# The Automorphism Groups of Circulant Digraphs of Degree 3

Qiongxiang Huang
Department of Mathematics, Xinjiang University,
Urumuqi, Xinjiang, 830046, P.R.China.

Jinjiang Yuan
Department of Mathematics, Zhengzhou University,
Zhengzhou, Henan, 450052, P.R.China.

ABSTRACT. In this paper, we discuss the automorphism groups of circulant digraphs. Our main purpose is to determine the full automorphism groups of circulant digraphs of degree 3.

## 1 Introduction

Let $Z_n$ be the cyclic group of integers modulo $n$ with operation "+" and zero element 0. Let $S$ be a subset of $Z_n$ not containing0. Denote by $C_n(S)$ a circulant digraph. Its vertex set is $Z_n$, and for $i \in Z_n$ and $s \in S$, $(i, i+s)$ is an arc of $C_n(S)$ from $i$ to $i+s$. We call $(i, i+s)$ an $s$-arc and $S$ the arc symbol set of $C_n(S)$.

There are many papers on circulant digraphs, one can refer to [1–5] in particular. In [3], B. Elspas and J. Turner posed the problem to characterize the automrphism groups of circulant digraphs. Up to now, there has been no decisive advancement on this subject. Let $S = \{s_1, s_2, ..., s_k\}$ and $\gcd(s_1, s_2, ..., s_k, n) = d$. It is easy to see that $C_n(S)$ has $d$ components and each of them is isomorphic to $C_{\frac{n}{d}}(\{\frac{s_1}{d}, \frac{s_2}{d}, ..., \frac{s_k}{d}\})$. In this case, the automorphism group of $C_n(S)$ is the wreath product of the automorphism groups of these $d$ copies of a component [7]. Thus we need only to characterize the automorphism groups of the strongly connected circulant digraphs. So $C_n(S)$ is always assumed to be strongly connected.

If $S$ contains only one integer, then $C_n(S)$ is an $n$-length directed cycle, and its automorphism group is clearly the cyclic group of order $n$. When $|S| = 2$, L. Sun [8] determined the automorphism groups of $C_n(S)$. B. Elspas and J. Turner [3] characterized the automorphism groups of circulant

ARS COMBINATORIA 44(1996), pp. 263-271

digraphs whose spectra have $n$ distinct eigenvalues. In this paper, we focus our attentions on the circulant digraphs of degree 3 (i.e., $|S| = 3$) and consider their automorphism groups.

Let $AutZ_n$ be the automorphism group of $Z_n$ and $Aut_S(Z_n) = \{\tau \in AutZ_n \mid \tau(S) = S\}$ be the stable subgroup of $S$ in $AutZ_n$. Let $Z_n^* = \{\lambda \in Z_n \mid \gcd(\lambda, n) = 1\}$. It is well known that $AutZ_n \cong Z_n^*$ and $C_n(S) \cong C_n(\lambda S)$ (where $\lambda S = \{\lambda s \mid s \in S\}$). Let $AutC_n(S)$ be the automorphism group of $C_n(S)$ and $L(Z_n) = \{\sigma_a : i \longmapsto a + i (\forall i \in Z_n) \mid a \in Z_n\}$. Then $L(Z_n)$ is clearly a subgroup of $AutC_n(S)$ that acts transitively on the vertices of $C_n(S)$. Define $\Omega(S) = \{\tau \in AutC_n(S) \mid \tau(0) = 0\}$.

**Lemma 1** [6]. $AutC_n(S) = L(Z_n)\Omega(S)$.

According to the above Lemma, to characterize $AutC_n(S)$ it is sufficient to characterize $\Omega(S)$. In view of the fact that $Aut_S(Z_n)$ is a subgroup of $\Omega(S)$, and $Aut_S(Z_n) \subseteq AutZ_n \cong Z_n^*$, we indeed know the automorphism group of $C_n(S)$ if $\Omega(S) = Aut_S(Z_n)$. Fortunately, in most situations, $\Omega(S)$ is actually equal to $Aut_S(Z_n)$. In the next section, we first give a necessary and sufficient condition for a circulant digraph to satisfy $\Omega(S) = Aut_S(Z_n)$. Based on this result, we fully characterize the automorphism groups of circulant digraphs of degree 3.

## 2 Circulant Digraphs with $\Omega(S) = Aut_S(Z_n)$

In this section, we give a necessary and sufficient condition for a circulant digraph to satisfy $\Omega(S) = Aut_S(Z_n)$ and derive some additional results in preparation for characterizing the automorphism groups of circulant digraphs of degree 3.

