Numerical Radius of a Multigraph

I. Andrew Affleck and D. R. Farenick
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2, Canada

ABSTRACT. Motivated by the spectral radius of a graph, we introduce the notion of numerical radius for multigraphs and directed multigraphs, and it is proved that, unlike the spectral radius, the numerical radius is invariant under changes in the orientation of a directed multigraph. An analogue of the Perron-Frobenius theorem is given for the numerical radius of a matrix with nonnegative entries.

A typical numerical value assigned to a finite multigraph is the spectral radius. By definition, the spectral radius of a multigraph is the spectral radius of the adjacency matrix of the graph. There are, however, other numeric quantities associated with finite square matrices, the spectral radius being just one of these. In this note our interest is in the quantity that is known as the numerical radius. Our aim is to assign to each finite multigraph a "numerical radius" and to describe in what sense the numerical radius is an invariant of the graph. Because the spectral radius and the numerical radius coincide for symmetric matrices, and hence for adjancency matrices, the definition of the numerical radius for a multigraph must necessarily involve nonsymmetric matrices.

Some linear algebra will be required. The complex vector space $\mathbb{C}^n$ carries a natural inner product $(\xi, \eta) = \sum_j \xi_j \overline{\eta_j}$ and an induced norm $||\xi|| = (\xi, \xi)^{1/2}$. The standard orthonormal basis vectors of $\mathbb{C}^n$ are denoted by $e_1, \ldots, e_n$ and the set of vectors in $\mathbb{C}^n$ for which each entry is nonegative will be denoted by $(\mathbb{R}^n)_0^+$. Every $T \in \mathcal{M}_n(\mathbb{C})$, an $n \times n$ matrix over $\mathbb{C}$, is a linear transformation on $\mathbb{C}^n$ and induces a quadratic form Q_T on $\mathbb{C}^n$ that is defined by $Q_T(\xi) = (T\xi, \xi)$, for $\xi \in \mathbb{C}^n$. The spectral radius $\rho(T)$ and

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the numerical radius $w(T)$ are

$$\rho(T) = \max\{|\lambda| \ : \ T\eta = \lambda\eta \text{ for some nonzero } \eta \in \mathbb{C}^n\}$$
$$w(T) = \max\{|(T\eta, \eta)| \ : \eta \in \mathbb{C}^n \text{ is such that } ||\eta|| = 1\}.$$

It is known from matrix analysis that the norm $||T||$ of T coincides with $\rho(T^*T)^{1/2}$, where T^* denotes the adjoint of T (the conjugate transpose of T). Furthermore $\rho(T) \leq w(T) \leq ||T||$, and equality holds throughout if T is normal (that is, if T and T^* commute). Denote the canonical matrix units for $\mathcal{M}_n(\mathbb{C})$ by E_{ij}, for $1 \leq i, j \leq n$, and the group of $n \times n$ unitary matrices by $\mathcal{U}_n$. The symmetric group $\mathcal{S}_n$ of permutations of $\{1, \ldots, n\}$ is a subgroup of $\mathcal{U}_n$: each $\sigma \in \mathcal{S}_n$ is represented by P_σ, the unique unitary whose action on the standard basis elements is $P_\sigma e_j = e_{\sigma(j)}$. The quantities $\rho(T)$ and $w(T)$ do not change if $T \in \mathcal{M}_n(\mathbb{C})$ is replaced by UTU^* for some $U \in \mathcal{U}_n$.

Perron-Frobenius theory is concerned with the spectral radius of finite square matrices over $\mathbb{R}_0^+$, the nonnegative real numbers. Before commencing with the study of multigraphs, we will indicate how the Perron-Frobenius theory also provides information about the numerical radius of matrices with nonnegative entries.

The first of the following definitions is classical, having been introduced by Frobenius for the study of spectral properties. The second definition is relevant to the numerical radius.

Definition. Suppose that $T \in \mathcal{M}_n(\mathbb{C})$.

