \infty_\ell, z_m], \quad [x_i, z_m, \infty_\ell],$$

where $\infty_\ell \in \{\infty_5, \ldots, \infty_f\}$, which cover exactly the same three differences as the original one does while a new fixed point is introduced.

Lemma 2.1 *If a (4, 3)-rotational $DTS(v-(f-4))$ on V_1 with $\pi|_{V_1}$ as an automorphism has t triplets of base blocks of form (1) and s base blocks of form (2) such that $3t + s \geq f - 4$, and if $s \geq 2$ when $f - 4 \equiv 2 \pmod 3$, then it is extendible to an $(f, 3)$-rotational $DTS(v)$ on V with π as an automorphism whenever $f \equiv 0, 1 \pmod 4$ and $f \geq 4$.*

Proof. By modifying, in the ways we described, a triplet of base blocks of form (1) of the $DTS(v-(f-4))$, we can introduce three new fixed points. Similarly, we can introduce one new fixed point with each base block of form (2). If $3t + s \geq f - 4$, and if $s \geq 2$ when $f - 4 \equiv 2 \pmod 3$, it is always possible to choose t_1 ($\leq t$) triplets of base blocks of form (1) and s_1 ($\leq s$) base blocks of form (2) such that $3t_1 + s_1 = f - 4$. Now modify these base blocks so that the new fixed points $\infty_5, \infty_6, \ldots, \infty_f$ are introduced, and replace the base blocks of the $DTS(v-(f-4))$ involved in only the fixed points $\infty_1, \ldots, \infty_4$ by the blocks of a $DTS(f)$ on $\{\infty_1, \infty_2, \ldots, \infty_f\}$. This results in the set of base blocks of an $(f, 3)$-rotational $DTS(v)$ on V. ⟋

We now give constructions for $(f, 3)$-rotational $DTS(v)$ in several cases through the following lemmas. For this, the following structures are required. An *(A, k)-system* (or a *(B, k)-system*, respectively) is a partition of the set $\{1, 2, \ldots, 2k\}$ (or $\{1, 2, \ldots, 2k+1\} \setminus \{2k\}$, respectively) into k ordered pairs (a_r, b_r), $r = 1, 2, \ldots, k$, such that $b_r - a_r = r$ for each r. It is proved in [8, 9] that an (A, k)-system exists if and only if $k \equiv 0, 1 \pmod 4$ and a (B, k)-system exists if and only if $k \equiv 2, 3 \pmod 4$.

Lemma 2.2 *If $n \equiv 0 \pmod 6$, $f \equiv 0, 1 \pmod 3$ and $4 \leq f \leq 3n - 1$, then there is an $(f, 3)$-rotational $DTS(3n + f)$.*

Proof. Let $n = 6t$ for $t \geq 1$. The set of the base blocks of a (4, 3)-rotational $DTS(3n + 4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1, \dots, \infty_4\}$,

$$(\text{ii})\quad \begin{array}{ll} [0_1, r_2, (2r)_3] & r = 0, 1, 2, \dots, 3t-1, \\ [0_1, r_2, (2r-1)_3] & r = 3t+1, 3t+2, \dots, 6t-1, \\ [0_3, r_2, (2r)_1] & r = 0, 1, 2, \dots, 3t-1, \\ [0_3, r_2, (2r-1)_1] & r = 3t+1, 3t+2, \dots, 6t-1, \end{array}$$

(iii) $[0_i, r_i, ((2t-1)+b_r)_i]$ $r = 1, 2, \dots, 2t-1$, and $i = 1, 2, 3$,
where (a_r, b_r) $(r = 1, 2, \dots, 2t-1)$ is an $(A, 2t-1)$-system when $2t-1 \equiv 1 \pmod 4$ or a $(B, 2t-1)$-system when $2t-1 \equiv 3 \pmod 4$,

$$(\text{iv})\quad \begin{array}{lll} [\infty_1, 0_1, (3t)_2], & [0_1, \infty_1, (6t-1)_3], & [0_2, (6t-1)_3, \infty_1], \\ [\infty_2, 0_3, (3t)_2], & [0_3, \infty_2, (6t-1)_1], & [0_2, (6t-1)_1, \infty_2], \end{array}$$

(v) if $2t-1 \equiv 1 \pmod 4$,

$$\begin{array}{ll} [0_i, \infty_3, (6t-2)_i] & i = 1, 2, 3, \\ [0_i, \infty_4, (6t-1)_i] & i = 1, 2, 3, \end{array}$$

or, if $2t-1 \equiv 3 \pmod 4$,

$$\begin{array}{ll} [0_i, \infty_3, (6t-3)_i] & i = 1, 2, 3, \\ [0_i, \infty_4, (6t-1)_i] & i = 1, 2, 3. \end{array}$$

There are $2t-1$ triplets of blocks in (iii), which are of form (1), and $12t-2$ blocks in (ii), which are of form (2). Note that $3(2t-1)+12t-2 = 18t-5 \geq f-4$. Therefore this system is extendible by Lemma 2.1.

