New Classes of Infinite 3-way Partition Identities

A.K. Agarwal
Department of Mathematics
Birla Institute of Technology and Science
Pilani - 333031
(Rajasthan)
India

ABSTRACT. We study four q-series. Each of which is interpreted combinatorially in three different ways. This results in four new classes of infinite 3-way partition identities. In some particular cases we get even 4-way partition identities. Our every 3-way identity gives us three Roders-Ramanujan Type identities and 4-way identity gives six. Several partition identities due to Gordon (1965), Hirschhorn (1979), Subbarao (1985), Blecksmith et.al. (1985), Agarwal (1988) and Subbarao and Agarwal (1988) are obtained as particular cases of our general theorems.

1 Introduction, definitions, notations and the main results

Let the sets S and T de defined by

$$S = \{-1, 1, 3, 5, 7, \ldots, k\} \tag{1.1}$$

and

$$T = \{-2, 0, 2, 4, 6, \ldots, \}. \tag{1.2}$$

For $|q| < 1$ and $1 \leq 1 \leq 4$, we define $f_i^k(q)$ as follow:

$$f_1^k(q) = \sum_{n=0}^{\infty} \frac{q^{n[1+(k+3)(n-1)/2]}}{(q, ; q)_{2n}}, \quad k \in S \tag{1.3}$$

where and in what follows, $(a;q)_n$ is the rising q-factorial defined by $(a;q)_n = \prod_{i=0}^{\infty}(1-aq^i)/(1-aq^{n+i})$.

$$f_2^k(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)(k+3)/2}}{(q;q)_{2n+1}}, \quad k \in S \tag{1.4}$$

$$f_3^k(q) = \sum_{n=0}^{\infty} \frac{q^{n[1+(n+1)(k+3)/2]}}{(q;q)_{2n+1}}, \quad k \in S \tag{1.5}$$

and

$$f_4^k(q) = \sum_{n=0}^{\infty} \frac{q^{n[2+(k+4)(n-1)/2]}}{(q;q)_{2n}}, \quad k \in T \tag{1.6}$$

In the next section we shall show that each $f_i^k(q)$ is a generating function of three different partition functions of which two are ordinary and one involves color-partitions introduced by Argarwal and Andrews in [2]. This eventually will lead to four new infinite classes of 3-way partition identities. For certain values of k our general theorems yield 4-way partition identities. To exhibit the importance of our theorems we obtain all results of Hirschhorn [6], Agarwal [1] and some results of Subbarao [7], Subbarao and Agarwal [8] and Blecksmith et.al. [9]. We first recall the following definitions from [3].

Definition 1: A partition with "$n+t$ copies of n", $t \geq 0$, is a partition in which a part of size n, $n \geq 0$ can come in $n+t$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Thus, for example, the partition of 2 with "$n+1$ copies of n" are

$$2_1, 2_1+0_1, 1_1+1_1, 1_1+1_1+0_1$$
$$2_2, 2_2+0_1, 1_2+1_1, 1_2+1_1+0_1$$
$$2_3, 2_3+0_1, 1_2+1_2, 1_2+1_2+0_1.$$

Note that zeros are permitted if and only if t is greater than or equal to one. And, in no partition are zeros permitted to repeat.

Definition 2: The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m-n-i-j$.

In the next section we shall prove the following theorems:

Theorem 1. *For $k \in S$, let $A_1^k(v)$ denote the number of partitions of v with "n copies of n" such that the weighted difference of each pair of parts is greater than k, even parts appear with even subscripts and odd with odd. Let $B_1^k(v)$ denote the number of ordinary partitions of v of the type $b_1+b_2+\ldots b_p$, where $b_i \geq b_{i+1}$, $b_i - b_{i+1} \geq k+3$ if $1 \leq i \leq [p-2/2]$, $b_{p/2} - b_{(p+2)/2} \geq 1$ if p is even, and $b_{(p-1)/2} - b_{(p+1)/2} \geq k+2$ if p is* odd. Let $c_1^k(v)$ denote the number of ordinary partitions of v of the type $c_1 + c_2 + \cdots + c_t$, where $c_i - c_{i+2} \geq (k+3)/2$ if $1 \leq i \leq t-2$, $c_{t-1} > c_t$ if t is even, and $c_{t-1} - c_t \geq (k+1)/2$ and $c_1 > c_2$ if t is odd.

Then

$$A_1^k(v) = B_1^k(v) = C_1^k(v), \text{ for all } v. \tag{1.7}$$

Theorem 2. For $k \in S$, let $A_2^k(v)$ denote the number of partitions of v with "$n+1$ copies of n" such that even parts appear with odd subscripts and odd with even and the weighted differences of each pair of parts is greater than k, for some i, i_{i+1} is a part. Let $B_2^k(v)$ denote the number of ordinary partitions of v of the type $b_1 + b_2 + \cdots + b_p$, where $b_i \geq b_{i+1}$, $b_i - b_{i+1} \geq k+3$ if $1 \leq i \leq [(p-1)/2]$, $b_p \geq (k+3)/2$ if p is even. Let $C_2^k(v)$ denote the nu,mber of ordinary partitions of v of the type $c_1 + c_2 + \cdots + c_t$ such that $c_{2i} - c_{2i+1} \geq (k+3)/2$ if $1 \leq i \leq [(t-1)/2]$ and $c_t \geq (k+3)/2$ if t is even.

Then

$$A_2^k(v) = B_2^k(v) = C_2^k(v), \text{ for all } v. \tag{1.8}$$

Theorem 3. Let $A_3^k(v)$ denote the number of partitions of v with "$n+2$ copies of n" such that the weighted difference of each pair of parts is greater than k for some i, i_{i+2} is a part and even parts appear with even subscripts and odd with odd. Let $B_3^k(v)$ denote the number of ordinary partitions of v of the type $b_1 + b_2 + \cdots + b_p$ such that $b_{i-1} \geq b_i$, $b_i - b_{i+1} \geq k+3$ if $1 \leq i \leq [(p-2)/2]$, $b_{p/2} - b_{(p+2)/2} \geq k+2$ if p is even and $b_{(p-1)/2} - b_{(p+1)/2} \geq k+4$ if p is odd.