1. T is said to be reducible if there is some $\sigma \in \mathcal{S}_n$ for which $P_\sigma T P_\sigma^t$ admits a block decomposition of the form

   $$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

   where A and C are square matrices. A matrix that is not reducible is called irreducible.

2. T is said to be $*$-reducible if there is some $\sigma \in \mathcal{S}_n$ for which $P_\sigma T P_\sigma^t$ admits a block decomposition of the form

   $$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

   where A and C are square matrices. A matrix that is not $*$-reducible is called $*$-irreducible.

Theorem and Definition: (Perron-Frobenius) If $A \in \mathcal{M}_n(\mathbb{R}_0^+)$, then there exists a unit vector $\xi \in (\mathbb{R}^n)_0^+$ such that $A\xi = \rho(A)\xi$. Futhermore, if

A is irreducible, then $\rho(A) < \rho(A + \epsilon E_{ij})$ for all i, j and all $\epsilon > 0$, and the vector ξ is the only unit vector $\psi \in (\mathbb{R}^n)_0^+$ that satisfies $A\psi = \rho(A)\psi$; in this case, where A is irreducible, each entry of the eigenvector ξ is positive. The spectral radius of A is called the *Perron value* of A; a unit vector $\xi \in (\mathbb{R}^n)_0^+$ satisfying the eigenvalue equation $A\xi = \rho(A)\xi$ is called a *Perron vector* of A and is unique if A is irreducible.

Our algebraic result, a version of the Perron-Frobenius theorem for the numerical radius of a matrix with nonnegative entries, is quite elementary, and although we could find no single reference for the result, it is no doubt well-known to many linear algebraists. (For example, the first portion of the theorem can be found either implicitly or explicitly in [2] and [3].) It seems that our focus on $*$-irreducibilty and, subsequently, graph theoretic interpretations is, however, new.

A Perron-Frobenius Theorem: If $A \in \mathcal{M}_n(\mathbb{R}_0^+)$, then there exists a unit vector $\xi \in (\mathbb{R}^n)_0^+$ such that $(A\xi, \xi) = w(A)$. In fact, a unit vector $\psi \in (\mathbb{R}^n)_0^+$ satisfies $(A\psi, \psi) = w(A)$ if and only if ψ is a Perron vector of $A^H = (1/2)(A + A^t)$ (the hermitian part of A). Furthermore, if A is $*$-irreducible, then $w(A) < w(A + \epsilon E_{ij})$ for all i, j and all $\epsilon > 0$, and the vector ξ is the only unit vector $\psi \in (\mathbb{R}^n)_0^+$ that satisfies $(A\psi, \psi) = w(A)$; in this case, where A is $*$-irreducible, each entry of ξ is positive. A unit vector $\psi \in (\mathbb{R}^n)_0^+$ for which $(A\psi, \psi) = w(A)$ will be called a *Perron w-vector* of A and is unique if A is $*$-irreducible.

Proof: To prove that the numerical radius of A is achieved at some unit vector $\xi \in (\mathbb{R}^n)_0^+$, it suffices to show that for every unit vector $\eta \in \mathbb{C}^n$ one can find a $\xi \in (\mathbb{R}^n)_0^+$ such that $|(A\eta, \eta)| \le (A\xi, \xi)$. Given such a vector η, simply let $\xi_j = |\eta_j|$ for each j and compute:

$$|(A\eta, \eta)| \le \sum_{i,j} a_{ij} |\eta_j|\, |\eta_i| = \sum_{i,j} a_{i,j}\xi_j\xi_i = (A\xi, \xi) \ .$$

We now proceed to explain the connection with the Perron vector of $A^H = (1/2)(A + A^t)$. As is shown above, to maximize the modulus of the quadratic form $Q_A : \mathbb{C}^n \to \mathbb{C}$ it suffices to restrict Q_A to $\mathbb{R}^n$ — in fact, to $(\mathbb{R}^n)_0^+$ — and to then maximize Q_A over real unit vectors. This allows us to symmetrize Q_A in the usual way. That is, consider Q_{A^H} instead of Q_A. For every $\xi \in (\mathbb{R}^n)_0^+$,

$$Q_{A^H}(\xi) = \sum_{i,j} (1/2)(a_{ij} + a_{ji})\xi_j\xi_i = \sum_{i,j} a_{ij}\xi_j\xi_i = Q_A(\xi) \ .$$

