

Holey Factorizations

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ABSTRACT. Holey factorizations of K_{v_1, v_2, \dots, v_n} are a basic building block in the construction of Room frames. In this paper we give some necessary conditions for the existence of holey factorizations and give a complete enumeration for nonisomorphic sets of orthogonal holey factorizations of several special types.

1 Introduction

The use of holes in designs is one of the most powerful tools in combinatorial design theory. The purpose of this paper is to study designs called holey factorizations. These objects are essentially one-factorizations of K_n with holes. They have been used implicitly in the construction of Room frames, but have never been studied systematically. We will give some general necessary conditions for the existence of holey factorizations, will discuss existence of holey factorizations of certain types and will give a complete enumeration of all nonisomorphic sets of orthogonal holey factorizations of several small types. We begin with the definitions.

Let V be a set of v vertices, and let $\{V_1, \dots, V_n\}$ be a partition of V , where $|V_i| = v_i$. Let K_{v_1, \dots, v_n} denote the complete multipartite graph with vertices V and with parts $\{V_1, \dots, V_n\}$. These parts are called the *holes* since $K_{v_1, \dots, v_n} = K_v \setminus \bigcup_{i=1}^n \{xy | x, y \in V_i\}$. A *holey factor* of K_{v_1, \dots, v_n} , missing hole V_i , is a one-factor of the graph $K_{v_1, \dots, v_n} \setminus V_i$ (i.e. a set of edges such that each vertex of $K_{v_1, \dots, v_n} \setminus V_i$ is on exactly one of these edges and there is no edge between any two vertices in the same hole). A *holey*

factorization of K_{v_1, \dots, v_n} is a partition of the edges of the graph into holey factors such that for each $1 \leq i \leq n$ there are exactly v_i holey factors missing hole V_i .

The *type* of a holey factorization with holes $\{V_1, V_2, \dots, V_n\}$ is the multiset $\{|V_1|, |V_2|, \dots, |V_n|\}$. We will say that a holey factorization has *type* $T = t_1^{u_1} t_2^{u_2} \dots t_k^{u_k}$ if there are u_i V_j 's of cardinality t_i , $1 \leq i \leq k$. If a holey factorization has type t^n for some t and n , then it is called *uniform*.

Holey factorizations represent a special case of a general object called a frame (not to be confused with a Room frame). A *frame* is a group-divisible design (X, G, B) whose block set admits a partition into *holey* parallel classes, each holey parallel class being a partition of $X \setminus G_i$ for some group $G_i \in G$. A holey factorization is a frame where *every* block has size 2. For further information pertaining to frames in general see [12].

A holey factorization of type $1^1 3^4$ is given in Figure 1. Note that the holes $V_1 = \{1\}$, $V_2 = \{2, 3, 4\}$, $V_3 = \{5, 6, 7\}$, $V_4 = \{8, 9, a\}$, and $V_5 = \{b, c, d\}$ are written on the left of each holey factorization in square brackets.

[1]	{9 c}	{5 b}	{2 d}	{3 6}	{7 a}	{4 8}
[234]	{8 b}	{a d}	{5 c}	{7 9}	{1 6}	
[234]	{1 c}	{9 b}	{5 d}	{6 a}	{7 8}	
[234]	{5 9}	{a c}	{1 b}	{7 d}	{6 8}	
[567]	{2 a}	{1 8}	{9 d}	{4 b}	{3 c}	
[567]	{a b}	{8 d}	{4 c}	{1 3}	{2 9}	
[567]	{8 c}	{1 d}	{2 b}	{4 9}	{3 a}	
[89a]	{6 d}	{7 c}	{3 b}	{1 2}	{4 5}	
[89a]	{6 b}	{1 7}	{2 c}	{4 d}	{3 5}	
[89a]	{3 d}	{7 b}	{6 c}	{1 4}	{2 5}	
[bcd]	{5 8}	{1 a}	{3 9}	{2 7}	{4 6}	
[bcd]	{6 9}	{4 a}	{2 8}	{3 7}	{1 5}	
[bcd]	{4 7}	{5 a}	{3 8}	{1 9}	{2 6}	

Figure 1: A holey factorization of type $1^1 3^4$

Two holey factorizations F and G , both of type T , are said to be *orthogonal* if

1. for any two edges of the underlying graph (say e_1 and e_2), if e_1 and e_2 are in the same holey factor in F , then they are different holey factors of G ; and
2. any holey factor in F and any holey factor in G which are missing the same hole have no edges in common.