Let $C_3^k(v)$ denote the number of ordinary partitions of v of the type $c_1 + c_2 + \cdots + c_t$ such that

$$\begin{aligned} c_{2i-1} - c_{2i} &\geq 1 \\ c_{2i-2} - c_{2i} &\geq (k+3)/2 \quad 1 \leq i \leq [t/2] \end{aligned}$$

$c_t \geq (k+3/2$ if t is even and $c_{t-1} - c_t \geq (k+3)/2$ if t is odd.

Then

$$A_3^k(v) = B_3^k(v) = C_3^k(v), \text{ for all } v. \tag{1.9}$$

Theorem 4. For k T, let $A_4^k(v)$ denote the number of partitions of v with "n copies of n" such that even parts appear with even subscripts and odd with odd subscripts greater than 1 and the weighted difference of each pair of parts is greater than or equal to k. Let $B_4^k(v)$ denote the number of *ordinary partitions of v of the type $b_1 + b_2 + \dots b_p$ such that $b_i \geq b_{i+1}$, $b_i \geq 2$, $b_i - b_{i+1} \geq (k+4)$ if $1 \leq i \leq [(p-2)/2]$ and $b_{(p-1)/2} - b_{(p+1)/2} \geq k+2$ if p is odd. Let $C_4^k(v)$ denote the number of ordinary partitions of v of the type $c_1 + c_2 + \cdots + c_t$ such that $c_t \geq 2$, $c_{2i} - c_{2i+1} \geq (k+4)/2$ if $1 \leq i \leq [(t-2)/2]$ and $c_{t-1} - c_t \geq (k+2)/2$ if t is odd.*

Then

$$A_4^k(v) = B_4^k(v) = C_4^k(v), \text{ for all } v. \tag{1.10}$$

Remark. We remark here that in the definitions of $B_i^k(v)$ the difference conditions are satisfied by about the "first half of the summands" where as in the definitions of $C_i^k(v)$ the difference conditions are satisfied by all summands.

In the next section we give detail proof of Theorem 1 and shortest possible proofs of the remaining theorems. $\pi_p(i,k;v)$ will denote the partition $b_1 + b_2 + \cdots + b_p$ enumerated by $B_i^k(v)$ and $\delta_t(i,k;v)$ will denote the partition $c_1 + c_2 + \cdots + c_t$ enumerated by $C_i^k(v)$. We shall write $f_i^k(z,q)$ $(1 \leq i \leq 4)$ for the right-hand sides of (1.j) $(3 \leq j \leq 6)$ with numerators multiplied by z^n where $1z1 < q^{-1}$. Thus for example

$$f_1^k(z,q) = \sum_{n=0}^{\infty} \frac{q^{n[1+(k+3)(n-1)/2]}}{(q;q)_{2n}} z^n. \tag{1.11}$$

Clearly,

$$f_i^k(1,q) = f_i^k(q), \quad 1 \leq i \leq 4. \tag{1.12}$$

Also, $A_i^k(m,v)$ will denote the number of partitions counted by $A_i^k(v)$ into m parts. Our method consists in proving that for each i all three partition functions viz., $A_i^k(v)$, $B_i^k(v)$ and $C_i^k(v)$ are generated by $f_i^k(q)$.

2 Proofs

We shall complete the proof of each theorem in three steps, viz.,

Step I. $\sum_{v=0}^{\infty} A_i^k(v) q^v = f_i^k(q)$

Step II. $\sum_{v=0}^{\infty} B_i^k(v) q^v = f_i^k(q)$

Step III. $\sum_{v=0}^{\infty} C_i^k(v) q^v = f_i^k(q)$

Proof of Theorem 1:

Step I We shall prove that

$$\sum_{v=0}^{\infty} A_1^k(v) q^v = f_1^k(q). \tag{2.1}$$

We split the partitions enumerated by $A_1^k(m,v)$ into three classes, viz.,

(i) those that do not contain 1_1 as a part,

(ii) those that contain 1_1 as a part, and

(iii) those that contain r_r $(r>1)$ as a part.

We now transform the partitions in class (i) by deleting 2 from each part ignoring the subscripts (we note that this is possible under the assumption that even parts appear with even subscripts and odd with odd subscripts). The transformed partition will be of the type enumerated by $A_1^k(m,v-2m)$. Next we transform the partitions in class (ii) by deleting the part 1_1 and then subtracting $k+3$ from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1^k(m-1,v-km-3m+k+2)$. Finally we transform the partitions in class (iii) by replacing r_r by $(r-1)_{(r-1)}$ and then subtracting 2 from all the remaining parts. This will produce a partition of $v-1-2(m-1)=v-2m+1$ into m parts. It is important to note here that by this transformation we get only those partitions of $v-2m+1$ into m parts which contain a part of the form r_r. Therefore the actual number of partitions which belong to class (iii) is obtained by subtracting those partitions of $v-2m+1$ into m parts which are free from the parts like r_r (the number of such partitions is $A_1^k(m,v-4m+1)$ by case (i)). Thus the number of partitions in class (iii) is

$$A_1^k(m,v-2m+1)-A_1^k(m,v-4m+1).$$

Since these three classes are mutually exclusive and exhaust all the partitions enumerated by $A_1^k(m,v)$, we get the identity.

$$\begin{aligned} A_1^k(m,v) &= A_1^k(m,v-2m)+A_1^k(m-1,v-(k+3)m+k+2) \\ &\quad + A_1^k(m,v-2m+1)-A_1^k(m,v-4m+1). \end{aligned} \tag{2.2}$$