**Lemma 2** [5]. *$C_n(S)$ is strongly connected if and only if $S$ generates $Z_n$ if and only if $\gcd(s_1, s_2, ..., s_k, n) = 1$, where $S = \{s_1, s_2, ..., s_k\}$.*

**Theorem 1.** *Let $C_n(S)$ be strongly connected and $S_0$ be a subset of $S$ that generates $Z_n$. Then $\Omega(S) = Aut_S(Z_n)$ if and only if for $\tau \in \Omega(S)$ and $a, b \in S_0$, $\tau(a+b) = \tau(a) + \tau(b)$.*

**Proof:** The necessity is obvious. To show the sufficiency, we need only confirm that $\Omega(S) \subseteq Aut_S(Z_n)$.

Let $\tau \in \Omega(S)$ and $a, b \in S_0$. Then $\sigma_u, \sigma_{-\tau(u)} \in L(Z_n)$ for $u \in Z_n$. It is easy to see that $\sigma_{-\tau(u)}\tau\sigma_u$ is an automorphism of $C_n(S)$. Since $\sigma_{-\tau(u)}\tau\sigma_u(0) = -\tau(u) + \tau(u) = 0$, $\sigma_{-\tau(u)}\tau\sigma_u \in \Omega(S)$. So by assumption, we have

$$\sigma_{-\tau(u)}\tau\sigma_u(a+b) = \sigma_{-\tau(u)}\tau\sigma_u(a) + \sigma_{-\tau(u)}\tau\sigma_u(b)$$

$$i.e., \quad -\tau(u) + \tau(u+a+b) = -\tau(u) + \tau(u+a) - \tau(u) + \tau(u+b)$$

$$i.e., \quad \tau(u+a+b) = \tau(u+a) - \tau(u) + \tau(u+b). \qquad (1)$$

Hence, if $u \in S_0$, we have $\tau(u+a+b) = \tau(u) + \tau(a) + \tau(b)$. In general, suppose $u = \sum_i s_i$ ($s_i \in S_0$). From (1), one can easily derive, by induction, that $\tau(\sum_i s_i) = \sum_i \tau(s_i)$. Since $S_0$ generates $Z_n$, for any $v_1$ and $v_2$ in $Z_n$, we have $v_1 = \sum_i s_{1i}$ and $v_2 = \sum_j s_{2j}$, where $s_{1i}, s_{2j} \in S_0$. Thus

$$\begin{aligned} \tau(v_1+v_2) &= \tau(\textstyle\sum_i s_{1i} + \sum_j s_{2j}) \\ &= \textstyle\sum_i \tau(s_{1i}) + \sum_j \tau(s_{2j}) \\ &= \tau(v_1) + \tau(v_2). \end{aligned}$$

This means $\tau \in Aut_S(Z_n)$ and completes our proof.

**Corollary 1.** *Suppose $a \in S$. If $a$ is the unique in-adjacency vertex of $2a$ in $S$, then for $u, v \in \langle a \rangle$ and $\tau \in \Omega(S)$, we have $\tau(u+v) = \tau(u) + \tau(v)$.*

**Proof:** By assumption, $\tau(a)$ is the unique in-adjacency vertex of $\tau(2a)$ in $S$. Furthermore, we have $\tau(2a) = 2\tau(a)$. Since otherwise, $\tau(2a) = \tau(a) + \tau(a')$ ($a' \neq a$), then $\tau(a)$ and $\tau(a')$ are two in-adjacency vertices of $\tau(2a)$ in $S$. Set $S_0 = \{a\}$. Our result follows immediately by the proof technique of Theorem 1.

**Corollary 2.** *Under the assumption of Corollary 1, if $a$ is also relatively prime to $n$, we have $\Omega(S) = Aut_S(Z_n)$.*

The condition $\tau(a+b) = \tau(a) + \tau(b)$ in Theorem 1 can be easily verified sometimes. Corollary 1 provides us with a criterion to check it when $a = b$. Another criterion is described in the following lemma.

**Lemma 3.** *Let $a, b \in S$. If $a$ and $b$ have a unique common out-adjacency vertex $a+b$, then for $\tau \in \Omega(S)$, we have $\tau(a+b) = \tau(a) + \tau(b)$.*

**Proof:** By assumption, $\tau(a+b)$ is the unique common out-adjacency vertex of $\tau(a)$ and $\tau(b)$. Since $\tau(a) + \tau(b)$ is also a common out-adjacency vertex of $\tau(a)$ and $\tau(b)$, we have $\tau(a+b) = \tau(a) + \tau(b)$.