Our interest in holey factorizations stems from their connection to Room frames. The following theorem gives this connection. (For results on Room frames see [4] or [5]).

Theorem 1.1 *The existence of a pair of orthogonal holey factorizations of type T is equivalent to the existence of a Room frame of type T .*

The proof is simple. Given a Room frame of type T , the rows and the columns are both holey factorizations of type T . Clearly the row holey factorization and the column holey factorization are orthogonal. This construction can be reversed also.

In Figure 2 we give a Room frame of type 1^13^4 to help clarify the above connection. Note that the rows of this frame contain the holey factors in the holey factorization given in Figure 1.

	9c		5b	2d			36		7a	48	
				8b	ad		5c			79	16
				1c	9b		5d			6a	78
59						ac	1b		7d	68	
2a	18	9d					4b	3c			
		ab	8d				4c			13	29
8c							1d	2b	49		3a
	6d		7c			3b			12		45
6b		17			2c	4d			35		
3d	7b	6c			14					25	
		58	1a	39				27	46		
			69	4a		28	37	15			
47	5a				38	19	26				

Figure 2: A Room frame of type 1^13^4

Another connection which we wish to point out is given in the following theorem. Note that in order for a holey factorization of type 1^n to exist, then necessarily n must be odd; each factor consists of $(n - 1)/2$ edges.

Theorem 1.2 *The existence of a holey factorization of type 1^n is equivalent to the existence of a one-factorization of K_{n+1} .*

Proof: Given the holey factorization F , say each holey factor f_i is missing the vertex x_i . Add the new vertex ∞ to the graph and let $h_i = f_i \cup$

$\{x_i, \infty\}$. Then $\{h_1, h_2, \dots, h_n\}$ is a one-factorization of the graph K_{n+1} . This construction is easily reversed. \square

Results on one-factorizations of K_n can be found in [10] or [20].

This paper is organized as follows: Section 2 describes some necessary conditions for the existence of a holey factorization of type T , in Section 3 we prove the existence of many types of holey factorizations, and in Section 4 we discuss some results on the enumeration of nonisomorphic holey factorizations with small types, and of sets of orthogonal holey factorizations of types 2^4 and 2^5 .

2 Necessary conditions for existence

A holey factorization must have at least three holes; if a holey factorization had two holes, then a factor missing hole V_i would need to contain edges connecting pairs of vertices in the other hole. The following theorem applies to holey factorizations with exactly three holes.

Theorem 2.1 *If a holey factorization has exactly three holes, then necessarily all the holes must be of the same size.*

The proof follows immediately from the observation that the edges in any of the factors missing hole V_i must form a matching between the vertices in the remaining two holes. The following is a necessary parity condition.

Theorem 2.2 *If there exists a holey factorization of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$, then $t_i \equiv v \pmod{2}$ for each i where $v = \sum_{i=1}^n (t_i \times u_i)$ is the number of vertices in the underlying graph.*

Proof: Since for each i there must be a holey factor on $v - t_i$ vertices, then $v - t_i$ must be even. Thus $t_i \equiv v \pmod{2}$. \square

The next theorem gives a condition on the sizes of the holes.

Theorem 2.3 *If there exists a holey factorization on v vertices of type $T = t_1 t_2 t_3 \dots t_n$, then $v \geq 2t_i + t_j$, for any i and j ,*

Proof: Let V_i be a hole of size t_i and let V_j be a hole of size t_j . A holey factor missing V_j must pair up the vertices in V_i with a subset of vertices from $G - V_i - V_j$. Thus $t_i \leq v - t_i - t_j$. \square

Theorems 2.2 and 2.3 are probably close to being sufficient conditions for the existence of holey factorizations. However we exhibit below an infinite class of holey factorizations that is allowed by the previous three theorems yet can not exist. We thank R. M. Wilson for helpful discussions concerning the next theorem.