Let

$$h^k(z,q)=\sum_{v,m=0}^{\infty} A_1^k(m,v)z^m q^v. \tag{2.3}$$

Substituting for $A_1^k(m,v)$ from (2.2) in (2.3) and then simplifying we get

$$\begin{aligned} h^k(z,q) &= h^k(zq^2,q)+zqh^k(zq^{k+3},q) \\ &\quad + q^{-1}h^k(zq^2,q)-q^{-1}h^k(zq^4,q). \end{aligned} \tag{2.4}$$

Setting

$$h^k(z,q) = \sum_{n=0}^{\infty} \alpha_{k,n}(q) z^n, \text{ in (2.4)} \tag{2.5}$$

and then comparing the coefficients of z^n in the resulting expression, we get

$$\alpha_{k,n}(q) = \frac{\alpha_{k,n-1}(q) q^{(k+3)(n-1)+1}}{(1-q^{2n})(1-q^{2n-1})} \tag{2.6}$$

Iterating (2.6) n times and observing that $\alpha_{k,0}(q) = 1$, we find that

$$\alpha_{k,n}(q) = \frac{q^{n[1+(k+3)(n-1)/2]}}{(q;q)_{2n}}, \tag{2.7}$$

and so

$$\begin{aligned} h^k(z,q) &= \sum_{n=0}^{\infty} \frac{q^{n[1+(k+3)(n-1)/2]}}{(q;q)_{2n}} z^n, \\ &= f_1^k(z,q). \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} \sum_{v=0}^{\infty} A_1^k(v) q^v &= \sum_{v=0}^{\infty} \left(\sum_{m=0}^{\infty} A_1^k(m,v) \right) q^v \\ &= f_1^k(1,q) \\ &= f_1^k(q). \end{aligned}$$

This proves (2.1).

Step II. We shall prove that

$$\sum_{v=0}^{\infty} B_1^k(v) q^v = f_1^k(q). \tag{2.9}$$

For some $s \geq 1$, $p = 2s$ or $2s-1$.

First suppose that $p = 2s$. Then $\pi_{2s}(1,k;v) = b_1 + b_2 + \cdots + b_{2s}$, with $1 \leq b_{2s} \leq b_{2s-1} \leq \cdots \leq b_{s+1}$, and

$$\begin{aligned} b_s &\geq 2 \\ b_{s-1} &\geq (k+3) + 2 \\ b_{s-2} &\geq 2(k+3) + 2 \\ &\cdots \\ b_1 &\geq (s-1)(k+3) + 2. \end{aligned}$$

We subtract 2, $(k+3)+2, 2(k+3)+2, \ldots, (s-1)(k+3)+2$ from $b_s, b_{s-1}, \ldots, b_2, b_1$ respectively and 1 from each of $b_{s+1}, \ldots, b_{2s}$. This produces a partition of $v-[s+\{2s+(k+3)s(s-1)/2\}]$ into almost $2s$ parts. Thus the partitions of the type $\pi_{2s}(1,k;v)$ are generated by

$$\frac{q^{2s+s[1+(k+3)(s-1)/2]}}{(q;q)_{2s}}.$$

Similarly, if $p = 2s-1$, then $\pi_{2s-1}(1,k;v) = b_1 + b_2 + \cdots + b_{2s-1}$ with $1 \le b_{2s-1} \le b_{2s-2} \le \cdots \le b_s$ and

$$\begin{aligned} b_{s-1} &\ge k+3 \\ b_{s-2} &\ge 2(k+3) \\ b_{s-3} &\ge 3(k+3) \\ &\cdots \\ b_1 &\ge (s-1)(k+3) \end{aligned}$$

Subtracting $(k+3), 2(k+3), \ldots, (s-1)(k+3)$ from $b_{s-1}, b_{s-2}, \ldots, b_1$, respectively and 1 from each of b_s, b_{s+1}, b_{2s-1}, we are left with a partition of $v - s[1+(k+3)(s-1)/2]$ into almost $2s-1$ parts.

This shows that the partitions of the type $\pi_{2s-1}(1,k;v)$ are generated by

$$\frac{q^{s[1+(k+3)(s-1)/2]}}{(q;q)_{2s-1}}.$$

Thus

$$\sum_{v=0}^{\infty} B_1^k(v)q^v = 1 + \sum_{s=1}^{\infty} \frac{q^{2s+s[1+(k+3)(s-1)/2]}}{(q;q)_{2s}} + \sum_{s=1}^{\infty} \frac{q^{s[1+(k+3)(s-1)/2]}}{(q;q)_{2s-1}} = f_1^k(q).$$

This proves (2.9)

Step III. We shall prove that

$$\sum_{v=0}^{\infty} C_1^k(v)q^v = f_1^k(q). \tag{2.10}$$

For some $s \ge 1$, $t = 2s$ or $t = 2s-1$.

First suppose that $t = 2s$. Then $\delta_{2s}(1, k; v) = c_1 + c_2 + \cdots + c_{2s}$, with

$$\begin{aligned} c_{2s} &\geq 1 \\ c_{2s-2} &\geq 1 + (k+3)/2 \\ c_{2s-4} &\geq 1 + 2(k+3)/2 \\ &\cdots \\ c_2 &\geq 1 + (s-1)(k+3)/2 \end{aligned}$$

and

$$\begin{aligned} c_{2s-1} &\geq 2 \\ c_{2s-3} &\geq 2 + (k+3)/2 \\ c_{2s-5} &\geq 2 + 2(k+3)/2 \\ &\cdots \\ c_1 &\geq 2 + (s-1)(k+3)/2 \end{aligned}$$

We subtract 1, $1 + (k+3)/2, 1 + 2(k+3)/2 \ldots, 1 + (s-1)(k+3)/2$ from $c_{2s}, c_{2s-2}, \ldots, c_2$ respectively, and $2, 2+(k+3)/2, 2+2(k+3)/2, \ldots, 2+(s-1)(k+3)/2$ from $c_{2s-1}, c_{2s-3}, \ldots, c_1$ respectively. This produces a partition of $v - [\{s + s(s-1)(k+3)/4\} + \{2s + 2(s-1)(k+3)/4\}]$ into at most $2s$ parts.

