Packing and covering of the complete graph, V: The forests of order six and their multiple copies

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ABSTRACT. It is shown that the maximal number of pairwise edge disjoint forests, F, of order six in the complete graph K_n, and the minimum number of forest of order six, whose union is K_n are $\lfloor\frac{n(n-1)}{2e(F)}\rfloor$ and $\lceil\frac{n(n-1)}{2e(F)}\rceil$, $n \geq 6$, respectively and $e(F)$ is the number of edges of F. ($\lfloor x\rfloor$ denotes the largest integer not exceeding: x and $\lceil x\rceil$ the least integer not less than x). Some generalizations to a multiple copies of that forests and of paths are also given.

1. Introduction

Graphs in our context are undirected, finite, and have no multiple edges or loops. We refer to [4] for the basic definitions.

We denote by $P(n, H)$, the *packing number*, namely, the maximal number of pairwise edge disjoint graphs H, in the complete graph K_n and by $C(n, H)$, the *covering number*, namely, the minimum number of graphs H whose union is K_n.

As usual $\lfloor x\rfloor$ will denote the largest integer not exceeding x and $\lceil x\rceil$ the least integer not less than x.

In [6]–[9] it was proved that:

$$P(n,T) = \left\lfloor \frac{n(n-1)}{2e(T)} \right\rfloor \tag{1}$$

and

$$C(n,T) = \left\lceil \frac{n(n-1)}{2e(T)} \right\rceil, \text{ for } n \geq n_0, \tag{2}$$

where T was any tree of order less than equal seven, $e(T)$ is the number of edges of T and n_0 was a constant determined in the various cases.

Definition: A graph H is said to have a *G-decomposition* if it is the union of edge disjoint subgraphs each isomorphic to G. We denote this fact by $G \mid H$.

The *G-decomposition problem*, for $H = K_n$ is to determine the set of naturals $N(G)$, such that K_n has a G-decomposition if and only if $n \in N(G)$.

Note that G-decomposition is actually an exact packing and covering.

In the proof of our problems of packing and covering, we make use of results obtained by Bialostocki and Roditty [2] for $3K_2$, and by Yin and Gong [12] for the rest of the forests of order six.

We denote $H = \bigcup_{i=1}^{t} G_i$ when the graph H is the union of t edge disjoint graphs G_i, $i = 1, 2, \ldots, t$.

The packing and covering results which will be discussed in details in the remainder of the paper, can be summarized in:

Main Theorem: (Packing and Covering).

(a) $P(n, F) = \lfloor \frac{n(n-1)}{2e(F)} \rfloor$, *$n \geq 6$ and F any forest order six.*

(b) $C(n, F) = \lceil \frac{n(n-1)}{2e(F)} \rceil$, *$n \geq 6$ and F any forest order six.*

Further we shall give some results concerning multiple copies of paths and the forgets of order six.

The relevant forests to our problems are:

(i) $F_1 = 3K_2$, the matching consists of three edges.

(ii) $F_2 = 2P_3$, two vertex disjoint path of order three, denoted$[(x, y, z)(u, v, w)]$.

(iii) $F_3 = K_{1,3} \cup K_2$,the star with three edges with a vertex disjoint edge, denoted $[(x; y, z, w)(u, v)]$.

(iv) $F_4 = P_4 \cup K_2$, the path on three edges with a vertex disjoint edge, denoted $[(x, y, z, w)(u, v)]$.

2. Results

Notation: The vertex set of K_n is defined to be Z_n, and addition of vertex labels are done $\mod n$. By $K_m(t)$ we denote the complete t-partite graph in which each color class is of size m. Also we define: $V(K_{2,4}) = \{a, b\} \cup \{0, 1, 2, 3\}$, $V(K_{3,4}) = \{a, b, c\} \cup \{0, 1, 2, 3\}$.

We start with F_1. In [2] we have a general theorem concerning F_1, namely,

Theorem 2.1. *The necessary conditions for a graph G to have a F_1-decomposition:*

$$e(G) = 3k, \tag{3}$$

$$\Delta(G) \leq k, \tag{4}$$

are also sufficient if $e(G) \geq 15$.

Remark 1:

1. In [12] we find that $N(F_1) = \{n \mid n \equiv 0, 1 \pmod 3, n \geq 6\}$.
2. In [2] one can find the list of all the exceptional graphs, namely, graphs satisfying conditions (3) and (4) but have no F_1-decomposition. The algorithm for finding those graphs is described there.