Theorem 2.4 *There exists a holey factorization of type $a^2t_1t_2t_3 \dots t_k$ where $a = t_1 + t_2 + t_3 + \dots + t_k$ if and only if there exists a holey factorization of type $t_1t_2t_3 \dots t_k$.*

Proof: Let U_1 and U_2 be the two holes of size a and let $V_1, V_2, V_3, \dots, V_k$ be the holes of sizes $t_1, t_2, t_3, \dots, t_k$, respectively. Now note that the $2a$ holey factors missing either U_1 or U_2 must contain every edge from $V_1, V_2, V_3, \dots, V_k$ to U_1 and U_2 . Thus, ignoring the edges between U_1 and U_2 , the holey factors missing the holes $V_1, V_2, V_3, \dots, V_k$ constitute a holey factorization of type $t_1t_2t_3 \dots t_k$. Thus, if there exists a holey factorization of type $a^2t_1t_2t_3 \dots t_k$, then there exists a holey factorization of type $t_1t_2t_3 \dots t_k$.

Conversely, begin with a holey factorization of type a^3 which exists by Theorem 3.1 below. Use Theorem 3.6 to fill in one hole of size a with a holey factorization of type $t_1t_2t_3 \dots t_k$. The resulting frame has type $a^2t_1t_2t_3 \dots t_k$. \square

Corollary 2.5 *There does not exist a holey factorization of type $a^2b^1c^1$ if $b + c \leq a$.*

Proof: If $b + c < a$, then this holey factorization does not exist by Theorem 2.3. If $b + c = a$, it does not exist by Theorem 2.4 since there is no holey factorization of type b^1c^1 . \square

Theorem 2.4 provides an infinite class of nonexistent holey factorizations which satisfy the necessary conditions of Theorems 2.2 and 2.3. Essentially, given a type T ($|T| = t$) for which no holey factorization exists, Theorem 2.4 states that there also does not exist a holey factorization of type t^2T . This process can then be repeated to show that there does not exist a holey factorization of type $(3t)^2t^2T$, etc. For example, since there is no holey factorization of type 2^2 , there is no holey factorization of type 4^22^2 . This in turn implies that there is no holey factorization of type $12^24^22^2$. There is also never a holey factorization of type t^2abc where $a + b + c = t$ and $a \neq b$, since holey factorizations of type abc exist only if $a = b = c$ (Theorem 2.1).

The next theorem extends Theorem 2.3 to the case of d orthogonal holey factorizations. The case of $d = 2$ was originally proven in [17].

Theorem 2.6 *If there exists a set of d orthogonal holey factorization on v vertices of type $T = t_1t_2t_3 \dots t_n$, then $v \geq (d + 1)t_i + t_j$, for any i and j .*

Proof: Let V_i be a hole of size t_i and let V_j be a hole of size t_j (containing the vertex x). Also let $\{F_1, F_2, \dots, F_d\}$ be a set of d pairwise orthogonal holey factorizations. Since the holey factors missing V_i must contain an edge on the vertex x , and since all of the holey factors in F_1, F_2, \dots and

F_d which are missing hole V_i contain no edges in common, we deduce that there must be at least $d \times t_i$ vertices other than the vertices in V_i or V_j . So $d \times t_i \leq v - t_i - t_j$ and hence $v \geq (d + 1)t_i + t_j$. \square

3 Existence of holey factorizations

In this section we will briefly discuss the existence of holey factorizations of certain specified types. Two particular types of Room frames have been extensively studied by researchers; these are frames of type h^u and of type $1^{u-v}v^1$. There have been numerous papers concerning the existence of these frames over the past 15 years. Recently, both spectra (for Room frames) have been essentially determined [19]. In contrast to the apparent difficulty in finding the spectra of the frames of these types, the first two theorems of this section give complete answers to the spectra of holey factorizations of these types. The remainder of this section will discuss some holey factorizations of other types.

The first result concerns uniform holey factorizations (type h^u). The proof can be found in [12] (Theorem 1.4). Note that the conditions are necessary by Theorem 2.2.

Theorem 3.1 (*Rees-Stinson*) *There exists a holey factorization of type h^u if and only if $u \geq 3$ and $h(u - 1) \equiv 0 \pmod{2}$.*

Holey factorizations of type $1^{u-v}v^1$ correspond to one-factorizations of K_{u+1} which contain sub one-factorizations of K_{v+1} . This problem has also been solved completely.