Thus like $\pi_{2s}(1, k; v)$ the partitions of the type $\delta_{2s}(1, k; v)$ are also generated by

$$\frac{q^{2s+s[1+(k+3)(s-1)/2]}}{(q;q)_{2s}}$$

Similarly, if $t = 2s - 1$, then $\delta_{2s-1}(1, k; v) = c_1 + c_2 + \cdots + c_{2s-1}$, with

$$\begin{aligned} c_{2s-1} &\geq 1 \\ c_{2s-3} &\geq 1 + (k+3)/2 \\ c_{2s-5} &\geq 1 + 2(k+3)/2 \\ &\cdots \\ c_1 &\geq 1 + (s-1)(k+3)/2 \end{aligned}$$

and

$$\begin{aligned} c_{2s-2} &\geq (k+3)/2 \\ c_{2s-4} &\geq 2(k+3)/2 \\ c_{2s-6} &\geq 3(k+3)/2 \\ &\cdots \\ c_2 &\geq (s-1)(k+3)/2 \end{aligned}$$

Subtracting $1, 1 + (k+3)/2, 1 + 2(k+3)/2, \ldots, 1 + (s-1)(k+3)/2$ from $c_{2s-1}, c_{2s-3}, \ldots, c_1$ respectively, and $(k+3)/2, 2(k+3)/2, \ldots, (s-1)(k+$

3)/2 from $c_{2s-2}, c_{2s-4}, \ldots, c_2$ respectively, we are left with a partition of $v - [\{s + s(s-1)(k+3)/4\} + s(s-1)(k+3)/4]$ into at most $2s-1$ parts. Thus like $\pi_{2s-1}(1,k;v)$ the partitions $\delta_{2s-1}(1,k;v)$ are also generated by

$$\frac{q^{s[1+(k+3)(s-1)/2]}}{(q;q)_{2s-1}}$$

Hence,

$$\sum_{v=0}^{\infty} C_1^k(v) q^v = f_1^k(q).$$

This completes the proof of Theorem 1.

Proof of Theorem 2:

Step 1. We shall prove that

$$\sum_{v=0}^{\infty} A_2^k(v) q^v = f_2^k(q). \tag{2.11}$$

From (2.8), we have

$$\begin{aligned} f_1^k(z,q) - f_1^k(zq^2,q) &= \sum_{n=1}^{\infty} \frac{q^{n[1+(k+3)(n-1)/2]}}{(q;q)_{2n-1}} z^n \\ &= zq \sum_{n=0}^{\infty} \frac{q^{n(n+1)(k+3)/2}}{(q;q)_{2n+1}} (zq)^n \end{aligned}$$

Thus

$$f_1^k(z;q) - f_1^k(zq^2;q) = zq f_2^k(zq;q). \tag{2.12}$$

Define $P(m,v)$ by

$$f_2^k(z;q) = \sum_{v,m=0}^{\infty} P^k(m,v) z^m q^v.$$

We see by coefficient comparison in (2,12) that

$$A_1^k(m,v) - A_1^k(m,v-2m) = P^k(m-1, v-m). \tag{2.13}$$

Equation (2.13) shows that $P^k(m,v)$ equals the number of partitions of $v+m+1$ with "n copies of n" into $m+1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k, for some i i_i is a part and even parts appear with even subscripts and odd with odd.

If we subtract 1 from each part of a partition enumerated by $P^k(m,v)$ ignoring the subscripts, we see that the resulting partition is enumerated by $A_2^k(m+1,v)$. Thus $P^k(m,v) = A_2^k(m+1,v)$, and so

$$\sum_{m,v=0}^{\infty} A_2^k(m+1,v)z^m q^v = f_2^k(z,q). \tag{2.14}$$

Now

$$\begin{aligned}\sum_{v=0}^{\infty} A_2^k(v)q^v &= \sum_{v=0}^{\infty}\left[\sum_{m=1}^{\infty} A_2^k(m,v)\right]q^v \\ &= \sum_{m,v=0}^{\infty} A_2^k(m+1,v)q^v \\ &= f_2^k(1,q) \\ &= f_2^k(q)\end{aligned}$$

This proves (2.11).

Step 2. We shall prove that

$$\sum_{v=0}^{\infty} B_2^k(v)q^v = f_2^k(q). \tag{2.15}$$

We write $\pi_{2s}(2,k;v) = b_1 + b_2 + \cdots + b_{2s}$ with $(k+3)/2 \le b_{2s} \le b_{2s+1} \le \cdots \le b_{s+1}$, and

$$\begin{aligned} b_s &\ge (k+3)/2 \\ b_{s-1} &\ge 3(k+3)/2 \\ b_{s-2} &\ge 5(k+3)/2 \\ &\cdots \\ b_1 &\ge (2s-1)(k+3)/2. \end{aligned}$$

It is easy to see that $\pi_{2s}(2,k;v)$ are generated by

$$\frac{q^{(s^2+s)(k+3)/2}}{(q;q)_{2s}}.$$

Similarly, writing $\pi_{2s-1}(2,k;v) = b_1 + b_2 + \cdots + b_{2s-1}$ with $1 \le b_{2s-1} \le b_{2s-2} \le \cdots \le b_s$, and

$$\begin{aligned} b_{s-1} &\ge (k+3)+1 \\ b_{s-2} &\ge 2(k+3)+1 \\ &\cdots \\ b_1 &\ge (s-1)(k+3)+1 \end{aligned}$$

and following the usual method we see that $\pi_{2s-1}(2,k;v)$ are generated by

$$\frac{q^{(2s-1)+(k+3)s(s-1)/2}}{(q;q)_{2s-1}}.$$

Thus

$$\sum_{v=0}^{\infty} B_2^k(v)q^v = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)(k+3)/2}}{(q;q)_{2s}} + \sum_{s=1}^{\infty} \frac{q^{(2s-1)+(k+3)s(s-1)/2}}{(q;q)_{2s-1}} = f_2^k(q).$$

