The spectrum of self-converse MTS

Yanxun Chang
Institute of Math., Hebei Normal College
Guihua Yang
Basic Teaching Bureau
Hebei Institute of Finance and Economics
Qingde Kang
Institute of Math., Hebei Normal College
(Shijiazhuang 050091, P. R. China)

ABSTRACT. A Mendelsohn triple system, $MTS(v) = (X, \mathcal{B})$, is called self-converse if it and its converse $(X, \mathcal{B}^{-1})$ are isomorphic, where $\mathcal{B}^{-1} = \{\langle z, y, x\rangle; \langle x, y, z\rangle \in \mathcal{B}\}$. In this paper, the existence spectrum of self-converse $MTS(v)$ is given, which is $v \equiv 0$ or $1 \pmod 3$ and $v \neq 6$.

1 Introduction

Let X be a v-set, $v \geq 3$. A *cyclic triple* from X is a collection of three ordered pairs (x, y), (y, z) and (z, x), where x, y, z are distinct elements of X. It is denoted by $\langle x, y, z\rangle$ (or $\langle y, z, x\rangle$ or $\langle z, x, y\rangle$). A *Mendelsohn triple system* on X is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of some cyclic triples from X such that each ordered pair of distinct elements of X belongs to exactly one cyclic triple of $\mathcal{B}$. This system is denoted by $MTS(v)$. It is easy to see that $|\mathcal{B}| = \frac{v(v-1)}{3}$, and it is well known that an $MTS(v)$ exists if and only if $v \equiv 0, 1 \pmod 3$ and $v \neq 6$ (see [1]). For an $MTS(v) = (X, \mathcal{B})$ define

$$\mathcal{B}^{-1} = \{B^{-1} = \langle z, y, x\rangle; B = \langle x, y, z\rangle \in B\}.$$

Obviously, $(X, \mathcal{B}^{-1})$ is also an $MTS(v)$, which is called the *converse* of $(X, \mathcal{B})$. If there exists an isomorphism mapping f from $(X, \mathcal{B})$ to $(X, \mathcal{B}^{-1})$, then the $MTS(v) = (X, \mathcal{B})$ is called *self-converse* and denoted by $SCMTS$ $(v) = (X, \mathcal{B}, f)$. To prove a system $(X, \mathcal{B}, f)$ is self-converse we only need to show that $f(B)^{-1} \in \mathcal{B}$ for any $B \in \mathcal{B}$.

For what orders do self-converse MTS exist? The problem was posed by C.J. Colbourn and A. Rosa in their survey (as open problem 5, see [2]). N.P. Carne [3] settles the problem when $n \equiv 0, 4 \pmod{12}$. In this paper, we will complete the existence spectrum of $SCMTS(v)$.

For convenience, we define some terminology. Let $(X, \mathcal{B}, f)$ be an $SCMTS(v)$. For $x \in X$, if $f^k(x) = x$ but $f^s(x) \neq x$ when $s < k$, then denote $p_f(x) = k$. When $p_f(x) = 1$, we call x a *fixed point* of the $SCMTS(v)$. If there exists an $SCMTS(u) = (Y, \mathcal{A}, g)$ such that $Y \subset X$, $\mathcal{A} \subset \mathcal{B}$ and $g = f|_Y$, then the sub-system is called a sub-$SCMTS(u)$ of the $SCMTS(v)$. An $SCMTS(v)$ is called

A-type if $p_f(x) \leq 2$ for any $x \in X$ and $|\{x \in X; p_f(x) = 1\}| \geq 2$;

B-type if $p_f(x) \leq 2$ for any $x \in X$ and there are a sub-$SCMTS(4)$ and another fixed point, which is not contained in the sub-$SCMTS(4)$;

C-type if $p_f(x) \leq 2$ for any $x \in X$ and there is a sub-$SCMTS(3)$, in which three points are all fixed.