Let $u \in Z_n$. We call $u$ a fixed vertex of $\Omega(S)$ if $\tau(u) = u$ for $\tau \in \Omega(S)$. The following theorem tells us that fixed vertices of $\Omega(S)$ generate a subgroup of $Z_n$ whose vertices are still fixed vertices.

**Theorem 2.** *If $C_n(S)$ is a circulant digraph and $T$ is a vertex set of $\Omega(S)$, then all the vertices in $\langle T \rangle$ are fixed vertices of $\Omega(S)$.*

**Proof:** For $u, v \in T$, $\sigma_{-\tau(u)}\tau\sigma_u \in \Omega(S)$. Hence by assumption, $\sigma_{-\tau(u)}\tau\sigma_u(v) = v$, and then $\tau(u+v) = u+v$. By using induction as in Theorem 1, one can easily prove that $\tau(\sum_i u_i) = \sum_i u_i($ where $u_i \in T)$. This implies our result.

**Corollary 3.** *If $T$ is a set of fixed vertices of $\Omega(S)$. Then for $\tau \in \Omega(S)$, $a \in \langle T \rangle$ and $u \in Z_n$, we have $\tau(a+u) = a + \tau(u)$.*

**Proof:** Since $\sigma_{-\tau(u)}\tau\sigma_u \in \Omega(S)$, then by Theorem 2, we have $\sigma_{-\tau(u)}\tau\sigma_u(a) = a$, i.e., $\tau(a+u) = a + \tau(u)$.

## 3 Main results

Suppose $S = \{a_1, a_2, a_3\}$ and $\gcd(a_1, a_2, a_3, n) = 1$. Then $C_n(S)$ is a strongly connected circulant digraph of degree 3.

Let $N^+(S)$ be the out-neighbour set of $S$ in $C_n(S)$. Then $N^+(S)$ contains at most six distinct vertices: $a_1+a_2, a_1+a_3, a_2+a_3, 2a_1, 2a_2, 2a_3$. The first three vertices are obviously different. For convenience, we classify $C_n(S)$ into one of four types: I-type, II-type, III-type, IV-type, where I-type, II-type, and III-type are given according to the following conditions (I), (II) and (III), respectively.

(I) $2a_1 = a_2 + a_3$ and $2a_2 = a_1 + a_3$;

(II) $2a_1 = a_2 + a_3$ and $2a_2 = 2a_3$;

(III) $2a_1 = 2a_2$ but $2a_3 \neq a_1 + a_2$.

$C_n(S)$ is said to be IV-type if none of the above three conditions is satisfied, even after renumbering $a_1$, $a_2$, and $a_3$. Denote by $C_n(T_1), C_n(T_2), C_n(T_3)$ and $C_n(T_4)$ the I-type, II-type, III-type and IV-type of $C_n(S)$, respectively. We first consider the automorphism group of IV-type of $C_n(S)$.

**(i) The Automorphism Group of $C_n(T_4)$**

Clearly, $|N^+(T_4)| \geq 3$. If $|N^+(T_4)| = 3$, we have $N^+(T_4) = \{2a_1 = a_2 + a_3, 2a_2 = a_1 + a_3, 2a_3 = a_1 + a_2\}$. This is in accordance with (I). If $|N^+(T_4)| = 4$, we may get $2a_1 = a_2 + a_3$ and $2a_2 = 2a_3$, this agrees with (II). So it suffices to consider the following two cases.

**Case 1.** $|N^+(T_4)| = 6$.

In this case, $N^+(T_4) = \{a_1 + a_2, a_1 + a_3, a_2 + a_3, 2a_1, 2a_2, 2a_3\}$. So for every pair $a_i$ and $a_j (a_i \neq a_j)$, $a_i + a_j$ is the unique common out-adjacency vertex of $a_i$ and $a_j$, and $2a_i$ has a unique in-adjacency vertex $a_i$ in $T_4$. By Lemma 3 and Corollary 1, for $\tau \in \Omega(T_4)$, we have $\tau(a_i+a_j) = \tau(a_i)+\tau(a_j)$. Then by Theorem 1, we obtain $\Omega(T_4) = Aut_{T_4}(Z_n)$.

**Case 2.** $|N^+(T_4)| = 5$.

In this case, $N^+(T_4)$ has two possiblities: $N^+(T_4) = \{2a_1 = a_2+a_3, a_1+a_2, a_1+a_3, 2a_2, 2a_3\}$ or $N^+(T_4) = \{a_1+a_2, a_1+a_3, a_2+a_3, 2a_1 = 2a_2, 2a_3\}$. But the latter coincides with (III). So we need only to deal with the former.