Now we are ready to prove the Main Theorem for F_1.

Theorem 2.2. *The Main Theorem is valid for F_1.*

Proof: By Theorem 2.1 and Remark 1 (1) we have to prove only the case when $n = 3m + 2$, $m \geq 2$. Let $G = K_{3m+2} \backslash e$, where e is any edges of K_{3m+2}. Then by Theorem 2.1, G has a F_1-decomposition, leaving e as a nonpacked edge for the covering. □

In fact we can have a general result than Theorem 2.2, namely, the Main Theorem is valid for tK_2. Indeed, using the well-known Hamilton cycle decomposition ([4] p.89) and a result due to Alon [1], we get:

Theorem 2.3. *The Main Theorem is valid for tK_2.*

Before proving the main theorem for F_2 we prove some general results concerning tP_3. First recall a theorem of Caro and Schönheim [3],

Theorem 2.4. *Necessary and sufficient condition for a graph G to have a P_3-decomposition is that each component of G has an even number of edges.*

A simple lemma in which the proof is omitted is:

Lemma 2.5. $tP_3 \mid K_4(t)$.

As a consequence of Lemma 2.5 we have,

Theorem 2.6. $tP_3 \mid K_{4t}; K_{4t+1}$.

Proof: Let $K_{4t} = tK_4 \cup K_4(t)$. Hence, tK_4 has an obvious tP_3-decomposition (since $P_3 \mid K_4$ by Theorem 2.4), and so does $K_4(t)$ by Lemma 2.5. For

K_{4t+1} we give the direct construction of the decomposition, namely, the $4t+1$ copies of tP_3 are:

$$[(0,4t,2t)(1,4t-1,2t+1),(2,4t-2,2t+2),\dots,(t-1,3t+1,3t-1)]$$
$$(\mathrm{mod}(4t+1))(t\geq 1).$$

□

Lemma 2.7. *$tP_3 \mid K_{ta,2tb}; K_{(t+1)c,2td}$, where a,b,c,d are positive integers.*

Proof: We shall prove the lemma in the case $a=b=c=d=1$. The general result follows immediately from that case.

Contract the color classes of size 2t into a color class of size t by contracting two vertices to one, so that we have $K_{t,2t}\to G_1=K_{t,t}$, $K_{t+1,2t}\to G_2=K_{t+1,t}$, where each edge in G_1,G_2 is a $K_{1,2}$ in the original graph. Hence, by a result of Alon [1] (see also [5]), we have tK_2-decomposition of G_1,G_2 which is a tP_3-decomposition of the required graphs. □

Now we are ready for,

Theorem 2.8. *$tP_3 \mid K_{4tm}; K_{4tm+1}$, $m\geq 1$, an integer.*

Proof: We use induction upon m. For $m=1$ it was proved in Theorem 2.6. Assume we have proved it for all $w\leq m-1$. Let,

$$K_{4tm+j}=K_{4t(m-1)}\cup K_{4t+j,4t(m-1)}\cup K_{4t+j}, j=0,1.$$

Then using the induction hypothesis together with Theorem 2.6 and Lemma 2.7, we are done. □

Remark 2: Theorem 2.8 is generalization of a result obtained in [12] for $t=2$ (for $t=1$ we have Theorem 2.4).

Now we are ready to prove the Main Theorem for F_2.

Theorem 2.9. *The Main Theorem is valid for F_2, for $n\geq 7$.*

Proof: For $n=7$ we have $P(7,F_2)=5$, $C(7,F_2)=6$, as the following F_2-packing of K_7 shows: $[(2,1,5)(0,4,3)]$, $[(2,3,0)(1,4,5)]$, $[(1,6,4)(2,0,5)]$, $[(2,6,5)(0,1,3)]$, $[(0,6,3)(4,2,5)]$. The edge $(0,5)$ is left for the covering.

Now by Theorem 2.8, for $t=2$, we have to prove the Main Theorem for the cases: $n=8m+j$, $j=2,3,\dots,7$. Let,

$$K_{8m+j}=K_{8m}\cup K_{j,8m}\cup K_j. \tag{5}$$

By Theorem 2.8 and Lemma 2.7 (for $t=2$), $F_2 \mid K_{8m}; K_{j,8m}$.

Denote the vertices of K_j by $8m, 8m+1,\dots,8m+j-1$.