Lemma 2.3 *If* $n \equiv 2 \pmod 6$, $f \equiv 0, 1 \pmod 3$ *and* $4 \leq f \leq 3n-1$, *then there is an* $(f, 3)$*-rotational* $DTS(3n+f)$.

Proof. Let $n = 6t+2$ for $t \geq 0$. The set of base blocks of $(4, 3)$-rotational $DTS(3n+4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1, \dots, \infty_4\}$,

$$(\text{ii})\quad \begin{array}{ll} [0_1, r_2, (2r)_3] & r = 0, 1, 2, \dots, 3t, \\ [0_1, r_2, (2r-1)_3] & r = 3t+2, 3t+3, \dots, 6t+1 \text{ (if } t > 0), \\ [0_3, r_2, (2r)_1] & r = 1, 2, 3, \dots, 3t \text{ (if } t > 0), \\ [0_3, r_2, (2r-1)_1] & r = 3t+2, 3t+3, \dots, 6t+1 \text{ (if } t > 0), \end{array}$$

(iii) $[0_i, r_i, (2t+b_r)_i]$ $r = 1, 2, \dots, 2t$ (if $t > 0$), and $i = 1, 2, 3$,
where (a_r, b_r) $(r = 1, 2, \dots, 2t)$ is an $(A, 2t)$-system when $2t \equiv 0 \pmod 4$ or a $(B, 2t)$-system when $2t \equiv 2 \pmod 4$,

$$(\text{iv})\quad \begin{array}{lll} [\infty_1, 0_1, (3t+1)_2], & [0_1, \infty_1, (6t+1)_3], & [0_2, (6t+1)_3, \infty_1], \\ [\infty_2, 0_3, (3t+1)_2], & [0_3, \infty_2, (6t+1)_1], & [0_2, (6t+1)_1, \infty_2], \\ [\infty_3, 0_3, 0_2], & [0_3, \infty_3, 0_1], & [0_2, 0_1, \infty_3], \end{array}$$

(v) if $2t \equiv 0 \pmod 4$,

$[0_i, \infty_4, (6t+1)_i] \quad i = 1, 2, 3,$

or, if $2t \equiv 2 \pmod 4$,

$[0_i, \infty_4, (6t)_i] \quad i = 1, 2, 3.$

There are $2t$ triplets of blocks in (iii), which are of form (1), and $12t+1$ base blocks in (ii), which are of form (2). Note that $3(2t) + 12t + 1 = 18t + 1 \geq f - 4$. Therefore this system is extendible by Lemma 2.1. ✓

Lemma 2.4 *If* $n \equiv 4 \pmod 6$, $f \equiv 0, 1 \pmod 3$ *and* $4 \leq f \leq 3n-1$, *then there is an* $(f, 3)$*-rotational* $DTS(3n+f)$.

Proof. Let $n = 6t+4$ for $t \geq 0$. The set of base blocks of a $(4,3)$-rotational $DTS(3n+4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1, \ldots, \infty_4\}$,

(ii)
$[0_1, r_2, (2r)_3] \quad r = 1, 2, \ldots, 3t+1,$
$[0_1, r_2, (2r-1)_3] \quad r = 3t+3, 3t+4, \ldots, 6t+3,$
$[0_3, r_2, (2r)_1] \quad r = 1, 2, \ldots, 3t+1,$
$[0_3, r_2, (2r-1)_1] \quad r = 3t+3, 3t+4, \ldots, 6t+3,$

(iii) $[0_i, r_i, (2t+b_r)_i] \quad r = 1, 2, \ldots, 2t$ (if > 0), and $i = 1, 2, 3$,

where (a_r, b_r) $(r = 1, 2, \ldots, 2t)$ is an $(A, 2t)$-system when $2t \equiv 0 \pmod 4$ or a $(B, 2t)$-system when $2t \equiv 2 \pmod 4$,