Note: $MTS(3)$ and $MTS(4)$ are all unique (up to isomorphic), i.e., $MTS(3) = \{\langle a, b, c\rangle, \langle c, b, a\rangle\}$ and $MTS(4) = \{\langle u, v, s\rangle, \langle u, s, t\rangle, \langle u, t, v\rangle, \langle v, t, s\rangle\}$. But as self-converse system, $MTS(3)$ has six mappings, i.e. all permutations on $\{a, b, c\}$; and $MTS(4)$ has twelve mappings, i.e. all permutations in the form $(^*)(^*)(^*,^*)$ or $(^*,^*,^*,^*)$ on $\{u, v, s, t\}$. Here, the mapping f in C-type sub-$SCMTS(3)$ is stipulated for identical mapping, and the mapping g in B-type sub-$SCMTS(4)$ can be only the form $(^*)(^*)(^*,^*)$ by $p_f(x) \leq 2$ for any x. Obviously, by definition, C-type implies A-type.

2 Case $v \equiv 1$, 3 and 4 (mod 6)

Theorem 1. *For $v \equiv 1, 3 \pmod 6$ there exists an $SCMTS(v)$, which is C-type.*

Proof: It is well known that there exists a Steiner triple system $STS(v) = (X, \mathcal{A})$ for $v \equiv 1, 3 \pmod 6$. Let $\mathcal{B} = \{\langle x, y, z\rangle, \langle z, y, x\rangle; \{x, y, z\} \in \mathcal{A}\}$ and f be an identical transformation on X. Then $(X, \mathcal{B}, f)$ is an $SCMTS(v)$. Obviously, this system is C-type. □

Lemma 1. *If there exist both $SCMTS(m)$ and $SCMTS(n)$ then there exists an $SCMTS(mn)$. Furthermore, when $SCMTS(m)$ is A-type, if $SCMTS(n)$ is A-type (or C-type) then the obtained $SCMTS(mn)$ is A-type (or C-type) too.*

Construction: Let X be a m-set and Y be a n-set. Given $SCMTS(m) = (X, \mathcal{A}, f)$ and $SCMTS(n) = (Y, \mathcal{B}, g)$. Construct a cyclic triple system J on the set $X \times Y$ as follows:

part 1. $\langle (x, u), (y, v), (z, w)\rangle$ with $\langle x, y, z\rangle \in \mathcal{A}$ and $\langle u, v, w\rangle \in \mathcal{B}$. This gives $mn(m-1)(n-1)/3$ cyclic triples.

part 2. $\langle (x,u),(x,v),(x,w)\rangle$ with $\langle u,v,w\rangle \in \mathcal{B}$ and $x \in X$. This gives $mn(n-1)/3$ cyclic triples.

part 3. $\langle (x,u),(y,u),(z,u)\rangle$ with $\langle x,y,z\rangle \in \mathcal{A}$ and $u \in Y$. This gives $mn(m-1)/3$ cyclic triples.

Define mapping $F: X \times Y \to X \times Y$ such that $F(x,y) = (f(x), g(y))$ for $x \in X$ and $y \in Y$. Then $(X \times Y, \mathcal{J}, F)$ is an $SCMTS(mn)$ as expected.

Proof: The system $\mathcal{J}$ contains

$$mn(m-1)(n-1)/3 + mn(n-1)/3 + mn(m-1)t3 = mn(mn-1)/3$$

cyclic triples, just the number as expected. For any $x \neq y \in X$, $u \neq v \in Y$, each ordered pair $((x,u),(x,v))$ appears in one block of part 2, each ordered pair $((x,u),(y,u))$ appears in one block of part 3 and each ordered pair $((x,u),(y,v))$ appears in one block of part 1. Thus $(X \times Y, \mathcal{J})$ is an $MTS(mn)$.