Step 3. We shall prove that

$$\sum_{v=0}^{\infty} C_2^k(v)q^v = f_2^k(q).$$

We write $\delta_{2s}(2,k;v) = c_1 + c_2 + \cdots + c_{2s}$, with

$$\begin{aligned} c_{2s} &\geq (k+3)/2 \\ c_{2s-1} &\geq (k+3)/2 \\ c_{2s-2} &\geq 2(k+3)/2 \\ c_{2s-3} &\geq 2(k+3)/2 \\ &\cdots \\ c_2 &\geq s(k+3)/2 \\ c_1 &\geq s(k+3)/2 \end{aligned}$$

Now it is easy to see that $\delta_{2s}(2,k;v)$ are generated by

$$\frac{q^{(s^2+s)(k+3)/2}}{(q;q)_{2s}}$$

Similarly, we write $\delta_{2s-1}(2,k;v) = c_1 + c_2 + \cdots + c_{2s-1}$, with

$$\begin{aligned} c_{2s-1} &\geq 1 \\ c_{2s-2} &\geq 1 + (k+3)/2 \\ c_{2s-3} &\geq 1 + (k+3)/2 \\ c_{2s-4} &\geq 1 + 2(k+3)/2 \\ c_{2s-5} &\geq 1 + 2(k+3)/2 \\ &\cdots \\ c_2 &\geq 1 + (s-1)(k+3)/2 \\ c_1 &\geq 1 + (s-1)(k+3)/2 \end{aligned}$$

and see that $\delta_{2s-1}(2,k;v)$ are generated by

$$\frac{q^{(2s-1)+s(s-1)(k+3)/2}}{(q;q)_{2s-1}}$$

As in Step 2, we can show that

$$\sum_{v=0}^{\infty} C_2^k(v)q^v = f_2^k(q).$$

Proof of Theorem 3:

Step I. We shall prove that

$$\sum_{v=0}^{\infty} A_3^k(v)q^v = f_3^k(q). \tag{2.17}$$

Equation (2.12) can also be written as

$$f_1^k(z,q) - f_1^k(zq^2,q) = zqf_3^k(z,q). \tag{2.18}$$

Define $Q^k(m,v)$ by $f_3^k(z,q) = \sum_{v,m=0}^{\infty} Q^k(m,v)z^m q^v$.

We see by coefficient comparison in (2.18) that

$$A_1^k(m,v) - A_1^k(m,v-2m) = Q^k(m-1,v-1). \tag{2.19}$$

Equation (2.19) shows that $Q^k(m,v)$ equals the number of partitions of $v+1$ with "n copies of n" into $m+1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k, the even parts appear with even subscripts and the odd with odd and for some i, i_i is a part. If we replace this part i_i by $(i-1)_{i+1}$, we see that the resulting partition is enumerated by $A_3^k(m+1,v)$. This implies that

$$Q^k(m,v) = A_3^k(m+1,v). \tag{2.20}$$

Therefore

$$\begin{aligned}\sum_{m,v=0}^{\infty} A_3^k(m+1,v)z^m q^v &= \sum_{m,v} \frac{q^{v[1+(v+1)(k+3)/2]}z^m}{(q;q)_{2v+1}} \\ &= f_3^k(z,q).\end{aligned} \tag{2.21}$$

Now

$$\begin{aligned}\sum_{v=0}^{\infty} A_3^k(v)q^v &= \sum_{v=0}^{\infty}\left[\sum_{m=0}^{\infty} A_3^k(m,v)\right]q^v \\ &= \sum_{v=0}^{\infty}\left[\sum_{m=0}^{\infty} A_3^k(m+1,v)\right]q^v \\ &= f_3^k(1,q) \\ &= f_3^k(q).\end{aligned}$$

This proves (2.17).

Step II. We shall prove that

$$\sum_{v=0}^{\infty} B_3^k(v) q^v = f_3^k(q). \tag{2.22}$$

We write $\pi_{2s}(3,k;v) = b_1 + b_2 + \cdots + b_{2s}$ with $1 \le b_{2s} \le b_{2s-1} \le \cdots \le b_{s+1}$ and

$$\begin{aligned} b_s &\ge k+3 \\ b_{s-1} &\ge 2(k+3) \\ &\cdots \\ b_1 &\ge s(k+3) \end{aligned}$$

By the usual argument it can be shown that $\pi_{2s}(3,k;v)$ are generated by

$$\frac{q^{s+s(s+1)(k+3)/2}}{(q;q)_{2s}}$$

Similarly, writing $\pi_{2s-1}(3,k;v) = b_1 + b_2 + \cdots + b_{2s-1}$, where $1 \le b_{2s-1} \le b_{2s-2} \le \cdots \le b_s$ and

$$\begin{aligned} b_{s-1} &\ge 2 + (k+3) \\ b_{s-2} &\ge 2 + 2(k+3) \\ &\cdots \\ b_1 &\ge 2 + (s-1)(k+3), \end{aligned}$$

We see that $\pi_{2s-1}(3,k;v)$ are generated by

$$\frac{q^{(2s-1)+(s-1)[1+(k+3)/2]}}{(q;q)_{2s-1}}$$

Therefore,

$$\begin{aligned} \sum_{v=0}^{\infty} B_3^k(v) q^v &= 1 + \sum_{s=1}^{\infty} \frac{q^{s+s(s+1)(k+3)/2}}{(q;q)_{2s}} \\ &\quad + \sum_{s=1}^{\infty} \frac{q^{2s-1+(s-1)[1+s(k+3)/2]}}{(q;q)_{2s-1}} = f_3^k(q). \end{aligned}$$