Thus,

$$\begin{aligned} w(A^H) &= \max\{Q_{A^H}(\xi) \ : \ \xi \in (\mathbb{R}^n)_0^+ \text{ and } \|\xi\| = 1\} \\ &= \max\{Q_A(\xi) \ : \ \xi \in (\mathbb{R}^n)_0^+ \text{ and } \|\xi\| = 1\} \\ &= w(A) \ . \end{aligned}$$

However, because A^H is hermitian, $\rho(A^H) = w(A^H) = \|A^H\|$; the Perron-Frobenius theorem states that $\rho(A^H)$ is an eigenvalue of A^H and that there is a corresponding unit eigenvector $\xi \in (\mathbb{R}^n)_0^+$. With this vector ξ, we have $w(A) = \rho(A^H) = (A^H\xi, \xi) = (A\xi, \xi)$. Hence, A attains its numerical radius on the Perron vectors of A^H. Conversely, suppose that $\psi \in (\mathbb{R}^n)_0^+$ satisfies $(A\psi, \psi) = w(A)$. Then ψ also satisfies $(A^H\psi, \psi) = w(A) = \|A^H\|$. The Cauchy-Schwarz inequality yields $\|A^H\| = (A^H\psi, \psi) \leq \|A^H\|\|\psi\| \leq \|A^H\|$; but this is a case of equality within the Cauchy-Schwarz inequality and, therefore, $A^H\psi$ must be a multiple of ψ. From $(A^H\psi, \psi) = \rho(A^H)$ it follows that $A^H\psi = \rho(A^H)\psi$ and so ψ is a Perron vector of A^H.

Assume henceforth that $A \in \mathcal{M}_n(\mathbb{R}_0^+)$ is $*$-irreducible. We claim that A^H is $*$-irreducible. If it is not, then there is some $\sigma \in \mathcal{S}_n$ for which $P_\sigma A^H P_\sigma^t$ is block upper-triangular. Because $P_\sigma A^H P_\sigma^t = (P_\sigma A P_\sigma^t)^H$, this block upper-triangular matrix is symmetric and hence block diagonal. But the only way for $(P_\sigma A P_\sigma^t)^H$ to be block diagonal is for $P_\sigma A P_\sigma^t$ itself to be block diagonal, which is in contradiction of the $*$-irreducibilty of A. Hence, A^H must be irreducible and so the remaining statements concerning the numerical radius of A follow from the corresponding ones for the spectral radius of A^H. □

If a matrix T is block diagonal, with diagonal blocks $S_1, \ldots, S_k$, then a straightforward computation reveals that $w(T)$ is the maximum of the $w(S_i)$, $1 \leq i \leq k$. If T is $*$-reducible, then there exists a $\sigma \in \mathcal{S}_n$ such that $P_\sigma T P_\sigma^t$ is block diagonal with each diagonal block being $*$-irreducible. Thus, for the numerical radius, one need only work with matrices that are $*$-irreducible.

Corollary 1. *If $A, B \in \mathcal{M}_n(\mathbb{R}_0^+)$ have the same hermitian parts, then they have the same numerical radius. If, further, one of A or B is $*$-irreducible, then ξ is a Perron w-vector for A if and only ξ is a Perron w-vector for B.*

Proof: The first assertion is an immediate consequence of the theorem and we move to the proof of the second statement. Assume that it is A that is $*$-irreducible. From our argument used in the final part of the theorm, we may conclude that A^H is irreducible. Therefore, A^H (which coincides with B^H) has a unique Perron vector. □

Corollary 2. *The numerical radius of a matrix over $\mathbb{Z}_0^+$, the nonnegative integers, is an algebraic number.*

We turn now to our principal objective: the introduction of a numerical value that is assigned to multigraphs and warrants the descriptive title "numerical radius." As with the spectral radius of a multigraph, we do this by defining a correspondence between multigraphs and matrices over $\mathbb{Z}_0^+$. There is a new twist however: the correspondence is made in a manner whereby several matrices, typically nonsymmetric, are affiliated with a single multigraph. In contrast, the definition of numerical radius for directed multigraphs is rather straightforward.