The triple $B = \langle (x,u),(y,v),(z,w)\rangle \in \mathcal{J}$ implies both $\langle x,y,z\rangle \in \mathcal{A}$ and $\langle u,v,w\rangle \in \mathcal{B}$. Thus $\langle f(z), f(y), f(x)\rangle \in \mathcal{A}$ and $\langle g(w), g(v), g(u)\rangle \in \mathcal{B}$, thereby $\langle (f(z),g(w)),(f(y),g(v)),(f(x),g(u))\rangle = \langle F(z,w),F(y,v),F(x,u)\rangle$ appears in part 1, i.e., $F(B)^{-1} \in \mathcal{J}$. The triple $B = \langle (x,u),(x,v),(x,w)\rangle \in \mathcal{J}$ implies $x \in X$ and $\langle u,v,w\rangle \in \mathcal{B}$. Thus $\langle g(w),g(v),g(u)\rangle \in \mathcal{B}$, thereby $\langle (f(x),g(w)),(f(x),g(v)),(f(x),g(u))\rangle = \langle F(x,w),F(x,v),F(x,u)\rangle$ appears in part 2, i.e., $F(B)^{-1} \in \mathcal{J}$. Similar for the triple $\langle (x,u),(y,u),(z,u)\rangle$. Therefore $(X \times Y, \mathcal{J}, F)$ is an $SCMTS(mn)$.

If $p_f(x) \le 2$ and $p_g(u) \le 2$, i.e. $f^2(x) = x$ and $g^2(u) = u$, for any $x \in X$ and $u \in Y$, then $F^2(x,u) = (f^2(x), g^2(u)) = (x,u)$, i.e. $p_F(x,u) \le 2$, for any $(x,u) \in X \times Y$. Suppose $(X, \mathcal{A}, f)$ is A-type, which has a fixed point x_0. If $(Y, \mathcal{B}, g)$ is also A-type, in which $g(u_0) = u_0$ and $g(v_0) = v_0$, then (x_0,u_0) and (x_0,v_0) are fixed points of $(X \times Y, \mathcal{J}, F)$. If $(Y, \mathcal{B}, g)$ is C-type, in which there exists a sub-$SCMTS(3) = \{\langle a,b,c\rangle, \langle c,b,a\rangle\}$ and $g(a) = a$, $g(b) = b$, $g(c) = c$, then both $\langle (x_0,a),(x_0,b),(x_0,c)\rangle$ and $\langle (x_0,c),(x_0,b),(x_0,a)\rangle$ appear in part 2 of $(X \times Y, \mathcal{J}, F)$ and $F(x_0,a) = (x_0,a)$, $F(x_0,b) = (x_0,b)$, $F(x_0,c) = (x_0,c)$. □

Theorem 2. *For* $v \equiv 4 \pmod 6$ *there exists an* $SCMTS(v)$*, which is A-types, and is still B-type when* $v > 4$.

Construction: Let $v = 6t + 4 = 3(2t+1) + 1$, $Z_{2t+1} = \{0, 1, \ldots, 2t\}$, $Z_3 = \{0,1,2\}$ and $\infty \notin Z_{2t+1} \times Z_3$. Construct a cyclic triple system $\mathcal{B}$ on the set $X = \{\infty\} \cup (Z_{2t+1} \times Z_3)$ as follows (The element (x,i) in $Z_{2t+1} \times Z_3$ is denoted by x_i, briefly):

part 1. $\cup\{MTS(4)$ on set $\{\infty\} \cup (\{x\} \times Z_3); x \in Z_{2t+1}\}$;

part 2. $\langle (x+y)_i, (x-y)_i, x_{i+1}\rangle$ with $x \in Z_{2t+1}$, $1 \le y \le t$ and $i \in Z_3$;

part 3. $\langle (x-y)_i, (x+y)_i, x_{i-1}\rangle$ with $x \in Z_{2t+1}$, $1 \le y \le t$ and $i \in Z_3$.

Define mapping f on X as follows:

$f(\infty) = \infty, f(x_0) = x_0, f(x_1) = x_2$ and $f(x_2) = x_1$ for any $x \in Z_{2t+1}$.