Step III. We shall prove that

$$\sum_{v=0}^{\infty} C_3^k(v) q^v = f_3^k(q). \tag{2.23}$$

We write

$$\begin{aligned} c_{2s} &\geq (k+3)/2 \\ c_{2s-2} &\geq 2(k+3)/2 \\ c_{2s-4} &\geq 3(k+3)/2 \\ &\dots \\ c_2 &\geq s(k+3)/2 \end{aligned}$$

and

$$\begin{aligned} c_{2s-1} &\geq 1+(k+3)/2 \\ c_{2s-3} &\geq 1+2(k+3)/2 \\ &\dots \\ c_1 &\geq 1+s(k+3)/2 \end{aligned}$$

By the usual argument it can be shown that $\delta_{2s}(3,k;v)$ are generated by

$$\frac{q^{s+s(s+1)(k+3)/2}}{(q;q)_{2s}}.$$

Similarly, writing $\delta_{2s-1}(3,k;v) = c_1 + c_2 + \cdots + c_{2s-1}$, with

$$\begin{aligned} c_{2s-1} &\geq 1 \\ c_{2s-3} &\geq 2+(k+3)/2 \\ c_{2s-5} &\geq 2+2(k+3)/2 \\ &\dots \\ c_1 &\geq 2+(s-1)(k+3)/2 \end{aligned}$$

and

$$\begin{aligned} c_{2s-2} &\geq 1+(k+3)/2 \\ c_{2s-4} &\geq 1+2(k+3)/2 \\ c_{2s-6} &\geq 1+3(k+3)/2 \\ &\dots \\ c_2 &\geq 1+(s-1)(k+3)/2. \end{aligned}$$

We see that $\delta_{2s-1}(3,k;v)$ are generated by

$$\frac{q^{2s-1+(s-1)\{1+s(k+3)/2\}}}{(q;q)_{2s-1}}$$

Now as in step II, we can show that

$$\sum_{v=0}^{\infty} C_3^k(v)q^v = f_3^k(v)$$

This completes the proof of Theorem 3.

Proof of Theorem 4:
Step 1. We shall prove that

$$\sum_{v=0}^{\infty} A_4^k(v) q^v = f_4^k(q). \tag{2.24}$$

Dividing the partitions enumerated by $A_4^k(m,v)$ into the following three classes:

(i) those that do not contain r_r as a part,

(ii) those that contain 2_2 as a part, and

(iii) those that contain $r_r (r > 2)$ as a part.

and then using a simple combinatorial argument, one can prove the following:

$$\begin{aligned} A_4^k(m,v) &= A_4^k(m, v-2m) + A_4^k(m-1, v-(k+4)m+k+2) \\ &\quad + A_4^k(m, v-2m+1) - A_4^k(m, v-4m+1). \end{aligned} \tag{2.25}$$

Set

$$g^k(z,q) = \sum_{m,v=0}^{\infty} A_4^k(m,v) z^m q^v. \tag{2.26}$$

Using (2.25) in (2.26), we get

$$\begin{aligned} g^k(z,q) &= g^k(zq^2, q) + zq^2 g^k(zq^{k+4}, q) \\ &\quad + q^{-1} g^k(zq^2, q) - q^{-1} g^k(zq^4, q). \end{aligned} \tag{2.27}$$

Set

$$g^k(z,q) = \sum_{n=0}^{\infty} \beta_{k,n}(q) z^n. \tag{2.28}$$

Then the coefficient comparison in (2.27) gives

$$\beta_{k,n}(q) = \frac{q^{(k+4)(n-1)+2}}{(1-q^{2n})(1-q^{2n-1})} \beta_{k,n-1}(q). \tag{2.29}$$

Iterating (2.29) n times and noting that $\beta_{k,0}(q) = 1$. We get

$$\beta_{k,n}(q) = \frac{q^{2n+n(k+4)(n-1)/2}}{(q;q)_{2n}} \tag{2.30}$$

Therefore

$$\begin{aligned} g^k(z,q) &= \sum_{n=0}^{\infty} \frac{q^{2n+n(n-1)(k+4)/2}}{(q;q)_{2n}} \\ &= f_4^k(z,q). \end{aligned} \tag{2.31}$$

Now

$$\begin{aligned} \sum_{v=0}^{\infty} A_4^k(v) q^v &= \sum_{v=0}^{\infty} \left[\sum_{m=0}^{\infty} A_4^k(m,v) \right] q^v \\ &= g^k(1,q) \\ &= f_4^k(1,q) \\ &= f_4^k(q). \end{aligned}$$

This proves (2.24).

Step II. We shall prove that

$$\sum_{v=0}^{\infty} B_4^k(v) q^v = f_4^k(q). \tag{2.32}$$

we write $\pi_{2s}(4,k;v) = b_1 + b_2 + \cdots + b_{2s}$, where $2 \le b_{2s} \le b_{2s-1} \le \cdots \le b_{s+1}$ and

$$\begin{aligned} b_s &\ge 2 \\ b_{s-1} &\ge 2 + (k+4) \\ b_{s-2} &\ge 2 + 2(k+4) \\ &\dots \\ b_1 &\ge 2 + (s-1)(k+4) \end{aligned}$$

It can now be shown that $\pi_{2s}(4,k;v)$ are generated by

$$\frac{q^{4s+s(s-1)(k+4)/2}}{(q;q)_{2s}}$$

Similarly writing $\pi_{2s-1}(4,k;v) = b_1 + b_2 + \cdots + b_{2s-1}$, where $2 \le b_{2s-1} \le b_{2s-2} \le \cdots \le b_s$, and

$$\begin{aligned} b_{s-1} &\ge k+4 \\ b_{s-2} &\ge 2(k+4) \\ &\dots \\ b_1 &\ge (s-1)(k+4), \end{aligned}$$