Definitions.

1. The directed multigraph $\tilde{\Gamma}(A)$ of a matrix $A \in \mathcal{M}(\mathbb{Z}_0^+)$ is the directed graph on n vertices $v_1, \ldots, v_n$ that has a_{ij} edges directed from v_i to v_j for all pairs i, j. Conversely, given a finite directed multigraph, there is a finite square matrix B over $\mathbb{Z}_0^+$ such that this graph is precisely $\tilde{\Gamma}(B)$. (We say that the directed graph induces the matrix B.) Plainly, this correspondence between finite directed multigraphs and finite square matrices over $\mathbb{Z}_0^+$ is a bijective correspondence.

2. The multigraph $\Gamma(A)$ of a matrix $A \in \mathcal{M}_n(\mathbb{Z}_0^+)$ is the graph on n vertices $v_1, \ldots, v_n$ that has $a_{ij} + a_{ji}$ edges between pairs of distinct vertices v_i and v_j, and that has a_{jj} loops at the vertices v_j. Note that several different matrices may share the same multigraph; in particular, $\Gamma(A^t) = \Gamma(A)$.

3. If $\tilde{\Gamma}$ is a finite directed multigraph, then Γ denotes the underlying (undirected) multigraph; equivalently, Γ is the multigraph of the matrix induced by $\tilde{\Gamma}$. On the other hand, an orientation of a multigraph Γ is a configuration of directions for the edges of Γ that result in a directed multigraph $\tilde{\Gamma}$; equivalently, an orientation of Γ is the directed multigraph of some matrix whose multigraph is Γ.

4. For a complex matrix $T \in \mathcal{M}_n(\mathbb{C})$, the digraph of T is the directed multigraph of the zero-one matrix whose (i, j)-entry is nonzero if and only if $t_{ij} \neq 0$, and the graph of T is the multigraph of this zero-one matrix determined by T.

5. (Numerical radius of a directed multigraph) If $\tilde{\Gamma}$ is a finite directed multigraph, then $w(\tilde{\Gamma})$, the numerical radius of $\tilde{\Gamma}$, is defined to be $w(A)$, the numerical radius of the matrix $A \in \mathcal{M}_n(\mathbb{Z}_0^+)$ that is induced by $\tilde{\Gamma}$. A Perron w-vector for $\tilde{\Gamma}$ is defined to be a Perron w-vector of A.

The first proposition addresses the combinatorial meaning of irreducibilty and $*$-irreducibilty. Recall that a directed graph is strongly connected if between any two vertices there exists a directed path connecting them, and that a graph is connected if between any two vertices there exist a path connecting them.

Proposition 1. *A complex matrix T is irreducible if and only if its digraph is strongly connected, and is $*$-irreducible if and only if its graph is connected.*

Proof: The relation between irreducibility and strong connectivity is well-known – see [1;3.2.1] – and we will prove only the second statement.

Let M denote the zero-one matrix whose (i,j)-entry is nonzero if and only if $t_{ij} \neq 0$. Observe that for $\sigma \in \mathcal{S}_n$, $P_\sigma T P_\sigma^t$ is block diagonal if and only if $P_\sigma M P_\sigma^t$ is block diagonal.