Then $(X, \mathcal{B}, f)$ is an $SCMTS(6t+4)$ as expected.

Proof: The system $\mathcal{B}$ contains

$$4(2t+1) + 3t(2t+1) + 3t(2t+1) = (6t+4)(6t+3)/3$$

cyclic triples, just the number as expected. Obviously, all ordered pairs (∞, x_i), (x_i, ∞) and (x_i, x_j) are contained in triples of part 1, where $x \in Z_{2t+1}$ and $i \ne j \in Z_3$. Now, let us consider any ordered pair $P = (u_i, v_j)$, $u \ne v \in Z_{2t+1}, i, j \in Z_3$.

i) When $P = (u_i, v_i)$, there is unique $x \in Z_{2t+1}$ such that $u + v = 2x$. If $u - x \le t$, let $u - x = y$, then $u = x + y$, $v = x - y$ and P appears in part 2. If $u - x > t$, then $x - u = y \le t$, thus $u = x - y$, $v = x + y$ and P appears in part 3.

ii) When $P = (u_i, v_{i+1})$, if $u - v = y \le t$ then $u = x$, $v = x - y$ and P appears in part 3; if $u - v > t$, let $v - u = y$, then $u = x - y$, $v = x$ and P appears in part 2.

iii) When $P = (u_i, v_{i-1})$, if $u - v = y \le t$ then $u = x + y$, $v = x$ and P appears in part 3; if $u - v > t$, let $v - u = y$, then $u = x$, $v = x + y$ and P appears in part 2.

Thus the system $(X, \mathcal{B})$ is an $MTS(6t+4)$.

Under the mapping f, ∞ and all x_0 are fixed points, and $p_f(x_1) = 2$, $p_f(x_2) = 2$ for any $x \in Z_{2t+1}$. Obviously, each $MTS(4)$ in part 1 is a sub-$SCMTS(4)$. And the mapping f gives the following correspondences:

$\langle (x+y)_0, (x-y)_0, x_1\rangle \longleftrightarrow$ the converse of $\langle (x-y)_0, (x+y)_0, x_2\rangle$,

$\langle (x+y)_1, (x-y)_1, x_2\rangle \longleftrightarrow$ the converse of $\langle (x-y)_2, (x+y)_2, x_1\rangle$,

$\langle (x+y)_2, (x-y)_2, x_0\rangle \longleftrightarrow$ the converse of $\langle (x-y)_1, (x+y)_1, x_0\rangle$,

Thus, the system $(X, \mathcal{B}, f)$ is an A-type $SCMTS(6t+4)$. Furthermore, when $t \ge 1$, besides a sub-$SCMTS(4)$ in part 1 there is another fixed point, thereby the system is still B-type. □

Corollary. *If there exists an $SCMTS(v)$ then there exist $SCMTS(3v)$, $SCMTS(4v)$ and $SCMTS(12v)$. And if the $SCMTS(v)$ is A-type (or C-type) then the $SCMTS(kv)$ above is the same type too, where $k = 3, 4, 12$.*

Proof: There exists a C-type $SCMTS(3)$ which is also A-type, by Theorem 1. There exists an A-type $SCMTS(4)$ by Theorem 2. Thus there exists an A-type $SCMTS(3 \times 4)$ by Lemma 1. Thereby, the conclusion holds by Lemma 1. □

3 Case $v \equiv 0 \pmod{18}$

Lemma 2. *There exists an A-type $SCMTS(18)$.*

Construction: Let $X = \{a, b, c, d\} \cup Z_{14}$. Define mapping $X \to X$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$ and $f(i) = 13 - i$ for $i \in Z_{14}$. The cyclic triple system $\mathcal{B}$ consists of the following triple:

part 1. $MTS(4)$ on $\{a, b, c, d\}$.

part 2. $\langle a, i, 13 - i\rangle$ and $\langle b, i, i + 8\rangle$, $i \in Z_{14}$.