(iv) $[0_2, 0_1, (6t+3)_1]$, $[0_2, (6t+3)_2, (6t+3)_3]$, $[0_3, (6t+3)_3, (6t+3)_1]$,

(v)
$[\infty_1, 0_1, (6t+3)_3]$, $[0_3, \infty_1, (3t+2)_2]$, $[0_1, (3t+2)_2, \infty_1]$,
$[\infty_2, 0_1, 0_3]$, $[0_3, \infty_2, 0_2]$, $[0_1, 0_2, \infty_2]$,

(vi) $[0_i, \infty_3, (6t+2)_i] \quad i = 1, 2, 3,$

and, if $2t \equiv 0 \pmod 4$,

$[0_i, \infty_4, (6t+1)_i] \quad i = 1, 2, 3,$

or, if $2t \equiv 2 \pmod 4$,

$[0_i, \infty_4, (6t)_i] \quad i = 1, 2, 3.$

This system is extendible. First note that, if necessary, the three blocks in (iv) can be modified to

$[\infty_p, 0_2, 0_3]$, $[0_3, \infty_p, 0_1]$, $[0_2, 0_1, \infty_p]$,
$[\infty_q, 0_2, (6t+3)_3]$, $[0_3, \infty_q, (6t+3)_1]$, $[0_2, (6t+3)_1, \infty_q]$,
$[0_i, \infty_m, (6t+3)_i] \quad i = 1, 2, 3,$

so that three new fixed points are introduced into the system. It remains to show that we can introduce $f-7$ new fixed points into the system. However, this is clear since there are $2t$ triplets of blocks in (iii), which are of form (1), and $12t+4$ blocks in (ii), which are of form (2), and $3(2t)+12t+4=18t+4\geq f-7$. ⁄

Lemma 2.5 *If* $n\equiv 1 \pmod 6$, $f\equiv 0,1 \pmod 3$ *and* $4\leq f\leq 3n-1$, *then there is an* $(f,3)$*-rotational* $DTS(3n+f)$.

Proof. Let $n=6t+1$ for $t\geq 1$. The set of base blocks of a $(4,3)$-rotational $DTS(3n+4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1,\ldots,\infty_4\}$,

(ii) $[0_1, r_2, (2r)_3]$ $r=1,2,\ldots,6t$,
$[0_3, r_2, (2r)_1]$ $r=0,1,2,\ldots,6t$,

(iii) $[0_i, r_i, (2t-1+b_r)_i]$ $r=1,2,\ldots,2t-1$, and $i=1,2,3$,
where (a_r,b_r) $(r=1,2,\ldots,2t-1)$ is an $(A,2t-1)$-system when $2t-1\equiv 1 \pmod 4$ or a $(B,2t-1)$-system when $2t-1\equiv 3 \pmod 4$,

(iv) $[\infty_1, 0_1, 0_2]$, $[0_2, \infty_1, 0_3]$, $[0_1, 0_3, \infty_1]$,

(v) $[0_i, \infty_2, (6t)_i]$ $i=1,2,3$,
$[0_i, \infty_3, (6t-1)_i]$ $i=1,2,3$,
and, if $2t-1\equiv 1 \pmod 4$,
$[0_i, \infty_4, (6t-2)_i]$ $i=1,2,3$,
or, if $2t-1\equiv 3 \pmod 4$,
$[0_i, \infty_4, (6t-3)_i]$ $i=1,2,3$.

There are $2t-1$ triplets of blocks of form (1) in (iii) and $12t+1$ blocks of form (2) in (ii). Note that $3(2t-1)+12t+1=18t-2\geq f-4$. Therefore this system is extendible by Lemma 2.1. ⁄

Lemma 2.6 *If* $n\equiv 3 \pmod 6$, $f\equiv 0,1 \pmod 3$ *and* $4\leq f\leq 3n-1$, *then there is an* $(f,3)$*-rotational* $DTS(3n+f)$.