By Corollary 1, for $\tau \in \Omega(S)$, we have $\tau(2a_2) = 2\tau(a_2)$ and $\tau(2a_3) = 2\tau(a_3)$. Because $(\tau(a_2), \tau(a_2 + a_3))$ is an arc of $C_n(T_4)$, there must be a certain $a_2' \in T_4$ such that $\tau(a_2 + a_3) = \tau(a_2) + \tau(a_2')$. Similarly, there is some $a_3' \in T_4$ such that $\tau(a_2 + a_3) = \tau(a_3) + \tau(a_3')$. Hence

$$\tau(a_2 + a_3) = \tau(a_2) + \tau(a_2') = \tau(a_3) + \tau(a_3').$$

We conclude that $a_2' = a_3$ and $a_3' = a_2$ (i.e., $\tau(a_2 + a_3) = \tau(a_2) + \tau(a_3)$). Otherwise, $a_2' = a_2$ or $a_3' = a_3$, say, $a_2' = a_2$. Then $\tau(a_2 + a_3) =$

$\tau(a_2) + \tau(a_2) = \tau(2a_2)$. This means $a_2 = a_3$, a contradiction. By the same technique, one can easily check that for any $\tau \in \Omega(T_4)$, $\tau(2a_1) = 2\tau(a_1), \tau(a_1 + a_2) = \tau(a_1) + \tau(a_2)$ and $\tau(a_1 + a_3) = \tau(a_1) + \tau(a_3)$. By Theorem 1, we have $\Omega(T_4) = Aut_{T_4}(Z_4)$.

By the above discussions, we get the following result.

**Theorem 3.** *The automorphism group of IV-type $C_n(T_4)$ is*

$$AutC_n(T_4) = L(Z_n)Aut_{T_4}(Z_n)$$

Since $Aut_{T_4}(Z_n) \cong \{\lambda \in Z_n^* \mid \lambda T_4 = T_4\}$, $Aut_{T_4}(Z_n)$ can be easily found if $T_4$ is given. In fact, $Aut_{T_4}(Z_n)$ can also be described more explicitly. Here we do not go into details.

**(ii) The Automorphism Group of $C_n(T_1)$**

Set $T_1 = \{a_1, a_2, a_3\}$. In this case, we have $3a_1 = 3a_2 = 3a_3$. By simple computation, we have $T_1 = \{a_1, \frac{n}{3} + a_1, \frac{2}{3}n + a_1\}$. Since $T_1$ generates $Z_n$, there is a certain $a_i \in T_1$, say $a_1$, such that $a_1$ is relatively prime to $n$. Let $T_1' = a_1^{-1}T_1 = \{1, \frac{n}{3} + 1, \frac{2}{3}n + 1\}$, then $C_n(T_1) \cong C_n(T_1')$. Now we depict $C_n(T_1')$ in Figure 1.

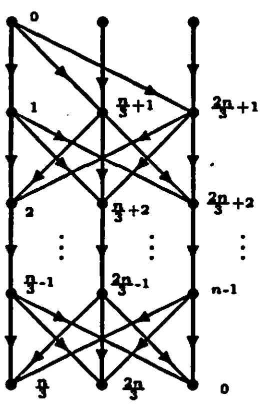

*Figure* 1

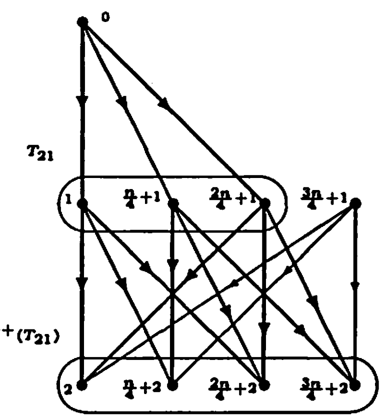


*Figure* 2

Let $V_i = \{i, \frac{n}{3} + i, \frac{2}{3}n + i\}(1 \le i < \frac{n}{3} - 1)$ and $V_{\frac{n}{3}-1} = \{\frac{n}{3}, \frac{2}{3}n\}$. Denote by $S_{V_i}$ the symmetric group on $V_i$, $i = 1, 2, ..., \frac{n}{3} - 1$. It is easy to see from Figure 1 that $\Omega(T_1') = S_{V_1} \times S_{V_2} \times \cdots \times S_{V_{\frac{n}{3}-1}}$. So we conclude that

**Theorem 4.** *The automorphism group of $C_n(T_1)$ is*

$$AutC_n(T_1) = L(Z_n)(S_{V_1} \times S_{V_2} \times \cdots \times S_{V_{\frac{n}{3}-1}}),$$

*and so* $|AutC_n(T_1)| = 6^{\frac{n}{3}-1}2n$.