part 3. $\langle c, i, i + 4\rangle$ and $\langle d, 9 - i, 13 - i\rangle$, $0 \le i \le 4$ and $10 \le i \le 11$.

part 4. $\langle c, i, i + 11\rangle$ and $\langle d, 2 - i, 13 - i\rangle$, $5 \le i \le 6$.

part 5. $\langle c, i, i + 3\rangle$ and $\langle d, 10 - i, 13 - i\rangle$, $7 \le i \le 9$.

part 6. $\langle c, 12, 13\rangle$, $\langle c, 13, 9\rangle$, $\langle d, 0, 1\rangle$, $\langle d, 4, 0\rangle$.

part 7. $\langle i, i{+}3, i{+}8\rangle$, $\langle i, i{+}10, i{+}1\rangle$, $\langle 13{-}i, 5{-}i, 10{-}i\rangle$, $\langle 13{-}i, 12{-}i, 3{-}i\rangle$, $i = 0, 11$.

part 8. $\langle i, i + 7, i + 2\rangle$ and $\langle 13 - i, 11 - i, 6 - i\rangle$, $i = 0, 5, 7, 12$.

part 9. $\langle j, j{+}i, j{+}2i\rangle$ and $\langle 13{-}2i{-}j, 13{-}i{-}j, 13{-}j\rangle$, where $i = 1, j = 1, 3$; $i = 2, j = 3$; $i = 6, j = 0, 3$; $i = 9, j = 1$; $i = 11, j = 2$ and $i = 13$, $j = 4, 6$.

part 10. $\langle 5, 6, 9\rangle$, $\langle 0, 2, 9\rangle$, $\langle 1, 11, 4\rangle$, $\langle 2, 7, 4\rangle$, $\langle 8, 4, 7\rangle$, $\langle 13, 4, 11\rangle$, $\langle 12, 9, 2\rangle$, $\langle 11, 9, 6\rangle$.

Proof: Firstly, $(X, \mathcal{B})$ is an $MTS(18)$ affirmatively, but this verification is tedious, which is deleted. For any triple B, it is easy to see that $f(B)^{-1}$ appears in the same part. Obviously, $f^2(x) = x$ holds for any $x \in X$ and a, b are fixed points. □

Theorem 3. *For $v \equiv 0 \pmod{18}$ there exists an $SCMTS(v)$, which is A-type.*

Proof: Let $v = 18t$, $t \ge 1$. Use induction method on t. When $t = 1$, there exists A-type $SCMTS(18)$ by Lemma 2. Suppose there exists an A-type $SCMTS(18r)$ for any $r < t$. Let us consider the next order $v = 18t$. When $t = 3r + 1$, $v = 18(3r + 1)$. Since $3r + 1 \equiv 1$ or $4 \pmod 6$, there exists an A-type $SCMTS(3r + 1)$ by Theorem 1 or 2. Since there exists an A-type $SCMTS(18)$ by Lemma 2, thus there exists an A-type $SCMTS(18(3r{+}1))$ by Lemma 1. When $t = 3r + 2$, $v = 18(3r + 2) = 9(6r + 4)$. There exist both A-type $SCMTS(9)$ and $SCMTS(6r + 4)$ by Thorem 1 and 2, thus there exists an A-type $SCMTS(9(6r{+}4))$ Lemma 1. Finally, when $t = 3r$,

$v = 18 \cdot 3r = 3 \cdot 18r$. By induction hypothesis, since $r < t$, there exists an A-type $SCMTS(18r)$, thereby there exists an A-type $SCMTS(3 \cdot 18r)$ by Corollary. □

4 Stronger $SCMTS(6k+4)$

Lemma 3. *If there exists an A-type $SCMTS(n+1)$ then there exists a C-type $SCMTS(3n+1)$.*