Proof. Let $n=6t+3$ for $t\geq 0$. The set of base blocks of a $(4,3)$-rotational $DTS(3n+4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1,\ldots,\infty_4\}$,

(ii) $[0_1, r_2, (2r)_3]$ $r=1,2,\ldots,6t+2$,
$[0_3, r_2, (2r)_1]$ $r=1,2,\ldots,6t+2$,

(iii) $[0_i, r_i, (2t+b_r)_i]$ $r = 1, 2, \ldots, 2t$ (if $t > 0$), and $i = 1, 2, 3$,
where (a_r, b_r) $(r = 1, 2, \ldots, 2t)$ is an $(A, 2t)$-system when $2t \equiv 0 \pmod 4$ or a $(B, 2t)$-system when $2t \equiv 2 \pmod 4$,

(iv) $[\infty_1, 0_1, 0_2]$, $[0_2, \infty_1, 0_3]$, $[0_1, 0_3, \infty_1]$,
$[\infty_2, 0_3, 0_2]$, $[0_2, \infty_2, 0_1]$, $[0_3, 0_1, \infty_2]$,

(v) $[0_i, \infty_3, (6t+2)_i]$ $i = 1, 2, 3$,
and, if $2t \equiv 0 \pmod 4$,
$[0_i, \infty_4, (6t+1)_i]$ $i = 1, 2, 3$,
or, if $2t \equiv 2 \pmod 4$,
$[0_i, \infty_4, (6t)_i]$ $i = 1, 2, 3$.

The extendibility is established by applying Lemma 2.1 since there are $2t$ triplets of blocks of form (1) in (iii) and $12t + 4$ blocks of form (2) in (ii) and $3(2t) + 12t + 4 = 18t + 4 \geq f - 4$. ✓

Lemma 2.7 *If* $n \equiv 5 \pmod 6$, $f \equiv 0, 1 \pmod 3$ *and* $4 \leq f \leq 3n - 1$, *then there is an* $(f, 3)$*-rotational* $DTS(3n + f)$.

Proof. Let $n = 6t + 5$ for $t \geq 0$. The set of base blocks of a $(4, 3)$-rotational $DTS(3n + 4)$ consists of

(i) the blocks of a $DTS(4)$ on $\{\infty_1, \ldots, \infty_4\}$,

(ii) $[0_1, r_2, (2r)_3]$ $r = 2, 3, \ldots, 6t + 4$,
$[0_3, r_2, (2r)_1]$ $r = 1, 2, \ldots, 6t + 4$,

(iii) $[0_i, r_i, (2t + 1 + b_r)_i]$ $r = 1, 2, \ldots, 2t + 1$, and $i = 1, 2, 3$,
where (a_r, b_r) $(r = 1, 2, \ldots, 2t + 1)$ is an $(A, 2t + 1)$-system when $2t + 1 \equiv 1 \pmod 4$ or a $(B, 2t + 1)$-system when $2t + 1 \equiv 3 \pmod 4$,

(iv) $[\infty_1, 0_1, 0_2]$, $[0_2, \infty_1, 0_3]$, $[0_1, 0_3, \infty_1]$,
$[\infty_2, 0_1, 1_2]$, $[0_2, \infty_2, 1_3]$, $[0_1, 2_3, \infty_2]$,
$[\infty_3, 0_3, 0_2]$, $[0_2, \infty_3, 0_1]$, $[0_3, 0_1, \infty_3]$,

(v) if $2t + 1 \equiv 1 \pmod 4$,
$[0_i, \infty_4, (6t + 4)_i]$ $i = 1, 2, 3$,
or, if $2t + 1 \equiv 3 \pmod 4$,
$[0_i, \infty_4, (6t + 3)_i]$ $i = 1, 2, 3$.

The extendibility of this system is established by applying Lemma 2.1 since there are $2t + 1$ triplets of blocks of form (1) in (iii) and $12t + 7$ blocks of form (2) in (ii), and $3(2t + 1) + 12t + 7 = 18t + 10 \geq f - 4$. ✓

Theorem 2.8 *There exists an $(f,3)$-rotational $DTS(v)$ whenever (i) $v \equiv 0 \pmod 3$ and $f \equiv 0 \pmod 3$, or $v \equiv 1 \pmod 3$ and $f \equiv 1 \pmod 3$, and (ii) $f \geq 4$ and $v \geq 2f+1$.*

Proof. Take $n = (v-f)/3$. Under the assumptions, we have $n \equiv 0,1,2,3,4$ or $5 \pmod 6$ and $4 \leq f \leq 3n-1$. The result is then established by applying Lemmas 2.2 - 2.7. ⁄