Assume that $\Gamma(M)$ – that is, the graph of T – is not connected. There exist vertices v_i and v_j in $\Gamma(M)$ that are not connected by a path within the graph. Partition the vertices of $\Gamma(M)$ into two sets: the set $\mathcal{I}$ will contain all vertices that are connected to v_i by a path, and the set $\mathcal{J}$ will contain all remaining vertices. Note that there can be no edges between a member of $\mathcal{I}$ and a member of $\mathcal{J}$. This absence of edges between the vertices of $\mathcal{I}$ and $\mathcal{J}$ immediately implies that $P_\sigma M P_\sigma^t$ is block diagonal, where $\sigma \in \mathcal{S}_n$ is any permutation that is induced by the patitioning of the vertices of $\Gamma(M)$ into the two sets $\mathcal{I}$ and $\mathcal{J}$. Hence, T is $*$-irreducible.

Conversely, assume that T is $*$-reducible. There is a $\sigma \in \mathcal{S}_n$ such that $P_\sigma M P_\sigma^t$ is block diagonal, with the first block being a $k \times k$ matrix. Two distinct vertices v_i, v_j in $\Gamma(P_\sigma M P_\sigma^t)$ share an edge only if $1 \leq i, j \leq k$ or $k < i, j$. Thus, a vertex $v \in \{v_1, \ldots, v_k\}$ does not share an edge with a vertex $w \in \{v_{k+1}, \ldots, v_n\}$. If $\tau = \sigma^{-1}$, then $v \in \{v_{\tau(1)}, \ldots, v_{\tau(k)}\}$ does not share an edge with $w \in \{v_{\tau(k+1)}, \ldots, v_{\tau(n)}\}$ in $\Gamma(P_\tau P_\sigma M P_\sigma^t P_\tau^t) = \Gamma(M)$; hence, the graph $\Gamma(M)$ of T is not connected. □

The second proposition concerns a weak form of equivalence for multigraphs and directed multigraphs. We say that two graphs Γ_1 and Γ_2 are equivalent, denoted by $\Gamma_1 \sim \Gamma_2$, if Γ_2 is obtained from Γ_1 by permuting the vertices but leaving the edges fixed. This is different from a graph isomorphism, which preserves valencies as well. The meaning of $\tilde{\Gamma}_1 \sim \tilde{\Gamma}_2$ is analogous.

Proposition 2. *For directed multigraphs $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ on n vertices, $\tilde{\Gamma}_1 \sim \tilde{\Gamma}_2$ by a permutation $\sigma \in \mathcal{S}_n$ that sends the vertex set of $\tilde{\Gamma}_1$ onto the vertex set of $\tilde{\Gamma}_2$ if and only if $A_2 = P_\sigma A_1 P_\sigma^t$, where $A_1, A_2 \in \mathcal{M}_n(\mathbb{Z}_0^+)$ are the matrices induced by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ respectively.*

Proof: Assume that $\tilde{\Gamma}_1 \sim \tilde{\Gamma}_2$ by the permutation σ. The group $\mathcal{S}_n$ is generated by transpositions and so it is enough to assume that σ itself is a transposition, say $\sigma = (i,j)$. The matrix $P_\sigma A_1$ is obtained by interchanging rows i and j of A_1. Thus, multiplication on the left by P_σ effects a change on where edges depart from but does not alter the destination; for example, in $\tilde{\Gamma}(A_1)$ an edge directed from the ith (respectively the jth) vertex to the jth (respectively the ith) vertex becomes a loop at the jth (respectively ith) vertex in $\tilde{\Gamma}(P_\sigma A_1)$. Exchanging columns i and j in $P_\sigma A_1$ results in the matrix $P_\sigma A_1 P_\sigma^t$. Thus multiplications of $P_\sigma A_1$ on the right by P_σ^t changes the destinations of the edges without altering the point of departure; thus, for example, the edges that had arrived at the ith (respectively, the jth) vertex of $\tilde{\Gamma}(P_\sigma A_1)$ now arrive at the jth (respectively, the ith) vertex of $\tilde{\Gamma}(P_\sigma A_1 P_\sigma^t)$. Hence, $\tilde{\Gamma}(P_\sigma A_1 P_\sigma^t)$ is precisely $\tilde{\Gamma}(A_2)$, whence $P_\sigma A_1 P_\sigma^t = A_2$.

