A note on the generalized Bernoulli sequenses

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ABSTRACT. Let n, s be positive integers, and let $r = 1 + 1/s$. In this note we prove that if the sequence $\{a_n(r)\}_{n=1}^{\infty}$ satisfies $ra_n(r) = \sum_{k=1}^{n} \binom{n}{k} a_k(r)$, $n > 1$, then $a_n(r) = na_1(r)[(n-1)!/(s+1)(\log r)^n + 1/r(s+1)]$. Moreover, we obtain a related combinatorial identity.

1 Introduction

Let r be a real number with $r \geq 1$, and let $\mathcal{A}(r) = \{a_m(r)\}_{m=0}^{\infty}$ be a sequence which satisfies the initial values $a_0(r)$, $a_1(r)$ and the recurrance

$$ra_n(r) = \sum_{k=0}^{n} \binom{n}{k} a_k(r), \quad n > 1. \tag{1}$$

When $a_0(1) = 0$ and $a_1(1) = 1$, $\mathcal{A}(1)$ is the usual Bernoulli sequrnce. Therefore, $\mathcal{A}(r)$ is called a generalized Bernoulli sequence for $r > 1$. In this note we give an explicit formula for $a_n(r)$ as follows:

Theorem 1. *If $r = 1 + 1/s$, s is a positive integer and $a_0(r) = 0$, then we have*

$$a_n(r) = na_1(r)\left[\frac{(n-1)!}{(s+1)(\log r)^n} + \frac{1}{r(s+1)}\right], \quad n > 1, \tag{2}$$

where $[x]$ denote the greatest integer not greater than x.

By the above theorem, we obtain a combinatorial identity as follows:

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Theorem 2. *For any positive integers n, s with $n > 1$,*

$$n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_i \geq 0, i=1,2,\ldots,n}} \binom{r_1+r_2+\cdots+r_n}{r_1, r_2, \ldots, r_n} \frac{s^{r_1+r_2+\cdots+r_n}}{(1!)^{r_1}(2!)^{r_2}\ldots(n!)^{r_n}}$$

$$= \left[\frac{n!}{(s+1)(\log(1+1/s))^{n+1}} + \frac{s}{(s+1)^2}\right],$$

where

$$\binom{r_1+r_2+\cdots+r_n}{r_1, r_2, \ldots, r_n} = \frac{(r_1+r_2+\cdots+r_n)!}{r_1!r_2!\ldots r_n!}$$

for any nonegative integers $r_1, r_2, \ldots, r_n$.

2 Proof of Theorem 1

We see from (1) that (2) holds for $a_1(r) = 0$. It suffices to consider the case that $a_1(r) \neq 0$. Let $\mathcal{A}(r, x)$ be the expontial generating function of $\mathcal{A}(r)$. Since $a_0(r) = 0$, we get from (1) that

$$r\mathcal{A}(r, n) - (r-1)a_1(r)x = e^x \mathcal{A}(r, x),$$

whence we obtain

$$\mathcal{A}(r, x) = \frac{(r-1)a_1(r)x}{r - e^x}. \tag{4}$$

Since $r = 1 + 1/s$, by (4), we get

$$\mathcal{A}(r,x) = \frac{(r-1)a_1(r)x}{r(1-e^x/r)} = \frac{a_1(r)x}{s+1}\sum_{m=0}^{\infty}\frac{e^{mx}}{r^m} = \frac{a_1(r)x}{s+1}\sum_{m=0}^{\infty}\frac{1}{r^m}\left(\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}\right).$$

It implies that

$$a_n(r) = \frac{na_1(r)}{s+1}\sum_{k=1}^{\infty}\frac{k^{n-1}}{r^k}, \quad n > 1. \tag{5}$$

For any nonnegative integer l, let $f_l(z) = z^l/r^z$. Then $f_l(z)$ is a continuously differentiable function in the interval $[0, \infty)$ and the integration $\int_1^\infty |f_l'(z)|dz$ is existed. By Euler's sum formula (see [1]), we have

$$\sum_{k=1}^{\infty} f_l(k) = \int_1^\infty f_l(z)dz - \frac{1}{2}f_l(z)|_1^\infty + \int_1^\infty (\{z\} - \frac{1}{2})f_l'(z)dz, \tag{6}$$

where $\{z\}$ is the fractional part of z. Since

$$\int f_l(z)dz = \frac{-l!}{r^z(\log r)^{l+1}}\sum_{j=0}^{l}\frac{(\log r)^j}{j!}z^j + C,$$

where C is an integral constant, we get

$$\int_1^{\infty} f_l(z)dz = \frac{l!}{r(\log r)^{l+1}}\sum_{j=0}^{l}\frac{(\log r)^j}{j!}. \tag{7}$$

