Lambda-fold 3-perfect 9-cycle systems

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ABSTRACT: The spectrum for the decomposition of λK_v into 3-perfect 9-cycles is found for all $\lambda > 1$. (The case $\lambda = 1$ was dealt with in an earlier paper by the authors and Lindner.) The necessary conditions for the existence of a suitable decomposition turn out to be sufficient.

1 Introduction

There are now many papers on decompositions of the complete graph K_v into cycle systems; a survey is given by Lindner and Rodger [8]. An *m-cycle system of* K_v is essentially a decomposition of the edges of K_v into disjoint m-cycles. So, formally, we say an m-cycle system of K_v is a pair (V, C) where V is the vertex set of K_v and C is a set of edge-disjoint cycles of length m, which partition the edge set of K_v.

Often, extra structure is required of an m-cycle system. Suppose that, in an m-cycle system of K_v, we take each m-cycle and replace it instead by the graph formed by joining all its vertices which are at distance i in the cycle. If, when this is done for each m-cycle, we again have a decomposition of K_v (so a partition of all the edges of K_v), we call the original m-cycle system an *i-perfect m-cycle system of* K_v. Previously most work has been done on 2-perfect m-cycle systems. (See [1, 6, 7, 9, 10] for example.) It is interesting to consider 3-perfect 9-cycle systems of K_v because the graph at distance 3 obtained from each 9-cycle consists of three triangles, and so the collection of all these triangles, from each 9-cycle, forms a Steiner triple system of order v. This case has been dealt with in [2].

Just as λ-fold triple systems are of interest, so we now consider λ-fold cycle systems here. So instead of the complete graph K_v, we take λ copies

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of each edge and consider λK_v. A λ-fold i-perfect m-cycle system of λK_v is again a pair (V, C) where V is the vertex set (of size v) and C is a collection of m-cycles which partition the edges of λK_v, in such a way that if the distance i graph in each m-cycle is taken, the collection of all these graphs also forms a partition of all the edges of λK_v. The case of 2-perfect 6-cycle systems of λK_v has been dealt with in [3] (when of course the distance 2 graphs in each 6-cycle, taken together, form a λ-fold triple system of λK_v). In this paper we find 3-perfect 9-cycle systems of λK_v, when again the distance 3 graphs in each 9-cycle together form a λ-fold triple system of λK_v. In particular we prove:

THEOREM 1.1 *The necessary and sufficient conditions for the existence of a λ-fold 3-perfect 9-cycle system of λK_v are as given in the table below.*

λ (mod 18)	$v \geqslant 9$
1, 5, 7, 11, 13, 17	1 or 9 (mod 18)
2, 4, 8, 10, 14, 16	0 or 1 (mod 9)
3, 15	1 or 3 (mod 6)
6, 12	0 or 1 (mod 3)
9	1 (mod 2)
18	all v

Simply using the fact that λK_v has $\lambda\binom{v}{2}$ edges, which must be a multiple of 9, and that the degree of each vertex in λK_v must be even, (and that $v \geqslant 9$ since we are forming 9-cycles), the above conditions on v are easily found to be necessary.

We may have a cycle repeated in a decomposition, so it is clear from the table that we need only consider six cases, namely $\lambda = 1, 2, 3, 6, 9$ or 18. Moreover, the case $\lambda = 1$ was covered in [2], so we essentially have only five values of λ to deal with.

2 The case $\lambda = 2$

When $\lambda = 2$ the number of vertices v must satisfy $v \equiv 0$ or 1 (mod 9). We have decompositions into 3-perfect 9-cycles for the small cases $2K_9$ (repeat one of K_9; see [2]), $2K_{10}$ and $2K_{18}$ (see the Appendix), $K_{9,9,9}$ (see [2], Example 2.2), and $2K_{19}$ (repeat one of K_{19}; see [2] or Lemma 2.2 of [7]).

Now suppose that $v = 9m$, and take the vertex set of $2K_v$ to be $\{(i, j) \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant 9\}$.

Now if $m \equiv 0$ or 1 (mod 3) and $m \geqslant 3$, it is well-known that there exists a two-fold triple system on m elements. Also if $m \equiv 2$ (mod 3) then a maximum two-fold packing of $2K_m$ with triangles exists which has leave consisting of one repeated edge. (See for instance [4] or [11].)

When $m \equiv 0$ or $1 \pmod 3$, a 3-perfect 9-cycle system of $2K_{9m}$ is given by the following:

(1) On $\{(i,j) \mid 1 \leqslant j \leqslant 9\}$, place a 3-perfect 9-cycle system of $2K_9$, for each $i = 1, 2, \ldots, m$.