The proof of the other direction is similar. □

Corollary. *If $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are connected directed multigraphs that are equivalent via the permutation $\sigma \in S_n$, then $w(\tilde{\Gamma}_1) = w(\tilde{\Gamma}_2)$ and the Perron w-vector ξ_2 for $\tilde{\Gamma}_2$ is given by $P_\sigma \xi_1$, where ξ_1 is the Perron w-vector for $\tilde{\Gamma}_1$.*

Proof: Let A_1 and A_2 denote the induced matrices. From $A_2 = P_\sigma A_1 P_\sigma^t$ and the invariance of the numerical radius under unitary similarity transformations, we have $w(A_2) = w(A_1)$. Since the graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are connected, the matrices A_1 and A_2 are $*$-irreducible, which implies that $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have unique Perron w-vectors, say ξ_1 and ξ_2 respectively. From the uniqueness of the Perron w-vectors and

$$w(\tilde{\Gamma}_2) = (A_2\xi_2, \xi_2) = (A_1 P_\sigma^t \xi_2, P_\sigma^t \xi_2) = w(\tilde{\Gamma}_1)$$

it follows that $\xi_1 = P_\sigma^t \xi_2$. □

Proposition 3. *Suppose that Γ is a finite multigraph and that $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are two orientations of Γ. Then $w(\tilde{\Gamma}_1) = w(\tilde{\Gamma}_2)$. If Γ is connected, then ξ is a Perron w-vector of $\tilde{\Gamma}_1$ if and only if ξ is a Perron w-vector of $\tilde{\Gamma}_2$.*

Proof: Let $A_1, A_2 \in \mathcal{M}_n(\mathbb{Z}_0^+)$ be the matrices induced by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ respectively. Because $\tilde{\Gamma}_2$ is simply $\tilde{\Gamma}_1$ with certain of the directions reversed, A_1 and A_2 must have the same hermitian parts. Hence, $w(A_1) = w(A_2)$ by Corollary 1. If Γ is connected, then A_1 and A_2 are $*$-irreducible and, again by Corollary 1, the unique Perron w-vector of $\tilde{\Gamma}_1$ coincides with the unique Perron w-vector of $\tilde{\Gamma}_2$. □

The proposition above indicates how to give a well-defined meaning to the notion of numerical radius of a multigraph.

Definition. (Numerical radius of a multigraph) If Γ is a finite multigraph, then the numerical radius $w(\Gamma)$ of Γ is defined to be $w(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is any orientation of Γ. If Γ is connected, then the Perron w-vector of Γ is defined to be the Perron w-vector of $\tilde{\Gamma}$, for any orientation $\tilde{\Gamma}$ of Γ.

We turn now to some properties of the numerical radius. The second statement of the proposition below demonstrates that it is only with multigraphs having loops that the numerical radius is indeed a new (numerical) invariant of graphs.

Proposition 4. *Suppose that Γ is a finite connected multigraph.*

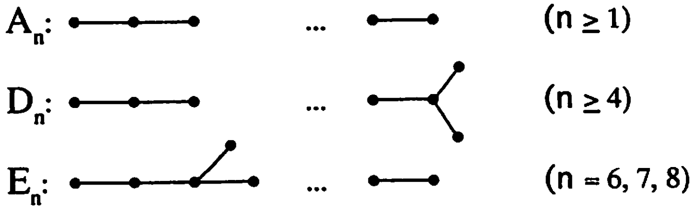
1. *If Γ' is a proper submultigraph of Γ, then $w(\Gamma') < w(\Gamma)$.*
2. *If ρ denotes the spectral radius of Γ, then $\rho \leq 2w(\Gamma)$ and equality holds if and only if Γ has no loops.*