Observe that by Theorem 2.4, K_j has either a P_3-decomposition or a P_3-packing leaving one non-packed edge.

We prove now according to the various cases of j.

j=2: The single edge K_2 in (5) is left for the covering.

j=3: Take any copy of F_2 from the decomposition of K_{8m} with some component P_3, say, (v, u, w). Replace this component by $(8m, 8m + 1, 8m + 2)$, so that the packing is accomplished, leaving the subgraph $(u, v, w) \cup (8m, 8m + 2)$ non-packed for the covering.

j=4: We use the same idea as in the previous case, by taking some copy of F_2 from the decomposition of K_{8m}. Since $P_3 \mid K_4$ such that there are three copies of P_3 in K_4, we replace one of the components of F_2 by one the P_3's and the other component with another copy of the P_3's, leaving exactly one P_3 non-packed, for the covering.

j=5: Using the same idea as in the previous case, by taking now two copies of F_2 from the decomposition of K_{8m} and replacing the appropriate components by others from the P_3-decomposition of K_5. This procedure leaves again one non-packed P_3 for the covering.

j=6: Observe that $G = K_6 \backslash K_{1,2} \cup K_2$, is connected with even number of edges. Hence by Theorem 2.4, $P_3 \mid G$, and we have six copies of P_3 in that decomposition. Thus, choose three copies of F_2 from K_{8m} and to each of two of the six P_3-copies of G, the same way as we did for $j = 5$. This completes the packing. The graph $K_{1,2} \cup K_2$ is left for the covering.

j=8: Take the F_2-packing and covering of K_7 from above and we are done.

This completes the proof of the theorem. □

Remark 3:

1. For $n = 6$ one can easily see that $P(6, F_2) = 3$, which proves the packing, but $C(6, F_2) = 5$.

2. Using similar ideas as in the proof of Theorem 2.9 we can prove the validity of the Main Theorem for tP_3, t even, say $t = 2k$, in particular cases, namely, $n = 4tm + j$, $2t \leq j \leq 4t - 1$. Put, $K_{4tm+j} = K_{4tm} \cup K_{j,4tm} \cup K_j$. By Lemma 2.7 and Theorem 2.8 $tP_3 \mid K_{4tm}; K_{j,4tm}$. Now take the kP_3-packing of K_j and only in the case where the number of kP_3 components in K_j is also even. For each two kP_3 components in K_j take some tP_3 component in K_{4tm} and match the two halves of it with the two mentioned components. Thus, the packing is completed and the subgraph left in K_j for the covering is good as well for K_{4tm}.

□

Before proving the Main Theorem for F_3 we start with some preliminary results from which we derive also some general results concerning tF_3, $t \geq 1$.

We start with a simple and almost straightforward lemma, where the proof is omitted:

Lemma 2.10. $tF_3 \mid K_{2ta,4tb}; K_{3tc,4td}$, *where* a, b, c, d *are positive integers.*

Theorem 2.11. $tF_3 \mid K_{8t}; K_{8t+1}$.

Proof: Let, $K_{8t} = tK_8 \cup K_8(t)$. From [12] we have $F_2 \mid K_8$, thus $tF_2 \mid tK_8$. To have $tF_2 \mid K_8(t)$, contract in each color class two vertices to obtain from $K_8(t)$ a graph $G = K_{2t} \backslash M$, where M is a complete matching and each edge in G is $K_{4,4}$ in $K_8(t)$. Then we have $tK_2 \mid G$ where each K_2 is $K_{4,4}$ in $K_8(t)$. Hence, the result follows using Lemma 2.10.

For K_{8t+1} take the $tK_{1,4}$-decomposition presented as follows:

$$(0; 1,2,3,4), (5; 10,11,12,13), (8; 17,18,19,20), \\ \ldots, (3t-1; 7t-4, 7t-3, 7t-2, 7t-1), \quad (\mathrm{mod}(8t+1)), t \geq 2,$$

For $t = 1$ take only the first term.