We see that $\pi_{2s-1}(4,k;v)$ are generated by

$$\frac{q^{2s+s(s-1)(k+4)/2}}{(q;q)_{2s-1}}$$

Thus

$$\sum_{v=0}^{\infty} A_4^k(v)q^v = 1 + \sum_{s=1}^{\infty} \frac{q^{4s+s(s-1)(k+4)/2}}{(q;q)_{2s}} + \sum_{s=1}^{\infty} \frac{q^{2s+2(s-1)(k+4)/2}}{(q;q)_{2s-1}} = f_4^k(q).$$

Step III. We shall prove that

$$\sum_{v=0}^{\infty} C_4^k(v)q^v = f_4^k(q). \tag{2.33}$$

We write $\delta_{2s}(4,k;v) = c_1 + c_2 + \cdots + c_{2s}$, where

$$\begin{aligned} c_{2s} &\geq 2 \\ c_{2s-2} &\geq 2 + (k+4)/2 \\ c_{2s-4} &\geq 2 + 2(k+4)/2 \\ &\cdots \\ c_2 &\geq 2 + (s-1)(k+4)/2 \end{aligned}$$

and

$$\begin{aligned} c_{2s-1} &\geq 2 \\ c_{2s-3} &\geq 2 + (k+4)/2 \\ c_{2s-5} &\geq 2 + 2(k+4)/2 \\ &\cdots \\ c_1 &\geq 2 + (s-1)(k+4)/2 \end{aligned}$$

It can now easily be seen that $\delta_{2s}(4,k;v)$ are generated by

$$\frac{q^{4s+s(s-1)(k+4)/2}}{(q;q)_{2s}}$$

Similarly, writing $\delta_{2s-1}(4,k;v) = c_1 + c_2 + \cdots + c_{2s-1}$, where

$$\begin{aligned} c_{2s-2} &\geq 1 + (k+4)/2 \\ c_{2s-4} &\geq 1 + (k+4)/2 \\ &\cdots \\ c_2 &\geq 1 + (s-1)(k+4)/2 \end{aligned}$$

and

$$c_{2s-1} \geq 2$$
$$c_{2s-3} \geq 1 + (k+4)/2$$
$$c_{2s-5} \geq 1 + 2(k+4)/2$$
$$\dots$$
$$c_1 \geq 1 + (s-1)(k+4)/2$$

we see that $\delta_{2s-1}(4,k;v)$ are generated by

$$\frac{q^{2s+s(s-1)(k+4)/2}}{(q;q)_{2s-1}}$$

As in step II, we can now show that

$$\sum_{v=0}^{\infty} A_4^k(v)(q^v) = f_4^k(q),$$

and the Theorem 4 is proved.

3 Particular cases

We shall discuss only those particular cases in which our theorems yield 4 way partition identities. We divide this section into four subsections and discuss the particular cases of each theorem seperately.

3.a Particular cases of Theorem 1

Case I. When $k = -1$

Let $D_1^{-1}(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0$, ± 2, ± 6, ± 8, $10 \pmod{20}$.

Then in view of the identity [5, (79)-(98)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1-q^{10n-2}\right)\left(1-q^{10n-8}\right) \left(1-q^{20n-14}\right)\left(1-q^{20n-6}\right)\left(1-q^{10n}\right) \tag{3.a.1}$$

Theorem 1 gives the following 4-way partition identity:

$$A_1^{-1}(v) = B_1^{-1}(v) = C_1^{-1}(v) = D_1^{-1}(v) \tag{3.a.2}$$

As mentioned earlier each 4-way identity gives 6 identities in the usual sense.

$$A_1^{-1}(v) = B_1^{-1}(v) \tag{3.a.3}$$
$$A_1^{-1}(v) = C_1^{-1}(v) \tag{3.a.4}$$
$$A_1^{-1}(v) = D_1^{-1}(v) \tag{3.a.5}$$
$$B_1^{-1}(v) = C_1^{-1}(v) \tag{3.a.6}$$
$$B_1^{-1}(v) = D_1^{-1}(v) \tag{3.a.7}$$
$$C_1^{-1}(v) = D_1^{-1}(v) \tag{3.a.8}$$

Identity (3.a.5) is due to the author [1, Theorem 1.1, p. 301], (3.a.7) is due to Subbarao [7, Theorem 2.2, p. 432] (3.a.8) is originally due to Gorden [4, Theorem 7] and was also proved by Hirschhorn [6, Theorem 1, p. 33] and Blecksmith et.al. [9, Theorem 8.1, $r = 8$, p. 748].

Case II When $k = 1$

Let $D_1^1(v)$ denote the number of ordinary partitions of v into odd parts then in view of the identity [5, (84)-(85), p. 161]

$$\sum_{n=0}^{\infty} \frac{q^{n2(n-1)}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \tag{3.a.9}$$

we get the following 4-way identity from our Theorem 1:

$$A_1^1(v) = B_1^1(v) = C_1^1(v) = D_1^1(v) \tag{3.a.10}$$

$A_1^1(v) = D_1^1(v)$ is Theorem 1.4 in [1].

3.b Particular cases of Theorem 2

Case I. When $k = -1$

Let $D_2^{-1}(v)$ denote the number of ordinary partition of v into parts $\not\equiv 0$, ± 3, ± 4, ± 7, 10 (mod 20). Then in view of the identity [5, (94)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1-q^{10n-3}\right)\left(1-q^{10n-7}\right) \left(1-q^{20n-16}\right)\left(q^{20n-4}\right)\left(1-q^{10n}\right). \tag{3.b.1}$$

We get the following 4-way identity from oue Theorem 2:

$$A_2^{-1}(v) = B_2^{-1}(v) = C_2^{-1}(v) = D_2^{-1}(v) \tag{3.b.2}$$

Identity $A_2^{-1}(v) = D_2^{-1}(v)$ is Theorem 1.2 in [1, p. 301]. Identity $B_2^{-1}(v) = D_2^{-1}(v)$ is due to Subbarao [7, Theorem 2.1, p. 432]. Identity $C_2^{-1}(v) = D_2^{-1}(v)$ is due to Hirschhorn [6, Theorem 2, p.33] and was also proved by Blecksmith et.al. [9, Theorem 8.1, $r = 7$, p. 748].