Theorem 3.2 (*Cruse [1]*) *There exists a holey factorization of type $1^{u-v}v^1$ if and only if u and v are odd and $u \geq 2v + 1$.*

Since the existence of a Room frame of type T implies the existence of a holey factorization of type T (via Theorem 1.1), results on the existence of Room frames apply to holey factorizations also. We summarize the list of known frames in the following theorem.

Theorem 3.3 *There exist Room frames (and hence holey factorizations) of the following types*

1. $1^a 3^b$ for all $a+b = 5, 7$ or 9 , except for $(a, b) = (2, 3), (3, 2), (4, 1), (5, 0), (5, 2)$ or $(6, 1)$, [5]
2. $2^a 4^b$ for all $a + b \in \{6, 7, \dots, 14, 31, 42, 43, 44\}$ or if $a + b \geq 48$, [5]
3. $2^n u^1$ if and only if u is even and $n \geq u + 1$, except possibly for $2^{19} 18^1$. [6] [19]

The following three constructions are recursive constructions for hole factorizations.

If T is the type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ and m is an integer, then mT is defined to be the type $(mt_1)^{u_1} (mt_2)^{u_2} \dots (mt_n)^{u_n}$. The following recursive construction is referred to as the Inflation Construction. It essentially “blows up” every hole by use of a Latin square of order m (thought of as a one-factorization of $K_{n,n}$). We leave the details to the reader.

Construction 3.4 *Suppose there exists a hole factorization of type T and suppose that m is a positive integer, then there exists a hole factorization of type mT .*

An interesting fact to note in the above construction is that m can be any positive integer. In the analogous theorem for frames there is a restriction that $m \neq 2$ or 6 .

The Fundamental Hole Factorization Construction follows immediately from the Fundamental Frame Construction [17].

Construction 3.5 (Fundamental Hole Factorization Construction)

Let $(X, \mathcal{G}, \mathcal{A})$ be a group divisible design having type T , and let $w : X \rightarrow Z^+ \cup \{0\}$ (we say that w is a weighting). For every $A \in \mathcal{A}$, suppose there is a hole factorization having type $\{w(x) : x \in A\}$. Then there is a hole factorization of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

The final recursive construction that we present here is a Filling in the Hole Construction. The proof is immediate.

Construction 3.6 *Suppose that there exists a hole factorization of type $T = t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ and a hole factorization of type $S = s_1^{v_1} s_2^{v_2} \dots s_m^{v_m}$ where $t_1 = \sum_{i=1}^m (s_i \times v_i)$, then there exists a hole factorization of type $t_1^{u_1-1} (s_1^{v_1} s_2^{v_2} \dots s_m^{v_m}) t_2^{u_2} \dots t_n^{u_n}$.*

There are many more recursive constructions that can be used to construct hole factorizations. Essentially, any recursive construction for Room frames will work for hole factorizations. This includes the Filling in the Holes and Fundamental Construction above, as well as the use of hole transversals, and the use of frame starters and intransitive frame starters. It is beyond our scope to discuss these additional constructions here. The interested reader is referred to [5], [17], [18], and [19].

The next two theorems follow from Theorem 3.3, where the missing cases were constructed using a modification of the hill-climbing algorithm for one-factorizations (described in [4]) to hole factorizations. This algorithm is extremely fast and effective in constructing hole factorizations. Files containing these factorizations are available from the authors.

Theorem 3.7 *There exists a holey factorization of type $1^a 3^b$ for all $a+b \in \{5, 7, 9\}$.*

Theorem 3.8 *There exists a holey factorization of type $2^a 4^b$ if and only if $a+b \geq 4$ with $(a, b) \neq (2, 2)$ or if $(a, b) = (3, 0)$ or $(0, 3)$.*

Note that the conditions in Theorem 3.8 are necessary by Theorems 2.1 and 2.2. The holey factorizations of types 2^3 and 4^3 exist by Theorem 3.1. The remaining holey factorizations which were missing from Theorem 3.3(2) were again constructed by the authors using the hill-climbing algorithm for holey factorizations.

We have also generated holey factorizations for other small orders by use of the hill-climbing algorithm for holey factorizations. We record these in the next theorem.