Proof: Orient Γ so that the resulting directed graph $\tilde{\Gamma}$ has its edges directed from v_i to v_j whenever $i \leq j$, and let $A \in \mathcal{M}_n(\mathbb{Z}_0^+)$ be the $*$-irreducible upper triangular matrix whose directed multigraph is $\tilde{\Gamma}$. The subgraph $\tilde{\Gamma}'$ induces a matrix $A' \in \mathcal{M}_n(\mathbb{Z}_0^+)$ such that $\tilde{\Gamma}' = \tilde{\Gamma}(A')$ and having the property that $a'_{kl} \leq a_{kl}$ for all pairs (k,l). Because $\tilde{\Gamma}'$ is a proper subgraph, $a'_{kl} < a_{kl}$ for at least one pair (k,l). By the Perron-Frobenius Theorem, $w(\Gamma') = w(A') < w(A) = w(\Gamma)$, which proves (1).

The spectral radius ρ of Γ is $\rho(adj\Gamma)$, the spectral radius of the adjacency matrix $adj\Gamma$ of Γ. The off-diagonal parts of $adj\Gamma$ and $2A^H$ agree, however the diagonal entries of $2A^H$ dominate the corresponding diagonal entries of $adj\Gamma$; hence $w(2A^H) \geq w(adj\Gamma) = \rho(adj\Gamma)$ and equality holds if and only if $2A^H = adj\Gamma$. But $2A^H = adj\Gamma$ if and only if the diagonal entries of $2A^H$ and $adj\Gamma$ are zero. □

A classical combinatorial problem is the determination of those real numbers that arise as the spectral radius of some graph (i.e., of some symmetric zero-one matrix). This problem dates back to Kronecker, to whom a good portion of the solution is attributed, and has only recently been settled to a certain level of satisfaction. Outside of small values, which is Kronecker's contribution, it is known from the works of Hoffman [5] and Shearer [6] that the set of spectral radii of graphs is dense in the set of real numbers beyond $\tau^{1/2}+\tau^{-1/2}$, where τ is the golden mean $2^{-1}(\sqrt{5}+1)$. (A thorough discussion of these developments is made in [4], Chapter 1 and Appendix 1.) Using Proposition 4, Kronecker's theorem immediately yields information concerning graphs with small numerical radii and those real numbers less than 1 that are the numerical radius of some graph.

Corollary.*(Kronecker) Suppose that Γ is a connected multigraph on n vertices. Then $w(\Gamma) < 1$ if and only if Γ is one of*



In these cases, $w(\Gamma) = \cos(\pi/h_\Gamma)$, where h_Γ is $n+1$ if $\Gamma = A_n$, $2n-2$ if $\Gamma = D_n (n \geq 4)$, or 12, 18, 30 if Γ is E_6, E_7, E_8 respectively.

Proof: If $w(\Gamma) < 1$, then the diagonal of any matrix induced by Γ must be zero, for if $A \in \mathcal{M}_n(\mathbb{Z}_0^+)$ is induced by Γ, then a_{ii} is given by (A_{e_i,e_i}) for each i, which are nonnegative integers less than 1. Hence, Γ has no loops and so by invoking Proposition 4, we find that $w(\Gamma) = 2\rho(\Gamma)$. The claimed conclusions now follow from Kronecker's theorem for the spectral radii of graphs [4:1.4.3].

On the other hand, if Γ is one of those graphs listed above, each of which is without loops, then we may again invoke Proposition 4 to obtain the claimed conclusions. □

More generally, one can study the set $\mathcal{W} = \{w(\Gamma) : \Gamma \text{ is a finite multigraph}\}$ by way of $\mathcal{E} = \{\rho(\mathcal{G}) : \mathcal{G} \text{ is a finite mutligraph without loops}\}$. If $A \in \mathcal{M}_n(\mathbb{Z}_0^+)$ is a matrix induced by some multigraph, then $2A^H$ is a symmetric matrix over $\mathbb{Z}_0^+$ and hence its spectral radius is given by the spectral radius of some zero-one matrix of trace zero [5:2.1]. Conversely, if M is a symmetric zero-one matrix of trace zero, then for the upper triangular part A of M, $2A^H = M^H = M$. It follows, therefore, that $\mathcal{W} = \frac{1}{2}\mathcal{E}$. Some of what is known about $\mathcal{E}$, and hence about $\mathcal{W}$, is summarised below (see Appendix 1 of [4] for further details).