Construction: Let $X = \{\infty\} \cup (Z_n \times Z_3)$, $\infty \notin Z_n \times Z_3$. By known condition, there exists an A-type $SCMTS(n+1) = (\{\infty\} \cup Z_n, \mathcal{A}, f)$, which contains two fixed points ∞ and 0, and $p_f(x) \leq 2$ for any $x \in Z_n \setminus \{0\}$. Now, construct a cyclic triple system $\mathcal{B}$ on the set X as follows:

part 1. $\langle (x,i),(y,i),(z,i) \rangle$ with $\langle x,y,z \rangle \in \mathcal{A}$ and $i \in Z_3$. (Whenever ∞ appears for x, y, z, omit the second coordinate i.)

part 2. $\langle (x,0),(y,1),(z,2) \rangle$ with $x,y,z \in Z_n$ and $x+y+z=0$.

part 3. $\langle (f(z),2),(f(y),1),(f(x),0) \rangle$ with $x,y,z \in Z_n$ and $x+y+z=0$.

Define $F(\infty) = \infty$, $F(x,i) = (f(x),i)$ for any $(x,i) \in Z_n \times Z_3$. Then $(X, \mathcal{B}, F)$ is an $SCMTS(3n+1)$ as expected.

Proof: $\mathcal{B}$ contains $3 \cdot \frac{n(n+1}{3} + n^2 + n^2 = 3n(3n+1)/3$ triples, just the number as expected. All ordered pairs $(\infty,(x,i)),((x,i),\infty)$ and $((x,i),(y,i))$ appears in cyclic triples of part 1. For ordered pair $P = ((x,i),(y,i+1))$, there is $z \in Z_n$ such that $x+y+z=0$, then P appears in triple $\langle (x,i),(y,i+1),(z,i-1) \rangle$ of part 2. For $P = ((x,i),(y,i-1))$, let $u = f(x)$, $v = f(y)$, $w = -(u+v)$ and $z = f(w)$. Since $f(u) = f^2(x) = x$ and $f(v) = f^2(y) = y$, thus P appears in triple $\langle (x,i),(y,i-1),(z,i+1) \rangle$ of part 3.

Triple $A = \langle (x,i),(y,i),(z,i) \rangle \in \mathcal{B}$ implies $\overline{A} = \langle x,y,z \rangle \in \mathcal{A}$, so $f(\overline{A})^{-1} = \langle f(z),f(y),f(x) \rangle \in \mathcal{A}$, thereby $\langle (f(z),i),(f(y),i),(f(x),i) \rangle \in$ part 1, i.e. $F(A)^{-1} \in \mathcal{B}$. Triple $B = \langle (x,0),(y,1),(z,2) \rangle \in \mathcal{B}$ implies $x+y+z=0$, so $\langle (f(z),2),(f(y),1),(f(x),0) \rangle \in$ part 3, i.e. $F(B)^{-1} \in \mathcal{B}$. For the triple $C = \langle (x,2),(y,1),(z,0) \rangle \in \mathcal{B}$, let $u = f(x)$, $v = f(y)$ and $w = f(z)$, then $f(u) = f^2(x) = x$, $f(v) = f^2(y) = y$ and $f(w) = f^2(z) = z$. Thus, $C = \langle (x,2),(y,1),(z,0) \rangle = \langle (f(u),2),(f(v),1),(f(w),0) \rangle \in B$ implies $u+v+w=0$ and $\langle (w,0),(v,1),(u,2) \rangle = \langle (f(z),0),(f(y),1),(f(x),2) \rangle \in$ part 2, that is $f(C)^{-1} \in \mathcal{B}$.