If t is even we match the components in pairs, say, A and B such that in order to get tF_2-decomposition we take from A a $K_{1,3}$ together with an endedge of B the remaining parts from A and B give also a F_2. In case of $t \geq 3$ odd we separate each time three components say, A, B, and C, such the remaining $t-3$ components are of even number so that we may apply the procedure described above. For the three copies of $K_{1,4}$ we match them as follows, $A \to B \to C \to A$, namely, $K_{1,3}$ from A with an endedge from B, the remaining of B with an endedge of C, and so on. Thus, we accomplish the tF_2-decomposition of K_{8t+1} □

Theorem 2.12. $tF_3 \mid K_{8tm}; K_{8tm+1}$, $m \geq 1$ *an integer.*

Proof: For K_{8tm} we use induction upon m. For $m = 1$ we have proved it in Theorem 2.11. Assume we have proved the theorem for all $w \leq m-1$. Let $K_{8tm} = K_{8t(m-1)} \cup K_{8t,8t(m-1)} \cup K_{8t}$. Then by the induction hypothesis Lemma 2.10 and Theorem 2.11 we have the required decomposition. For K_{8tm+1} we take a tmF_3-decomposition promised by Theorem 2.11. Since each component gives m copies of tF_4 we are done. □

Remark 4: Theorem 2.12 is a generalization of a result obtained in [12] for $t = 1$.

Now we prove the Main Theorem for F_3.

Theorem 2.13. *The Main Theorem is valid for* F_3, $n \geq 7$.

Proof: First we start with $n = 7$. Let the packing of K_7 be $[(0+j; 1+j, 2+j, 3+j)(4+j, 5+j)]$, $j = 0,1,2$, $[(6; 1,2,3)(0,5)]$, $[(4,0,3,6)(1,5)]$.

The edge $(3,5)$ is left for the covering. Let K_{8m+j} $j = 2, 3, \ldots, 7$ be as in (5). By Theorem 2.12 (for $t = 1$) and Lemma 2.10, $F_3 \mid K_{8m}; K_{j,8m}$.

Denote the vertices of K_j by $8m, 8m+1, \ldots, 8m+j-1$. We prove now according to the various cases of j.

j=2: The single edges in (5) is left for the covering.

j=3: Take some copy of F_3 from the decompositions of K_{8m} with the single edge say (u, v). Replace that edge by the edge $(8m, 8m+1)$. The packing is left with no changes and the subgraph $[(8m, 8m+2, 8m+1)(u, v)]$ is left for the covering.

j=4: Take some copy of F_3 from the decomposition of K_{8m}, say, $[(x; y, z, w)\ (u, v)]$. Instead of the single edge take $(8m, 8m+1)$, and the edge (u, v) together with $(8m+2; 8m+1, 8m, 8m+3)$ created a new F_3, leaving $(8m+1, 8m+3, 8m)$ for the covering.

j=5: Take two copies of F_3 from the decomposition of K_{8m}, say, $[(x; y, z, w)\ (u, v)]$, $[(a; b, c, d)(e, f)]$. With the edges of K_5 we create the following new F_3's. $[(x; y, z, w)(8m, 8m+1)]$, $[(a; b, c, d)(8m+1, 8m+2)]$, $[(8m; 8m+2, 8m+3, 8m+4)(u, v)]$, $[(8m+3; 8m+1, 8m+2, 8m+4)(e, f)]$. We are left with the non-packed P_3: $(8m+1, 8m+4, 8m+2)$ for the covering.

j=6: Let be a F_3-packing of K_6: $[(8m; 8m+1, 8m+2, 8m+3)(8m+4, 8m+5)]$, $[(8m+1; 8m+2, 8m+3, 8m+4)(8m, 8m+5)]$, $[(8m+5; 8m+1, 8m+2, 8m+3)(8m, 8m+4)]$, leaving the triangle $(8m+2, 8m+3, 8m+4)$ nonpacked. Take some copy of F_3 from the decomposition of K_{8m}, say, $[(x; y, z, w)(u, v)]$. Instead the edge (u, v) take the edge $(8m+2, 8m+3)$, so that we are left with the nonpacked subgraph $(8m+2, 8m+4, 8m+3)(u, v)$ for the covering.

j=7: Take the packing and covering of K_7 from above.

This completes the proof of the theorem. □

Remark 5: Partial resutls concerning tF_3-packing can be obtained using the same ideas as in Remark 3, for tF_2.

We start with some preliminaries concerning F_4. The following Lemma has a simple and straightforward proof.