Proposition 5. *(Kronecker-Hoffman-Shearer) The set $\mathcal{W} \cap [0,1]$ is the closure of $\{\cos(\pi/q)\}_{q\geq 2}$. For $\tau = (\sqrt{5}+1)/2$, the set of limit points of $\mathcal{W}\cap[1,(\tau^{1/2}+\tau^{-1/2})/2]$ is countable and is given by the closure of $\{\frac{1}{2}(\beta_k^{1/2}+\beta_k^{-1/2})\}_{k\geq 1}$, where β_k is the positive root of the polynomial $p_k(x) = x^{k+1} - \Sigma_{j=0}^{k-1} x_j$, for $k \geq 1$. Every real number exceeding $(\tau^{1/2}+\tau^{-1/2})/2$ is the limit of numerical radii of multigraphs.*

We conclude with some further examples.

EXAMPLE 1. Let K_n denote the complete graph on n vertices. A tournament $\mathcal{T}$ with n players is any orientation of K_n. Thus, the numerical radius of any tournament $\mathcal{T}$ is $w(K_n)$, by Proposition 3. The computation of $w(K_n)$ is simple. Let $A \in \mathcal{M}_n(\{0,1\})$ be any tournament matrix. Then $A^H = (1/2)(ee^t - I)$, where $e = \Sigma_i e_i$. The matrix ee^t is a rank-1 matrix with eigenvalues 0 and n; hence, the eigenvalues of A^H are $-1/2$ and $(n-1)/2$. Therefore $w(K_n) = w(A^H) = (n-1)/2$. The Perron w-vector for K_n is given by $n^{-1/2}e$.

EXAMPLE 2. If Γ has a cycle, then $w(\Gamma) \geq 1$. To see this, assume first that Γ itself is a cycle with n-vertices. Then the cyclic shift $A \in \mathcal{M}_n(\{0,1\})$ with the action $Ae_n = e_1$, $Ae_j = e_{j+1}$ for $1 \leq j < n$, is a unitary matrix with graph Γ. Hence, $w(\Gamma) = w(A) = 1$ and the Perron w-vector for the cycle is given by $n^{-1/2}e$, as in Example 1. In general, if Γ contains an n-cycle, then this cycle is a subgraph of Γ and hence $w(\Gamma) \geq 1$ (the inequality is strict if Γ is connected but is not a cycle).

EXAMPLE 3. It is simple to determine the numerical radius and the Perron

w-vector of any multigraph on two vertices. This amounts to computing the numerical radius of any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ having nonnegative entries. We are to maximize $(A\xi, \xi)$ for $\xi = \cos\theta e_1 + \sin\theta e_2$ as θ varies from 0 to $\pi/2$. Written explicitly,

$$(A\xi, \xi) = \frac{a+d}{2} + \cos 2\theta(\frac{a-d}{2}) + \sin 2\theta(\frac{b+c}{2}).$$

An application of the Cauchy-Schwarz inequality to the final two summands reveals that

$$w(A) = \frac{a+d}{2} + \sqrt{\left(\frac{a-d}{2}\right)^2 + \left(\frac{b+c}{2}\right)^2}$$

with $D\sin 2\theta = (b+c)/2$ and $D\cos 2\theta = D(a-d)/2$, where $D = \sqrt{(\frac{a-d}{2})^2 + (\frac{b+c}{2})^2}$. Solving for $\cos\theta$ and $\sin\theta$ yields the Perron w-vector

$$\xi = \sqrt{\frac{1}{2} + \frac{a-d}{4D}} e_1 + \sqrt{\frac{1}{2} + \frac{d-a}{4D}} e_2.$$

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