Finally, since $f(0) = 0$ and $0+0+0=0$, thus $F(0,i) = (0,i)$, $i \in Z_3$ and the $(X,\mathcal{B},f)$ contains a sub-$SCMTS(3) = \{\langle (0,0),(0,1),(0,2) \rangle, \langle (0,2),(0,1),(0,0) \rangle\}$ by part 2 and part 3 . And for any element $(x,i) \in Z_n \times Z_3$, $F^2(x,i) = (f^2(x),i) = (x,i)$, thus $p_F((x,i)) \leq 2$. The element ∞ is another fixed point. Thus, $(X,\mathcal{B},F)$ is C-type. □

Lemma 4. *If there exists a B-type $SCMTS(n+4)$ then there exists a C-type $SCMTS(3n+4)$, for $n > 0$.*

Construction: Given a B-type $SCMTS(n+4) = (\{u,v,s,t\} \cup Z_n, \mathcal{A}, f)$, which contains a sub-$SCMTS(4)$ on the sub-set $\{u,v,s,t\}$, and satisfies $p_f(x) \leq 2$ for any $x \in Z_n$, where $x = 0$ is fixed point. Construct a cyclic triple system $\mathcal{B}$ on set $X = \{u,v,s,t\} \cup (Z_n \times Z_3)$ as follows:

part 1. sub-$SCMTS(4)$ on $\{u,v,s,t\}$, say D.

part 2. $\langle (x,i),(y,i),(z,i)\rangle$ with $\langle x,y,z\rangle \in \mathcal{A} \setminus D$ and $i \in Z_3$. (whenever u,v,s,t appears for x,y,z, omit the second coordinate i.)

part 3. $\langle (x,0),(y,1),(z,2)\rangle$ and $\langle (f(x),2),(f(y),1),(f(x),0)\rangle$ with $x,y,z \in Z_n$ and $x+y+z=0$.

Define $F(u) = f(u)$, $F(v) = f(v)$, $F(s) = f(s)$, $F(t) = f(t)$ and $F(x,i) = (f(x),i)$ for any $(x,i) \in Z_n \times Z_3$. Then $(X,\mathcal{B},F)$ is an $SCMTS(3n+4)$ as expected.

Proof: It is not difficult to show that $(X,\mathcal{B})$ is an $MTS(3n+4)$, similar to Lemma 3. If B is a triple from $\{u,v,s,t\}$, then $F(B)^{-1} = f(B)^{-1}$ appear in part 1. As sub-$SCMTS(4)$ of $\mathcal{A}$, D is also a sub-$SCMTS(4)$ of $\mathcal{B}$, thereby $p_F(z) \leq 2$ for $z = u,v,s,t$. Other statements of the proof are almost the same to that in Lemma 3. □

Lemma 5. *If there exists a C-type $SCMTS(n+3)$ then there exists a C-type $SCMTS(3n+3)$.*

Construction: Given a C-type $SCMTS(n+3) = (\{a,b,c\} \cup Z_n, \mathcal{A}, f)$, which contains a sub-$SCMTS(3)$, $D = \{\langle a,b,c\rangle, \langle c,b,a\rangle\}$, and satisfies $p_f(x) \leq 2$ for any $x \in Z_n$. Let us construct a cyclic triple system $\mathcal{B}$ on set $X = \{a,b,c\} \cup (Z_n \times Z_3)$ as follows:

part 1. D.

part 2. $\langle (x,i),(y,i),(z,i)\rangle$ with $\langle x,y,z\rangle \in \mathcal{A} \setminus D$ and $i \in Z_3$. (Whenever a,b,c appears for x,y,z, omit the second coordinate i.)

part 3. $\langle (x,0),(y,1),(z,2)\rangle$ and $\langle (f(z),2),(f(y),1),(f(x),0)\rangle$ with $x,y,z \in Z_n$ and $x+y+z=0$

Define $F(a) = a$, $F(b) = b$, $F(c) = c$ and $F(x,i) = (f(x),i)$ for any $(x,i) \in Z_n \times Z_3$. Then $(X,\mathcal{B},F)$ is an $SCMTS(3n+3)$ as expected.

Proof: Similar to Lemma 3 and Lemma 4. □

Lemma 6. *There exists a C-type $SCMTS(v)$ for $v \equiv 4 \pmod 6$ and $v > 4$.*

Proof: Let $v = 6k+4$, all possibilities are classified as follows.