By the way, we mention that $AutC_n(T_1)$ is isomorphic to the wreath product $Z_{\frac{n}{3}}$ wr $S_{V_1}$.

**(iii) The Automorphism Group of $C_n(T_2)$**

Set $T_2 = \{a_1, a_2, a_3\}$, then $2a_1 = a_2 + a_3$, $2a_2 = 2a_3$. This implies $4a_1 = 4a_2 = 4a_3$. Set $T_{21} = \{1, \frac{n}{4}+1, \frac{2}{4}n+1\}$, $T_{22} = \{1, \frac{n}{4}+1, \frac{3}{4}n+1\}$ and $T_{23} = \{1, \frac{2}{4}n+1, \frac{3}{4}n+1\}$. Since $4x = 4a_1 \bmod(n)$ has four solutions: $a_1, \frac{n}{4}+a_1, \frac{2}{4}n+a_1$ and $\frac{3}{4}n+a_1$, it is not difficult to see that $C_n(T_2)$ is isomorphic to one of $C_n(T_{21}), C_n(T_{22})$ and $C_n(T_{23})$. Hence it suffices to determine $AutC_n(T_{2i})$ $(i = 1, 2, 3)$ because $T_2$ generates $Z_n$, we can assume that $\gcd(a_1, n) = 1$.

First we consider $C_n(T_{21})$. To be explicit, we depict $T_{21} \cup N^+(T_{21})$ in Figure 2 along with the edges of $C_n(T_{21})$ in it. For $\tau \in \Omega(T_{21})$, since $\tau(0) = 0$, we have $\tau(T_{21}) = T_{21}$ and $\tau(N^+(T_{21})) = N^+(T_{21})$. So $\tau$ maps the in-neighbour set of $N^+(T_{21})$ to itself, i.e., $\tau(T_{21} \cup \{\frac{3}{4}n+1\}) = T_{21} \cup \{\frac{3}{4}n+1\}$. Hence $\tau(\frac{3}{4}n+1) = \frac{3}{4}n+1$. Thus we claim that

**Conclusion 1.** $\frac{3}{4}n+1$ is a fixed vertex of $\Omega(T_{21})$.

By similar method, we have

**Conclusion 2.** $\frac{2}{4}n+1$ is a fixed vertex of $\Omega(T_{22})$.

**Conclusion 3.** $\frac{1}{4}n+1$ is a fixed vertex of $\Omega(T_{23})$.

Under the assumption of the $T_2$, we have $4|n$, so $\gcd(\frac{2}{4}n+1, n) = 1$. If $8|n$, we also have $\gcd(\frac{3}{4}n+1, n) = \gcd(\frac{n}{4}+1, n) = 1$. Thus by Theorem 2, we deduce that every vertex of $C_n(T_{22})$ is fixed vertex of $\Omega(T_{22})$ and so $\Omega(T_{22}) = \{I\}$ (the identical element group). Similarly, if $8|n$, $\Omega(T_{21}) = \Omega(T_{23}) = \{I\}$.

In the following, we assume that $2^2||n$ (i.e., $2^2|n$ but $2^3\nmid n$). We continue to consider $C_n(T_{21})$. According to assumption, $\gcd(\frac{3}{4}n+1, n) = 2$ or $4$. Hence we have to consider two situations.

**Case 1.** $\gcd(\frac{3}{4}n+1, n) = 2$.

In this case, $\frac{3}{4}n+1$ generates the subgroup $\langle 2\rangle$ of $Z_n$. Since $\frac{3}{4}n+1$ is a fixed vertex of $\Omega(T_{21})$, by Theorem 2, each vertex of $\langle 2\rangle$ is fixed. Thus by Corollary 3, for $\tau \in \Omega(T_{21})$ and $u+1 \in \langle 2\rangle + 1$ (or $u + (\frac{2}{4}n+1) \in \langle 2\rangle + (\frac{2}{4}n+1)$), we have $\tau(u+1) = u + \tau(1)$ (or $\tau(u + (\frac{2}{4}n+1)) = u + \tau(\frac{2}{4}n+1)$). Additionally, note that $\tau(\{1, \frac{2}{4}n+1\}) = \{1, \frac{2}{4}n+1\}$ and $Z_n = \langle 2\rangle \cup (\langle 2\rangle + 1) = \langle 2\rangle \cup (\langle 2\rangle + (\frac{2}{4}n+1))$, $\tau$ is uniquely determined by its action on $A = \{1, \frac{2}{4}n+1\}$. Define

$$\pi_0(v) = \begin{cases} u + \frac{2}{4}n + 1, & \text{if } v = u+1 \in \langle 2\rangle + 1 \\ v, & \text{otherwise.} \end{cases}$$