Theorem 3.9 *There exist holey factorizations of the following types: $2^4 6^1$, $2^3 6^2$, $2^2 6^3$, $2^1 6^4$, and $2^3 4^1 6^1$.*

In the next theorem we will be concerned with holey factorizations of type $2^n u^1$. From Theorems 2.2 and 2.3 we have that necessarily u must be even and $n \geq \frac{1}{2}u + 1$. From Theorem 3.3(3) there is a holey factorization of type $2^n u^1$ if $n \geq u + 1$. In the theorem below we will show essentially that if $n \geq \frac{2}{3}u$, then there exists a holey factorization of type $2^n u^1$.

Theorem 3.10 *There exists a holey factorization of type $2^n u^1$ if*

1. $n \geq \frac{1}{2}u + 1$, $u \equiv 2 \pmod{4}$ and n is even,
2. $n \geq \frac{2}{3}u$ and $u \equiv 0 \pmod{6}$,
3. $n \geq \frac{2}{3}(u + 2)$ and $u \equiv 4 \pmod{6}$, and
4. $n \geq \frac{2}{3}(u + 4)$ and $u \equiv 2 \pmod{6}$.

Proof: Statement 1 follows from Theorem 3.2 and Theorem 3.4 with $m = 2$.

To prove statement 2, give weight 2 to every point in the first three groups of a transversal design $TD(5, t)$, give weights 2, 4 or 6 to every point in the fourth group and give weight 6 to every point in the last group. (Note that $t \neq 2, 3, 6, 10$). Apply Construction 3.5, using the fact that holey factorizations of type $2^4 6^1$, $2^3 4^1 6^1$ and $2^3 6^2$ all exist by Theorem 3.9. Now fill in the first three groups with holey factorizations of type 2^t . Also fill in the fourth group with a holey factorization of type 2^a for $t \leq a \leq 3t$. Thus for every $4t \leq s \leq 6t$ we have constructed a holey factorization of

type $2^a(6t)^1$. Let $u = 6t$ to get that there is a holey factorization of type $2^a u^1$ for all $\frac{2}{3}u \leq s \leq u$, except for the cases $2^i 12^1$ for $8 \leq i \leq 12$, $2^i 18^1$ for $12 \leq i \leq 18$, $2^i 36^1$ for $24 \leq i \leq 36$, and $2^i 60^1$ for $40 \leq i \leq 60$.

To handle the case $u = 36$, first begin with transversal designs TD(5,12). Give weight 1 to every point in the first three groups, give weight either 1 or 3 to the points in the fourth group and give weight 3 to every point in the last group. Apply Construction 3.5, using the fact that holey factorizations of type $1^a 3^{5-a}$ exist for $a = 3, 4$ by Theorem 3.7. The construction now proceeds as above by filling in the first four groups with holey factorizations of type 2^a for $6 \leq a \leq 18$. The case $u = 60$ is done similarly beginning with a TD(5, 20). All of the missing cases for $u = 12$ and $u = 18$ (including $2^{19} 18^1$) were constructed on the computer by use of the hill-climbing algorithm. The final result now follows from Theorem 3.3(3).

Statement 3 is proven in a similar manner. Now give all the points in the last group of the TD(5,t) weight 6 except for one point which receives weight 4. The proof is then the same as above. Again the exceptional cases occur when $t = 2, 3, 6, 10$. The cases for $t = 2$ and $t = 3$ ($u = 10$ and $u = 16$, respectively) were handled by the computer. For $t = 6$ and $t = 10$ ($u = 34$ and $u = 58$, respectively) use the same construction as above for $u = 36$ and $u = 60$ except in the last group give weight 1 to exactly one point.

Statement 4 follows this same pattern. Here we give all the points in that last group weight 6 except for one point which receives weight 2. The proof is again the same as above where once more the exceptional cases occur when $t = 2, 3, 6, 10$. The cases for $t = 2$ and $t = 3$ ($u = 8$ and $u = 14$, respectively) were handled by the computer. For $t = 6$ and $t = 10$ ($u = 32$ and $u = 56$, respectively) use the same construction as above for $u = 36$ and $u = 60$ except in the last group give weight 1 to exactly two points. \square

4 Enumerating nonisomorphic holey factorizations

In this section we will enumerate the nonisomorphic holey factorizations of types 2^3 , 3^3 , 2^4 and 2^5 . In addition we will enumerate nonisomorphic sets of orthogonal holey factorizations of type 2^5 . We begin by defining isomorphic holey factorizations.