Lemma 2.14. *$F_4 \mid K_{3a,4b}; K_{4c,4d}; K_{3e+4f,4g}$ where a, b, c, d, g are positive integers and e, f are integers with at most one being zero.*

Corollary 2.15. *$tF_4 \mid K_{3t,4t}; K_{4t,4t}$.*

Theorem 2.16. *$tF_4 \mid K_{8t}; K_{8t+1}$.*

Proof: The proof is based upon the proof for $t = 1$ and the same arguments as in the proof of Theorem 2.11. Again we use for K_{8t} the same ideas as in Theorem 2.11. For K_{8t+1} take the tP_5-decomposition presented as follows

$$(0;4,1,3,2),(8;8t,9,8t-1,10),(11;8t-2,12,8t-3,13),$$
$$\dots,(3t+2;6t+4,3t+3,6t+3,3t+4), \quad (\bmod(8t+1)), t \geq 2,$$

For $t = 1$ take only the first term.

If t is even we match the components in pairs, say, A and B such that in order to get tF_4-decomposition we take from A a P_4 together with an endedge of B the remaining parts from A and B give also a F_4. In case of $t \geq 3$ odd we separate each time three components, say, A, B, and C, such that the remaining $t-3$ components are of even number so that we may apply the procedure described above. For the three copies of P_5 we match them as follows, $A \to B \to C \to A$, namely, P_4 from A with an endedge from B, the remaining B with an endedge of C, and so on.

Thus, we accomplish the tF_4-decomposition of K_{8t+1}. □

Corollary 2.17. $tF_4 \mid K_{8tm}; K_{8tm+1}$.

Now we are ready for the proof of the Main Theorem for F_4.

Theorem 2.18. *The Main Theorem is valid for F_4.*

Proof: The proof will take care of several cases according to the various values of n.

n	Packing	Remain for Covering
6	$[(0,1,3,4)(2,5)$ (mod 6)	$(2,0,4) \cup (1,5)$
7	$[(1,0,2,6)(3,4)]$, $[(1,2,4,0)(5,6)]$	
	$[(2,3,5,1)(0,6)]$, $[(4,1,3,6)(2,5)]$	$(4,6)$
	$[(4,5,0,3)(1,6)]$	
$8m, 8m+1$	F_4-decomposition [12]	
10	$[(0,1,3,6)(5,9)]$ (mod 10)	$(3,8)$
	$[(2,5,1,6)(4,9)]$, $[(7,2,0,9)(4,8)]$	
13	$[(0,1,3,6)(7,11)]$ (mod 13)	
	$[(0+j,5+j,10+j,2+j)(6+j,12+j)]$(mod 13)	
	$j=0,1,2$	$(3,9,4)$
	$[(12,5,11,4)(1,7)]$, $[(1,9,2,8)(3,10)]$	
	$[(6,0,8,3)(4,10)]$	

Let K_{8m+j} be as in (5). We shall use (5) in all cases of j but $j = 2,5$. Observe that in the cases of $j \neq 2,5$ we have a result of [12] and Lemma 2.14 $F_4 \mid K_{8m}; K_{j,8m}$.

j=2: Let,

$$K_{8m+2} = K_{8(m-1)} \cup K_{10,8(m-1)} \cup K_{10}, m \geq 2.$$

Then by the table the packing and covering of $K_{8(m-1)}$, K_{10} is completed and by Lemma 2.14 we have the decomposition of $K_{10,8(m-1)}$.

j=3: Take some copy of F_4 from the decomposition of K_{8m}, say, $[(x; y, z, w)\ (u, v)]$ and replace the edge (u, v) by $(8m, 8m + 1)$. Hence, the packing is not changed and we are left with $(8m+1, 8m+2, 8m) \cup (u, v)$ for the covering.

j=4: Take again some copy of F_4 from the decomposition of K_{8m}, say, $[(x, y, z, w)\ (u, v)]$ and with the edges of K_4 we create the following two F_4's: $[(x, y, z, w)(8m, 8m+1)]$, $[(8m+1, 8m+2, 8m, 8m+3)(u, v)]$, leaving $(8m + 1, 8m + 3, 8m + 2)$ for the covering.

j=5: Let,

$$K_{8m+5} = K_{8(m-1)} \cup K_{13,8(m-1)} \cup K_{13}, m \geq 2.$$

Then by the table the packing and covering of $K_{8(m-1)}$, K_{13} is completed and by Lemma 2.14 we have the decomposition of $K_{13,8(m-1)}$.

j=6,7: Take the packing and covering of K_6 and K_7 from the table and we are done.

This completes the proof of the Theorem. □

Final Remark

A complete version of the paper in which a detailed proofs are represented is in [10].

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