One can simply verify that $\pi_0$ is an automorphism of $C_n(T_{21})$. Notice that $\pi_0(u + (\frac{2}{4}n+1)) = u + \frac{2}{4}n + \pi_0(1) = u + 1$ $(\forall u \in \langle 2\rangle)$, $\pi_0^2 = I$. Thus we conclude that $\Omega(T_{21}) = \{I, \pi_0\} \cong S_A \cong S_2$ (where $S_2$ is the symmetric group on 2 element set).

**Case 2.** $\gcd(\frac{3}{4}n+1,n)=4$.

In this case, $\frac{3}{4}n+1$ generates a subgroup $\langle 4\rangle$ whose vertices are all the fixed vertices of $\Omega(T_{21})$. Since $4\nmid(\frac{n}{4}+1)-1$, $(\frac{2}{4}n+1)-1$ and $(\frac{2}{4}n+1)-(\frac{n}{4}+1)$, it is clear that $\langle 4\rangle+1, \langle 4\rangle+(\frac{n}{4}+1)$ and $\langle 4\rangle+(\frac{2}{4}n+1)$ must be distinct cosets of $\langle 4\rangle$. Hence

$$Z_n=\langle 4\rangle\cup((\langle 4\rangle+1)\cup((\langle 4\rangle+(\frac{n}{4}+1))\cup((\langle 4\rangle+(\frac{2}{4}n+1)).$$

According to Corollary 3, for $\tau\in\Omega(T_{21})$, $\tau$ is uniquely determined by its action on $T_{21}=\{1,\frac{n}{4}+1,\frac{2}{4}n+1\}$. Let $\mathcal{S}_{T_{21}}$ be the symmetric group on $T_{21}$. For $\pi\in\mathcal{S}_{T_{21}}$ and $v\in Z_n$, define

$$\tau(v)=\begin{cases} v, & \text{if } v\in\langle 4\rangle\\ u+\pi(1), & \text{if } v=u+1\in\langle 4\rangle+1\\ u+\pi(\frac{n}{4}+1), & \text{if } v=u+\frac{n}{4}+1\in\langle 4\rangle+(\frac{n}{4}+1)\\ u+\pi(\frac{2}{4}n+1), & \text{if } v=u+\frac{2}{4}n+1\in\langle 4\rangle+(\frac{2}{4}n+1).\end{cases}$$

One can directely check that $\tau$ is an automorphism of $\Omega(T_{21})$. So we have $\Omega(T_{21})\cong\mathcal{S}_{T_{21}}\cong\mathcal{S}_3$.

Now we turn to consider $\Omega(T_{23})$. Remember that $2^2||n$, $\gcd(\frac{n}{4}+1,n)=2$ or 4. According to Conclusion 3, $\frac{n}{4}+1$ is a fixed vertex of $\Omega(T_{23})$. By using the same method as in Case 1, we claim that $\Omega(T_{23})\cong\mathcal{S}_{\{1,\frac{3}{4}n+1\}}\cong\mathcal{S}_2$ if $\gcd(\frac{n}{4}+1,n)=2$, or $\Omega(T_{23})\cong\mathcal{S}_{T_{23}}\cong\mathcal{S}_3$ otherwise.

Now we summarize the above discussions in the following theorem.

**Theorem 5.** *Let $T_{21}=\{1,\frac{n}{4}+1,\frac{2}{4}n+1\}$, $T_{22}=\{1,\frac{n}{4}+1,\frac{3}{4}n+1\}$ and $T_{23}=\{1,\frac{2}{4}n+1,\frac{3}{4}n+1\}$. Then $C_n(T_1)$ is isomorphic to the one of $C_n(T_{21}), C_n(T_{22})$ and $C_n(T_{23})$, and so*

$$AutC_n(T_1)=\begin{cases} L(Z_n)\mathcal{S}_2, & \text{if } 2^2||n,\ T_1=T_{21} \text{ and } \gcd(\frac{3}{4}n+1,n)=2\\ L(Z_n)\mathcal{S}_2, & \text{if } 2^2||n,\ T_1=T_{23} \text{ and } \gcd(\frac{n}{4}+1,n)=2\\ L(Z_n)\mathcal{S}_3, & \text{if } 2^2||n,\ T_1=T_{21} \text{ and } \gcd(\frac{3}{4}n+1,n)=4\\ L(Z_n)\mathcal{S}_3, & \text{if } 2^2||n,\ T_1=T_{23} \text{ and } \gcd(\frac{n}{4}+1,n)=4\\ L(Z_n), & \text{otherwise.}\end{cases}$$