Case 1. $k = 3t$, $t \geq 1$, then $v = 18t + 4 = 3 \times 6t + 4$. There exists a B-type $SCMTS(6t+4)$ by Theorem 2, thus there exists a C-type $SCMTS(3 \times 6t + 4)$ by Lemma 4.

Case 2. $k = 3t + 1$, then $v = 18t + 10 = 3 \times (6t + 3) + 1$. There exists an A-type $SCMTS(6t + 3 + 1)$ by Theorem 2, thus there exist a C-type $SCMTS(3 \times 6t + 3) + 1)$ by Lemma 3.

Case 3. $k = 3t + 2$, then $v = 18t + 16$.

Subcase 1. $t = 2s$, then $v = 36s + 16 = 4(9s + 4)$. There exists a C-type $SCMTS(9s+4)$ by Theorem 1 (when s odd) or Case 1 (when s even), thus there exists a C-type $SCMTS(4(9s + 4))$ by Corollary.

Subcase 2. $t = 2s+1$, then $v = 36s+34 = 3(12s+11)+1$. If there exists an A-type $SCMTS(12s + 12)$ then there exists a C-type $SCMTS(v)$ by Lemma 3.

Below, we will discuss the existence of A-type $SCMTS(12s+12)$. Firstly, since there exist C-type $SCMTS(18p+4)$ and $SCMTS(18p+10)$ by Case 1 and Case 2, thus, by Lemma 5, there exist C-type

$$SCMTS(54p + 6) \text{ and } SCMTS(54p + 24). \qquad (*)$$

Now, we can get the existence of A-type $SCMTS(12s + 12)$ (classified by $s = 6m + i$, $i \in Z_6$):

(i) $72m + 12 = 12(6m + 1)$: by Theorem 1 and Corollary.

(ii) $72m + 24$, let $m = 3p + i$ ($i \in Z_3$):

$$\begin{cases} 216p + 24 = 4(54p + 6) & \text{by (*) and Corollary;} \\ 216p + 96 = 4(54p + 24) & \text{by (*) and Corollary;} \\ 216p + 168 = 54(4p + 3) + 6 & \text{by (*).} \end{cases}$$

(iii) $72m + 36 = 4(18m + 9)$: by Theorem 1 and Corollary.

(iv) $72m + 48 = 12(6m + 4)$: by Theorem 2 and Corollary.

(v) $72m + 60 = 4(18m + 15)$: by Theorem 1 and Corollary.

(vi) $72m + 72 = 4(18m + 18)$: by Theorem 3 and Corollary.

This completes the proof. □

5 The spectrum of $SCMTS(v)$

Theorem 5. *There exists an $SCMTS(v)$ if and only if $v \equiv 0$ or $1 \pmod 3$ and $v \neq 6$.*

Proof: The necessity is obvious due to the spectrum of $MTS(v)$. When $v \equiv 1$ or $3 \pmod 6$ and $v \equiv 4 \pmod 6$, the sufficiency holds by Theorem 1 and Theorem 2. When $v \equiv 0 \pmod 6$, if $v = 18t$ the conclusion has already been given by Theorem 3; if $v = 18t + 6$, $t \geq 1$, there exists a C-type $SCMTS(6t+4)$ by Lemma 6, thus there exists an $SCMTS(18t+6)$ by Lemma 5; if $v = 18t+12 = 3(6t+4)$, then the conclusion can be gotten by Theorem 2 and Corollary. □

References

[1] N.S. Mendelssohn, A natural generalization of Steiner triple systems, *Computers in Number Theory*, 323–338, Academic Press, New York, 1971.

[2] C.J. Colbourn and A. Rosa, Directed and Mendelsohn triple systems, in Chapter 4 of *Contemporary Design Theory*, A Wiley Interscience Publication (1992), 97–136.

[3] Neil P. Carne, Cyclic automorphisms of Mendelsohn triple systems, to appear in *Discrete Mathematics*.