3 (f,k)-rotational $DTS(v)$

We first look at the necessary condition for the existence of (f,k)-rotational $DTS(v)$s. Obviously, if such a system exists, then $v \equiv 0,1 \pmod 3$ and $v - f \equiv 0 \pmod k$. Since the set of fixed points forms a subsystem it also follows that $f \equiv 0,1 \pmod 3$ and $v = f$ or $v \geq 2f+1$. Furthermore, suppose that $v \neq f$ and that there exists such a system on $V = \{\infty_1,\dots,\infty_f\} \bigcup (Z_n \times \{1,\dots,k\})$ with $\pi = (\infty_1)\dots(\infty_f)(0_1\dots(n-1)_1)\dots(0_k\dots(n-1)_k)$ as an automorphism, where $v = kn+f$. Then each of its base blocks is one of the following types:

1. $[\infty_i,\infty_j,\infty_m]$, where i,j,m are distinct and $i,j,m \in \{1,\dots,f\}$;
2. $[x_i,\infty_m,y_i]$, where $m \in \{1,\dots,f\}$, $x_i,y_i \in Z_n \times \{1,\dots,k\}$;
3. $[\infty_m,x_i,y_j]$, $[x_i,\infty_m,y_j]$, $[x_i,y_j,\infty_m]$, where $m \in \{1,\dots,f\}$, $i \neq j$, and $x_i,y_j \in Z_n \times \{1,\dots,k\}$;
4. $[x_i,y_j,z_m]$, where $x_i,y_j,z_m \in Z_n \times \{1,\dots,k\}$.

Each base block of the first type induces an orbit of length one, containing the block itself, while each of the other types induces an orbit of length n. There are altogether $v(v-1)/3$ blocks in the $DTS(v)$, and the subsystem on the fixed points contains $f(f-1)/3$ blocks in total. Thus the set of blocks in the $DTS(v)$ which are not in the subsystem must be partitioned into several orbits of length n. This implies that $v(v-1)/3 - f(f-1)/3 \equiv 0 \pmod{(v-f)/k}$, or equivalently $k(v+f-1) \equiv 0 \pmod 3$.

With the above discussion, we have proved the following lemma.

Lemma 3.1 *If an (f,k)-rotational $DTS(v)$ exists, where $v = kn+f$, then the following conditions hold: (i) $v,f \equiv 0,1 \pmod 3$, (ii) $v-f \equiv 0 \pmod k$, (iii) $v = f$ or $v \geq 2f+1$, and (iv) $k(v+f-1) \equiv 0 \pmod 3$.*

These conditions turn out to be also sufficient for the existence of (f,k)-rotational $DTS(v)$. To be precise, we have the following result.

Theorem 3.2 *There exists an (f,k)-rotational $DTS(v)$, where $v = kn+f$, if and only if (i) $v, f \equiv 0, 1 \pmod 3$, (ii) $v - f \equiv 0 \pmod k$, (iii) $v = f$ or $v \geq 2f+1$, and (iv) $k(v+f-1) \equiv 0 \pmod 3$.*

Proof. The necessity follows from Lemma 3.1. For the sufficiency, the result is certainly true in the case $v = f$. Furthermore, it is also true in the cases $f = 1$ and $f = 3$ by Lemma 1.1 and Lemma 1.2, respectively. So we assume $v \geq 2f+1$, $f \geq 4$ and prove in the following the existence of an (f,k)-rotational $DTS(v)$ under these restrictions.

When $k \equiv 1, 2 \pmod 3$, the conditions (i) and (iv) are equivalent to $v \equiv 0 \pmod 3$ when $f \equiv 1 \pmod 3$ or $v \equiv 1 \pmod 3$ when $f \equiv 0 \pmod 3$. This fact, combined with condition (iii), implies the existence of a $(v-f)$-cyclic $DTS(v)$ $(V, \mathcal{B})$ by Lemma 1.3. Suppose π is an automorphism of this system which has f fixed points and a single cycle of length $v-f$. Then π^k is also an automorphism of $(V, \mathcal{B})$. Since π^k consists of f fixed points and k cycles each of length $(v-f)/k$ by condition (ii), $(V, \mathcal{B})$ is (f,k)-rotational.

When $k \equiv 0 \pmod 3$, the condition (ii) implies $v - f \equiv 0 \pmod 3$ and so $v \equiv 0 \pmod 3$ when $f \equiv 0 \pmod 3$ or $v \equiv 1 \pmod 3$ when $f \equiv 1 \pmod 3$. There is an $(f,3)$-rotational $DTS(v)$ in this case by Theorem 2.8. We can prove, with an argument similar to that in the above case, that this $DTS(v)$ is also (f,k)-rotational. ✓

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