Two holey factorizations F and H of a graph G , say $F = \{f_1, f_2, \dots, f_k\}$, $H = \{h_1, h_2, \dots, h_k\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{f_1\phi, f_2\phi, \dots, f_k\phi\} = \{h_1, h_2, \dots, h_k\}$. Here $f_i\phi$ is the set of all the edges $\{x\phi, y\phi\}$ where $\{x, y\}$ is an edge in f_i .

The exact number of nonisomorphic one-factorizations (OFs) of K_n is known only for even $n \leq 12$. It is easy to see that there is a unique one-factorization of K_2 , K_4 , and K_6 . There are exactly six for K_8 ; these were found by Dickson and Safford [2] and a full exposition is given in [21]. In

1973, Gelling [7] proved that there are exactly 396 isomorphism classes of OFs of K_{10} . Recently, Dinitz, Garnick, and McKay [3] determined that there are 526,915,620 nonisomorphic OFs of K_{12} .

It is easy to check that there is a unique holey factorization of each of the types 1^1 , 1^3 and 1^5 . The automorphism group of each of the six one-factorizations of K_8 is transitive on points (see [21]). Thus by deleting a point to form a holey factorization of type 1^7 , we see that there are also exactly six nonisomorphic holey factorizations of this type.

In the next theorem we will count the number of distinct holey factorizations of type t^3 . The underlying graph here is $K_{t,t,t}$. Let V_1, V_2 , and V_3 be the three holes. The number of distinct ways to pick the holey factors missing hole V_i is obviously the number of distinct one-factorizations of $K_{t,t}$. This in turn is the number of distinct latin squares of side t with the first column fixed. We get the following theorem.

Theorem 4.1 *The number of distinct holey factorizations of type t^3 , with fixed holes, is $[L(t)]^3$, where $L(t)$ is the number of distinct Latin squares of order t with the first column fixed.*

Since $L(2) = 1$ and $L(3) = 2$, then it follows that there is a unique holey factorization of 2^3 and 8 distinct holey factorizations of type 3^3 .

Theorem 4.2 *There are 2 nonisomorphic holey factorizations of type 3^3 .*

Proof: Without loss of generality, one can fix the first six factors, as well as one edge in the seventh factor, in a holey factorization of type 3^3 . There are two choices for the second edge in the seventh factor, and that choice determines the rest of the factorization. Thus, there are at most two isomorphism classes of holey factorizations of type 3^3 ; representatives, A and B , of the classes are shown in Figure 3, where the holes are indicated as triples in square brackets. If we choose any three holey factors f , g , and h from A such that each of the three is missing a different hole, then $f \cup g \cup h$ forms either three disjoint triangles, or a single 9-cycle. However, if we choose any three holey factors f , g , and h from B , again such that each is missing a different hole, then $f \cup g \cup h$ forms a triangle and a disjoint 6-cycle. Thus, A and B are nonisomorphic, and there are two equivalence classes of 3^3 holey factorizations. \square

In order to count holey factorizations on a larger number of holes we use the computer. We construct nonisomorphic holey factorizations by use of an *orderly algorithm*; it generates the nonisomorphic holey factorizations of type T in a lexicographic order defined in the following way. First, if $F = \{f_1, f_2, \dots, f_n\}$ is a holey factorization, then we must have $f_i < f_j$ for all $i < j$. Then we say that $F > H = \{h_1, h_2, \dots, h_n\}$ if there is some k ,

A:	[123] 47 58 69	B:	[123] 47 58 69
	[123] 48 59 67		[123] 48 59 67
	[123] 49 57 68		[123] 49 57 68
	[456] 17 28 39		[456] 17 28 39
	[456] 18 29 37		[456] 18 29 37
	[456] 19 27 38		[456] 19 27 38
	[789] 14 25 36		[789] 14 26 35
	[789] 15 26 34		[789] 15 24 36
	[789] 16 24 35		[789] 16 25 34

Figure 3: The two nonisomorphic holey factorizations of type 3^3

$1 \leq k \leq n$, such that $f_i = h_i$ for all $i < k$ and $f_k > h_k$. We define a similar ordering on partial holey factorizations consisting of fewer than n factors.