(2) Whenever $\{(x,j),(y,j),(z,j)\}$ is a triple of a two-fold triple system on the set $\{(i,j) \mid 1 \leqslant i \leqslant m\}$, we place on the set

$$\{(x,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(y,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(z,j) \mid 1 \leqslant j \leqslant 9\}$$

a 3-perfect 9-cycle system of $K_{9,9,9}$. (Here we are generalising the definition of a 3-perfect 9-cycle system of a complete graph, in the obvious way, to $K_{9,9,9}$.)

When $m \equiv 2 \pmod 3$, a 3-perfect 9-cycle system of $2K_{9m}$ is given by the following:

(1′) On $\{(i,j) \mid 1 \leqslant j \leqslant 9\}$, place a 3-perfect 9-cycle system of $2K_9$, for each $i = 3, 4, \ldots, m$.

(2′) Whenever $\{(x,j),(y,j),(z,j)\}$ is a triple of a two-fold maximum packing into triples on the set $\{(i,j) \mid 1 \leqslant i \leqslant m\}$, with leave being the double edge between the two vertices $(1,j)$ and $(2,j)$, we place on the set

$$\{(x,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(y,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(z,j) \mid 1 \leqslant j \leqslant 9\}$$

a 3-perfect 9-cycle system of $K_{9,9,9}$.

(3′) Finally, on the set $\{(1,j),(2,j) \mid 1 \leqslant j \leqslant 9\}$, we place a 3-perfect 9-cycle system of $2K_{18}$.

Now suppose that $v = 9m + 1$. The only change in the above is to include the vertex $\{\infty\}$, and to replace $2K_9$ and $2K_{18}$ respectively by $2K_{10}$ and $2K_{19}$. Note that since the two-fold triple system and the two-fold maximum packing into triples described above, of order m, exist for all $m \geqslant 3$, we need no more than these four cases. So this completes the case $\lambda = 2$.

3 The construction for higher λ

For $\lambda = 3$, 6, 9 and 18, the construction described below is used. Let the vertices of λK_v, where $v = 18m + \delta$, and $0 \leqslant \delta \leqslant 17$, be

$$\{\infty_i\}_{i=1}^{\delta} \cup \{(i,j) \mid 1 \leqslant i \leqslant 2m, \ 1 \leqslant j \leqslant 9\}.$$

(If $\delta = 0$, there are no elements ∞_i.)

We shall use the following well-known result (essentially due to Hanani [5], Lemma 6.3, and Wilson [12], page 276; see Lemma 2.3 of [2]).

LEMMA 3.1 *There is a group divisible design on $2m$ elements with block size three and group size two when $2m \geqslant 6$ and $2m \equiv 0$ or $2 \pmod 6$, and there is a group divisible design on $2m$ elements, block size three, one group of size four and the rest of size two, when $2m \geqslant 10$ and $2m \equiv 4 \pmod 6$.*

When $2m \equiv 0$ or $2 \pmod 6$ on $\{(i,j) \mid 1 \leqslant i \leqslant 2m\}$ we take a group divisible design with groups $\{(2i-1,j),(2i,j)\}$ for $i = 1,2,\ldots,m$, and blocks of size three. Now on $\lambda K_{18m+\delta}$ we take the following decomposition into 3-perfect 9-cycles:

(1) On $\{\infty_i\}_{i=1}^{\delta} \cup \{(1,j),(2,j) \mid 1 \leqslant j \leqslant 9\}$ we place a 3-perfect 9-cycle decomposition of $\lambda K_{18+\delta}$.

(2) On $\{\infty_i\}_{i=1}^{\delta} \cup \{(2i-1,j),(2i,j) \mid 1 \leqslant j \leqslant 9\}$, for $i = 2,3,\ldots,m$, we place a 3-perfect 9-cycle decomposition of the graph $\lambda(K_{18+\delta} \setminus K_\delta)$, that is, λ copies of the complete graph on $18+\delta$ vertices with a hole of size δ. This graph has all the $\lambda\binom{\delta}{2}$ edges between vertices labelled ∞_i removed, but all other edges remain.

[Of course, if $\delta = 0$ (or 1) then we combine (1) and (2) and simply place a 3-perfect 9-cycle decomposition of λK_{18} (or λK_{19}) on $\{(2i-1,j),(2i,j) \mid 1 \leqslant j \leqslant 9\}$, for $i = 1,2,\ldots,m$, (or on this set together with $\{\infty_1\}$).]

(3) For each block $\{(x,j),(y,j),(z,j)\}$ of the group divisible design, we place on the vertex set

$$\{(x,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(y,j) \mid 1 \leqslant j \leqslant 9\} \cup \{(z,j) \mid 1 \leqslant j \leqslant 9\}$$

λ copies of a 3-perfect 9-cycle system of $K_{9,9,9}$.