Case II. When $k = 1$

Let $D_2^1(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0$, ± 2, ± 3, ± 5, 8 (mod 16) then in view of the identity [5, (38)-(86)]

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - q^{8n-3}\right)\left(1 - q^{8n-5}\right) \left(1 - q^{16n-14}\right)\left(1 - q^{16n-2}\right)\left(1 - q^{8n}\right) \tag{3.b.3}$$

we get the following 4-way identity from our Theorem 2:

$$A_2^1(v) = B_2^1(v) = C_2^1(v) = D_2^1(v) \tag{3.b.4}$$

Identity $A_2^1(v) = D_2^1(v)$ is Theorem 1.7 of [1]. We remark here that $A_2^1(v) = A_7(v)$ of Theorem 1.7 in [1]. This can easily be seen by increasing the subscripts in the partitions enumerated by $A_2(v)$ by 1.

Identity $B_2^1(v) = D_2^1(v)$ seems to be new, but has a striking resemblance with Subbarao's Theorem 2.4 in [7]. Identity $C_2^1(v) = D_2^1(v)$ is due to Hirschhorn [6, Theorem 3, p. 33] and was also obtained by Blecksmith et.al. [9, Theorem 8.1, $r = 5$, p. 748].

3.c Particular cases of Theorem 3

We consider the case when $k = -1$.

Let $D_3^{-1}(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0$, ± 2, ± 4, ± 6, 10 (mod 20). Then in view of the identity [5, (96)].

$$\sum_{n=0}^{\infty} \frac{q^n(n+2)}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - q^{10n-4}\right)\left(1 - q^{10n-6}\right) \left(1 - q^{20n-18}\right)\left(1 - q^{20n-2}\right)\left(1 - q^{10n}\right) \tag{3.c.1}$$

we get the following 4-way identity from our Theorem 3:

$$A_3^{-1}(v) = B_3^{-1}(v) = C_3^{-1}(v) = D_3^{-1}(v) \tag{3.c.2}$$

identity $A_3^{-1}(v) = D_3^{-1}(v)$ is Theorem 1.3 of [1]. Identity $C_3^{-1}(v) = D_3^{-1}(v)$ is due to Blecksmith et.al [9, Theorem 8.1, $r = 9$, p. 748].

3.d Particular cases of Theorem 4

Case I. When $k = -2$

Let $D_4^{-2}(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0$, ± 1, ± 8, ± 9, 10 (mod 20).

Then in view of the identity [5, (99)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1-q^{10n-1}\right)\left(1-q^{10n-9}\right) \left(1-q^{20n-8}\right)\left(1-q^{20n-12}\right)\left(1-q^{10n}\right) \tag{3.d.1}$$

We get the following 4-way identity from Theorem 4

$$A_4^{-2}(v) = B_4^{-2}(v) = C_4^{-2}(v) = D_4^{-2}(v). \tag{3.d.2}$$

Identity $A_4^{-2}(v) = D_4^{-2}(v)$ Theorem 1.5 of [1]. Identity $B_4^{-2}(v) = D_4^{-2}(v)$ is due to Subbarao and Agarwal [8, Theorem 1.4, p. 211] and identity $C_4^{-2}(v) = D_4^{-2}(v)$ is due to Blecksmith et.al. [9, Theorem 8.1, $r = 10$, p. 748].

Case II. When $k = 0$

Let $D_4^0(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0$, ± 1, ± 6, ± 7, 8 (mod 16). Then in view of the identity [5, (39)-(83)]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1-q^{8n-1}\right)\left(1-q^{8n-7}\right) \left(1-q^{16n-10}\right)\left(1-q^{16n-6}\right)\left(1-q^{8n}\right) \tag{3.d.3}$$

we get the following 4-way identity from Theorem 4

$$A_4^0(v) = B_4^0(v) = C_4^0(v) = D_4^0(v). \tag{3.d.4}$$

Identity $B_4^0(v) = D_4^0(v)$ is believed to be new but is very similar to Subbarao's Theorem 2.3 in [7]. Identity $C_4^0(v) = D_4^0(v)$ is due to Hirschhorn [6, Theorem 4, p. 34] and was also obtained by Blecksmith et.al. [9, Theorem 8.1, $r = 6$, p. 748].

4 Conclusion

The most obvious questions which arise from this work are

1. Is it possible to give 3-way correspondence for our Theorems 1-4?
2. For certain values of k we have interpreted $f_i^k(q)$ $(1 \le i \le 4)$ combinatorially in four different ways, is it possible to do this for general value of k?

References

[1] A.K. Agarwal, Rogers-Ramanujan identities for n-color partitions, *J. Number Theory* 28(1988), 299–305.

[2] A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N", *J. Combin. Theory Ser. A* 45,(1)(1987), 40–49.

[3] A.K. Agarwal and D.M. Bressoud, Lattice paths and multiple basic hypergeometric series, *Pacific J. Math.* 136(1989), 209–228.

[4] B. Gordon, Some continued fractions of the Rogers-Ramanjan type, *Duke Math. J*, 32(1965), 741–748.

[5] L.J. Slater, Further identities of the Rogers-Ramanjan type, *Proc. London Math. Soc.* 54(1951-52), 147–167.

[6] M.D. Hirschhorn, Some partition theorems of the Rogers-Ramanjan type, *J. Combin. Theory Ser a.* 27,(1)(1979), 33–37.

[7] M.V. Subbarao, Some Rogers-Ramanjan type partition theorems, *Pacific J. Math.* 120(1985), 431–435.

[8] M.V. Subbarao and A.K. Agarwal, Further theorems of the Rogers-Ramanjan type theorems, *Canada. Math. Bull.* 31,(2)(1988), 210–214.

[9] R. Blecksmith, J. Brillhart and I. Gerst, A computer-assisted investigation of Ramanujan pairs, *Math. Comp.* 46, (174)(1986), 731–749.