The algorithm builds up each factorization by adding one factor at a time and rejects a partial factorization if it is not the lowest representative (lexicographically) of all the partial factorizations in its isomorphism class. In this way, the algorithm generates only the lowest representative of any isomorphism class of factorizations and as such never generates any holey factorizations which are isomorphic to each other. This approach saves both time and space over algorithms which first generate distinct (but possibly isomorphic) factorizations and then use methods to winnow isomorphs.

This type of algorithm has been used in other combinatorial searches including enumerating Latin squares [11], strong starters [8], one-factorizations of small graphs [13, 15], perfect one-factorizations of K_{14} [14], and Howell designs of small order [13]. Our algorithm below is essentially the one that was described in [3] to count the nonisomorphic one-factorizations of K_{12} . However, our algorithm here reduces the search space by assuming a first fixed factor consisting of the hyper-edges that correspond to the holes.

Theorem 4.3 *There are 2 nonisomorphic holey factorizations of type 2^4 and, for a fixed set of holes, there are 40 distinct holey factorizations of type 2^4 .*

Using the orderly algorithm we generated the lexicographically lowest representatives of the two isomorphism classes of holey factorizations of type 2^4 . The computation required about .1 seconds at a rate of 20 mips. The two factorizations are shown in Figure 4.

We determined the orders of the automorphism groups by counting the number of permutations that mapped a factorization onto itself while checking that the factorization was the lexicographically lowest representative of

<p>1: $Aut = 12$</p> <ul style="list-style-type: none"> [1 2] 35 47 68 [1 2] 36 48 57 [3 4] 15 28 67 [3 4] 17 26 58 [5 6] 14 27 38 [5 6] 18 24 37 [7 8] 13 25 46 [7 8] 16 23 45 	<p>2: $Aut = 48$</p> <ul style="list-style-type: none"> [1 2] 35 47 68 [1 2] 38 46 57 [3 4] 15 28 67 [3 4] 17 26 58 [5 6] 13 27 48 [5 6] 18 24 37 [7 8] 14 25 36 [7 8] 16 23 45
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Figure 4: The two nonisomorphic holey factorizations of type 2^4

its isomorphism class. We checked the results by using *nauty* [9] to determine the order of the automorphism group of the line graph corresponding to the factorization. If we label the nonisomorphic holey factorizations of type 2^4 as F_i , $1 \leq i \leq 2$, then the number of distinct holey factorizations of type 2^4 , with fixed holes, can be computed as

$$\sum_{i=1}^2 \frac{4! 2^4}{|Aut(F_i)|} = 40$$

We checked this result by fixing the holes, and using backtracking to generate all distinct factorizations.

Theorem 4.4 *There are 747 nonisomorphic holey factorizations of type 2^5 and, for a fixed set of holes, there are 2, 253, 312 distinct holey factorizations of type 2^5 .*

Again, we used the orderly algorithm to generate the lexicographically lowest representatives of the isomorphism classes of holey factorizations of type 2^5 . The computation required about 20 minutes at a rate of 20 mips. The file containing these 747 holey factorizations is available from the authors.

We computed the orders of the automorphism groups in the same ways as for the 2^4 factorizations. We computed the number of distinct holey factorizations of type 2^5 , with fixed holes, as

$$\sum_{i=1}^{747} \frac{5! 2^5}{|Aut(F_i)|} = 2, 253, 312$$

where $\{F_i | 1 \leq i \leq 747\}$ is the set of nonisomorphic holey factorizations of type 2^5 . Checking this result with backtracking required 5.5 hours of cpu time at 20 mips.

We now turn our attention to nonisomorphic sets of orthogonal hole factorizations. There does not exist a pair of orthogonal hole factorizations of type 2^4 [16]. We will show below that there are pairs (in fact triples) of orthogonal hole one-factorizations of type 2^5 .

Two sets of orthogonal hole factorizations \bar{F} and \bar{H} of type T , say $\bar{F} = \{F_1, F_2, \dots, F_s\}$, $\bar{H} = \{H_1, H_2, \dots, H_s\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{F_1\phi, F_2\phi, \dots, F_s\phi\} = \{H_1, H_2, \dots, H_s\}$. When $s = 2$, this corresponds to the notion of isomorphic Room frames of type T .