### (iv) The Automorphism Group of $C_n(T_3)$

Let $T_3=\{a_1,a_2,a_3\}$, then $2a_1=2a_2$ but $2a_3\neq a_1+a_3$. In this case, $N^+(T_3)=\{a_1+a_2,a_1+a_3,a_2+a_3,2a_1=2a_2,2a_3\}$. Since $2a_3$ is the unique vertex in $N^+(T_3)$ which has only one in-adjacency vertex $a_3$ in $T_3$, $2a_3$ must be a fixed vertex of $\Omega(T_3)$. This implies that $a_3$ is also a fixed vertex of $\Omega(T_3)$. Then by Theorem 2 and Corollary 2, $AutC_n(T_4)=L(Z_n)$ if $\gcd(a_3,n)=1$. In the following, we assume that $\gcd(a_3,n)=\alpha$. Let $\gcd(a_1,a_2,n)=\beta$. Notice that $2a_1=2a_2 \bmod(n)$, it is not difficult to show that $\gcd(a_1,n)=\beta$ or $\gcd(a_2,n)=\beta$. Without loss of generality, we assume that $\gcd(a_1, n) = \beta$, then $\langle a_1, a_2 \rangle = \langle a_1 \rangle = \langle \beta \rangle$. Recall that $C_n(T_3)$ is strongly connected, we have $\gcd(\alpha, \beta) = 1$ and

$$Z_n = \langle \alpha \rangle \cup (\langle \alpha \rangle + a_1) \cup (\langle \alpha \rangle + 2a_1) \cup \cdots \cup (\langle \alpha \rangle + (\alpha - 1)a_1).$$

Since each vertex of $\langle \alpha \rangle$ is fixed vertex of $\Omega(T_3)$, by Corollary 3, for $\tau \in \Omega(T_3)$ and $u + ia_1 \in \langle \alpha \rangle + ia_1$, we have $\tau(u + ia_1) = u + \tau(ia_1)$. Thus $\tau$ is uniquely determined by its action on $\{a_1, 2a_1, ..., (\alpha - 1)a_1\}$. Now we distinguish two cases.

**Case 1.** $\langle \alpha \rangle + a_1 = \langle \alpha \rangle + a_2$.

Clearly, for $\tau \in \Omega(T_3)$, $\tau$ maps every $a_3$-arc to itself. Thus $\tau$ is also an automorphism of $C_n(\{a_1, a_2\})$. We illustrate $C_n(T_3)$ in Figure 3 (Note that $2a_1 = 2a_2$, $a_2 = (\frac{n}{2\beta} + 1)a_1$).

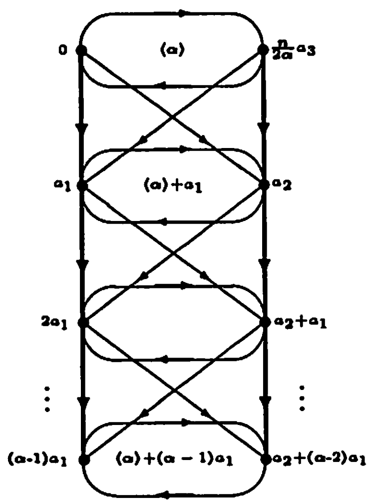


*Figure* 3

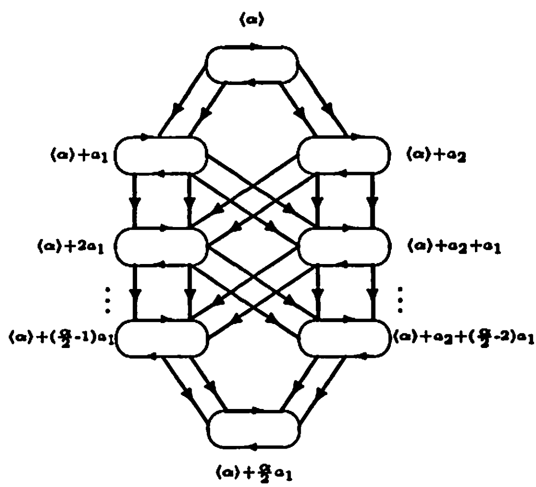


*Figure* 4

From Figure 3, it is easy to see that $\tau(\{ia_1, a_2+(i-1)a_1\}) = \{ia_1, a_2+(i-1)a_1\}$. So $\tau$ acting on $\{a_1, a_2;\ 2a_1, a_2+a_1;\ ...,\ (\alpha-1)a_1, a_2+(\alpha-2)a_1\}$ is the product of some transpositions as follows: $(a_1\ a_2), (2a_1\ a_2 + a_1), (3a_1\ a_2 + 2a_1), ..., ((\alpha - 1)a_1\ a_2 + (\alpha - 2)a_1)$.