On the other hand, since $|\{z\} - 1/2| \leq 1/2$, we have

$$0 \leq |\int_1^{\infty}(\{z\} - \frac{1}{2})f_l'(z)dz| \leq \frac{1}{2}|\int_{1/r}^{\infty} df_l(z)| = \frac{1}{2r}. \tag{8}$$

Substitue (7) and (8) into (6), we get

$$\frac{l!}{r(\log r)^{l+1}}\sum_{j=0}^{l}\frac{(\log r)^j}{j!} \leq \sum_{k=1}^{\infty} f_l(k) \leq \frac{1}{r} + \frac{l!}{r(\log r)^{l+1}}\sum_{j=0}^{l}\frac{(\log r)^j}{j!}. \tag{9}$$

We see from (5) that

$$a_n(r) = \frac{na_1(r)}{s+1}\sum_{k=1}^{\infty} f_{n-1}(k). \tag{10}$$

Therefore, by (9) and (10), we get

$$\frac{n!}{r(\log r)^n}\sum_{j=0}^{n-1}\frac{(\log r)^j}{j!} \leq (s+1)\frac{a_n(r)}{a_1(r)} \leq \frac{n}{r} + \frac{n!}{r(\log r)^n}\sum_{j=0}^{n-1}\frac{(\log r)^j}{j!}. \tag{11}$$

Notice that $1 < r < 2$, $0 < \log r < 1$, $r = \sum_{m=0}^{\infty}(\log r)^m/m!$ and

$$r - \frac{2(\log r)^n}{n!} < \sum_{l=0}^{n-1}\frac{(\log r)^j}{j!} = r - \sum_{j=n}^{\infty}\frac{(\log r)^j}{j!} < r.$$

We get from (11) that

$$\frac{n!}{(\log r)^n} - \frac{2}{r} < (s+1)\frac{a_n(r)}{a_1(r)} < \frac{n!}{(\log r)^n} + \frac{n}{r}. \tag{12}$$

Since $a_0(r) = 0$, we find from (1) that

$$a_n(r) = s\sum_{k=1}^{n-1}\binom{n}{k}a_k(r), \quad n > 1. \tag{13}$$

Since $a_1(r) \neq 0$, we see from (13) that $a_n(r)/a_1(r)$ is a positive integer for any $n \geq 1$. When $n = 2$, we have $a_2(r)/a_1(r) = 2s \equiv 0 \pmod{2s}$. For $n > 2$, we assume that

$$\frac{a_t(r)}{a_1(r)} \equiv 0 \pmod{ts}, \quad t = 2, \ldots, n-1. \tag{14}$$

By (13) and (14), we get

$$\frac{a_n(r)}{a_1(r)} = s\sum_{k=1}^{n-1}\binom{n}{k}\frac{a_k(r)}{a_1(r)} = s\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{na_k(r)}{ka_1(r)} \equiv 0 \pmod{ns}.$$

Thus, by the deductive method, we have $a_n(r)/a_1(r) \equiv 0 \pmod{ns}$ for any $n > 1$. Notice that the difference of both sides of (12) is less than ns. From (12), we obtain (2). The theorem is proved.

3 Proof of Theorem 2

We get from (4) that

$$\mathcal{A}(r,x) = \frac{(r-1)a_1(r)x}{(r-1) - \sum_{n=1}^{\infty} x^n/n!}. \tag{15}$$

By the properties of Bell's polynomial (see [2]), if $r = 1 + 1/s$ and s is a positive integer, then from (15) we get

$$\mathcal{A}(r,x) = a_1(r)x\left(1 + \sum_{n=1}^{\infty}\left(\sum_{\substack{r_1+2r_2+\cdots+nr_n=n\\ r_i\geq 0, i=1,2,\ldots,n}}\binom{r_1+r_2+\cdots+r_n}{r_1,r_2,\ldots,r_n}\frac{s^{r_1+r_2+\cdots+r_n}}{(1!)^{r_1}(2!)^{r_2}\ldots(n!)^{r_n}}\right)x^n\right).$$

whence we obtain

$$a_n(r) = n!a_1(r)\sum_{\substack{r_1+2r_2+\cdots+nr_n=n\\ r_i\geq 0, i=1,2,\ldots,n}}\binom{r_1+r_2+\cdots+r_n}{r_1,r_2,\ldots,r_n}\frac{s^{r_1+r_2+\cdots+r_n}}{(1!)^{r_1}(2!)^{r_2}\ldots(n!)^{r_n}},$$
$$n > 1. \tag{16}$$

Put $a_1(r) = 1$. By Theorem 1, we get (13) by (16). The theorem is proved.

References

[1] N.G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, 1958.

[2] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley and Sons, New York, 1958.