Now when $2m \equiv 4 \pmod 6$, let the group divisible design on $\{(i,j) \mid 1 \leqslant i \leqslant 2m\}$ have $\{(1,j),(2,j),(3,j),(4,j)\}$ as its group of size four, with the other groups of size two being $\{(2i-1,j),(2i,j)\}$ for $i = 3,4,\ldots,m$. Then 9-cycles are taken as follows:

(1′) On $\{\infty_i\}_{i=1}^{\delta} \cup \{(i,j) \mid 1 \leqslant i \leqslant 4,\ \ 1 \leqslant j \leqslant 9\}$ we place a 3-perfect 9-cycle system of $\lambda K_{36+\delta}$.

(2′) On $\{\infty_i\}_{i=1}^{\delta} \cup \{(2i-1,j),(2i,j) \mid 1 \leqslant j \leqslant 9\}$, for $i = 3,4,\ldots,m$, we place a 3-perfect 9-cycle system of $\lambda(K_{18+\delta} \setminus K_\delta)$.

(3′) As in (3) above.

As a result of the above constructions, when $v = 18m + \delta$ and $2m \equiv 0$ or $2 \pmod 6$, we only need find 3-perfect 9-cycle systems of $\lambda K_{18+\delta}$, $\lambda(K_{18+\delta} \setminus K_\delta)$, and λK_δ (if $\delta > 9$).

Also when $2m \equiv 4 \pmod 6$, we need a 3-perfect 9-cycle system of $\lambda K_{36+\delta}$.

It is clear from the table of necessary conditions on λ and v that, having dealt with the cases $\lambda = 1$ and 2, we only need consider $\lambda = 3$, 6, 9 and 18. Moreover, by using a combination of decompositions of $\lambda_1 K_v$ and $\lambda_2 K_v$ for suitable λ_1 and λ_2 to give a decomposition of $(\lambda_1 + \lambda_2)K_v$, it can be seen that decompositions of the multi-graphs in the following table are all that is needed to complete the proof of Theorem 1.1.

$\lambda = 3$	$\delta = 3$	$3K_{21}$, $3(K_{21} \setminus K_3)$, $3K_{39}$
	$\delta = 7$	$3K_{25}$, $3(K_{25} \setminus K_7)$, $3K_{43}$
	$\delta = 13$	$3K_{13}$, $3K_{31}$, $3(K_{31} \setminus K_{13})$, $3K_{49}$
	$\delta = 15$	$3K_{15}$, $3K_{33}$, $3(K_{33} \setminus K_{15})$, $3K_{51}$
$\lambda = 6$	$\delta = 4$	$6K_{22}$, $6(K_{22} \setminus K_4)$, $6K_{40}$
	$\delta = 6$	$6K_{24}$, $6(K_{24} \setminus K_6)$, $6K_{42}$
	$\delta = 12$	$6K_{12}$, $6K_{30}$, $6(K_{30} \setminus K_{12})$, $6K_{48}$
	$\delta = 16$	$6K_{16}$, $6K_{34}$, $6(K_{34} \setminus K_{16})$, $6K_{52}$
$\lambda = 9$	$\delta = 5$	$9K_{23}$, $9(K_{23} \setminus K_5)$, $9K_{41}$
	$\delta = 11$	$9K_{11}$, $9K_{29}$, $9(K_{29} \setminus K_{11})$, $9K_{47}$
	$\delta = 17$	$9K_{17}$, $9K_{35}$, $9(K_{35} \setminus K_{17})$, $9K_{53}$
$\lambda = 18$	$\delta = 2$	$18K_{20}$, $18(K_{20} \setminus K_2)$, $18K_{38}$
	$\delta = 8$	$18K_{26}$, $18(K_{26} \setminus K_8)$, $18K_{44}$
	$\delta = 14$	$18K_{14}$, $18K_{32}$, $18(K_{32} \setminus K_{14})$, $18K_{50}$

For a decomposition of $9K_p$ where p is an odd prime, we have a cyclic solution.

LEMMA 3.2 *Let p be an odd prime not less than m. Then there exists an i-perfect m-cycle system of mK_p for any i (that is, a Steiner m-cycle system).*

Proof Let x be a multiplicative generator of the field GF(p). Then take the following $(p-1)/2$ starter m-cycles, and cycle them modulo p:

$$\begin{array}{l}(1, x, x^2, \ldots, x^{m-1}),\\ (x, x^2, x^3, \ldots, x^m),\\ (x^2, x^3, \ldots, x^{m+1}),\\ \quad\vdots\\ (x^{(p-3)/2}, x^{(p-1)/2}, \ldots, x^{(p-3)/2+m-1}).\end{array}$$

This lemma gives us decompositions of $9K_{23}$, $9K_{41}$, $9K_{11}$, $9K_{29}$, $9K_{47}$, $9K_{17}$ and $9K_{53}$, all into 3-perfect 9-cycle systems.

The remaining cases in the table are given in the Appendix.

4 Concluding remarks.

In this paper we have shown how to obtain a 3-perfect 9-cycle system of λK_v for any λ and any admissible v. The necessary conditions for the existence of a decomposition are sufficient in every case; the case $\lambda = 1$ was dealt with in [2].

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