For $u + ja_1 \in \langle \alpha \rangle + ja_1$ $(0 \le j \le \alpha - 1)$ and $1 \le i \le \alpha - 1$, define

$$\tau_i(u + ja_1) = \begin{cases} u + a_2 + (j-1)a_1, & \text{if } j = i \\ u + ja_1, & \text{if } j \neq i. \end{cases}$$

Notice that $a_2 = (\frac{n}{2\beta} + 1)a_1$ and $a_2 - a_1 \in \langle \alpha \rangle$, we can directely examine that $\tau_i$ is an automorphism of $\Omega(T_3)$ and $\tau_i^2 = I$. Therefore, $\Omega(T_3) =$

$\langle\tau_1\rangle \times \langle\tau_2\rangle \times \cdots \times \langle\tau_{\alpha-1}\rangle \cong S_2^{\alpha-1}$ (where $S_2$ is the symmetric group on a two element subset).

**Case 2.** $\langle\alpha\rangle + a_1 \neq \langle\alpha\rangle + a_2$.

In this case, by Corollary 3, for any $\tau \in \Omega(T_3)$, $\tau$ must permute the cosets of $\langle\alpha\rangle$. We can regard the cosets of $\langle\alpha\rangle$ as vertices and $\{a_1 + \langle\alpha\rangle, a_2 + \langle\alpha\rangle\}$ as arc symbol set. Since $2a_1 + \langle\alpha\rangle = 2a_2 + \langle\alpha\rangle$, we have $a_2 = (\frac{\alpha}{2}+1)a_1 + \langle\alpha\rangle$. Thus $\tau$ acting on $C_n(T_3)$ induces an action of $\tau$ on $C_\alpha(\{1, \frac{\alpha}{2}+1\})$. From Figure 4, we claim that $\tau$ is the product of some transpositions of cosets as follows: $(\langle\alpha\rangle + a_1\ \ \langle\alpha\rangle + a_2), (\langle\alpha\rangle + 2a_1\ \ \langle\alpha\rangle + a_2 + a_1), \ldots, (\langle\alpha\rangle + (\frac{\alpha}{2} - 1)a_1\ \ \langle\alpha\rangle + a_2 + (\frac{\alpha}{2} - 2)a_1)$. This means that $\Omega(T_3) \cong S_2^{\frac{\alpha}{2}-1}$. Notice that, if $\alpha = 1$, $a_1 + \langle\alpha\rangle$ is certainly equal to $a_2 + \langle\alpha\rangle$, we have $\Omega(T_3) \cong S_2^{\alpha-1} = \{I\}$. By combining the discussions in Case 1 and Case 2, we have

**Theorem 6.** *Let* $\gcd(a_3, n) = \alpha$. *Then the automorphism group of* $C_n(T_3)$ *is*

$$AutC_n(T_3) \cong \begin{cases} L(Z_n) \times S_2^{\alpha-1}, & \text{if } a_1 + \langle\alpha\rangle = a_2 + \langle\alpha\rangle \\ L(Z_n) \times S_2^{\frac{\alpha}{2}-1}, & \text{otherwise.} \end{cases}$$

## References


[1] A. Ádám , Research Problem 2-10, *J. Combinatorial Theory*, 2 (1967), 393.

[2] B. Alspach and T.D. Parsons, Isomorphism of Circulant Graphs and Digraphs, *Discrete Mathematics*, **25** (1979), 97–108.

[3] B. Elspas and J. Turner, Graphs with Circulant Adjacency Matrices, *J. Combinatorial Theory*, **9** (1970), 297–307.

[4] D. Witte and J.A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, *Discrete Mathematics*, **51** (1984), 283–304.

[5] F. Boesch and R. Tindell, Circulants and Their Connectivities, *J. Graph Theory*, **8** (1984), 487–499.

[6] N.L. Biggs, *Algebraic Graph Theory*, Cambridge Univ., Press (1974).

[7] P.J. Cameron, Autmorphism Groups, in: *Selected Topics in Graph Theory Vol.II* (edited by L.W. Beineke and R.J. Wilson), Academic Press, London, 1983.

[8] Sun Liang, Isomorphisms of circulant digraphs with degree 2, *J. Beijing Polytechnic Institute*, (1984)