We extend the ordering used for hole factorizations to an ordering of sets of hole factorizations. First, if $\bar{F} = \{F_1, F_2, \dots, F_s\}$ is a set of orthogonal hole factorizations, then we must have $F_i < F_j$ for all $i < j$. Then we say that $\bar{F} > \bar{H}$ if there is some t , $1 \leq t \leq s$, such that $F_i = H_i$ for all $i < t$ and $F_t > H_t$. Using this ordering we obtained the following results.

Theorem 4.5 1. *There are 64 nonisomorphic sets of two orthogonal hole factorizations of type 2^5 .*

2. *There are 28 nonisomorphic sets of three orthogonal hole factorizations of type 2^5 .*

3. *There are no nonisomorphic sets of four orthogonal hole factorizations of type 2^5 .*

We generated the 64 mutually orthogonal pairs of factorizations in the following way. Let F_i , $1 \leq i \leq 747$ be the lexicographically lowest nonisomorphic factorizations as described in Theorem 4.4. We attempted to pair each F_i , $1 \leq i \leq 747$, with each possible permutation of F_j , $i \leq j \leq 747$. We checked each successful pairing to see if it was the lexicographically lowest such pair in its isomorphism class.

The Appendix displays the 64 pairs as Room frames of type 2^5 . We counted the sets of three and four orthogonal hole factorizations in a similar way. The Appendix also lists the sets of three orthogonal factorizations of type 2^5 .

References

- [1] A.B. Cruse, On embedding incomplete symmetric Latin squares, *J. Combin. Theory (A)* 16 (1974), 18–22.
- [2] L.E. Dickson and F.H. Safford, Solution to problem 8 (group theory). *Amer. Math. Monthly* 13 (1906), 150–151.
- [3] J.H. Dinitz, D.K. Garnick, and B.D. McKay, There are 526,915,620 nonisomorphic one-factorizations of K_{12} , *J. Comb. Des.*, 2 (1994), 273–285.
- [4] J.H. Dinitz and D.R. Stinson, “Room Squares and Related Designs”, *Contemporary Design Theory*, J.H. Dinitz and D.R. Stinson, eds., Wiley, New York, 1992, pp. 137–204.
- [5] J.H. Dinitz and D.R. Stinson, A few more Room frames, *Graphs Matrices and Designs*, R. Rees (Editor), Dekker, New York, 1993, pp. 133–146.
- [6] G. Ge and L.Zhu, Existence of Room frames of type $2^n u^1$, *J. Comb. Math. and Comb. Comput.*, 18 (1995), 65–82.
- [7] E.N. Gelling and R.E. Odeh, On 1-factorizations of the complete graph and the relationship to round-robin schedules. *Congressus Num.* 9 (1974), 213–221.
- [8] W.L. Kocay, D.R. Stinson and S.A. Vanstone, On strong starters in cyclic groups, *Discrete Math.* 56 (1985), 45–60.
- [9] B.D. McKay, *nauty* User’s Guide, Computer Science Dept., Australian National University, Canberra, Australia.
- [10] E. Mendelsohn and A. Rosa, One-factorizations of the complete graph – A survey, *J. Graph Theory* 9 (1985), 43–65.
- [11] R.C. Read, Every one a winner, *Annals of Discrete Math.* 2 (1978), 107–120.
- [12] R.S. Rees and D.R. Stinson, Frames with block size four, *Can. J. Math.* 44 (1992), 1030–1049.
- [13] E. Seah and D.R. Stinson, An enumeration of non-isomorphic one-factorizations and Howell designs for the graph K_{10} minus a one-factor, *Ars Combin.* 21 (1986), 145–161.
- [14] E. Seah and D.R. Stinson, Some perfect one-factorizations for K_{14} . *Ann. Discrete Math.* 34 (1987), 419–436.

- [15] E. Seah and D.R. Stinson, On the enumeration of one-factorizations of the complete graph containing prescribed automorphism groups. *Math Comp.* 50 (1988), 607–618.
- [16] D.R. Stinson, The non-existence of a $(2,4)$ -frame, *Ars Combin.* 11 (1981), 99–106.
- [17] D.R. Stinson, Some constructions for frames, Room squares, and subsquares, *Ars Combin.* 12 (1981), 229–267.
- [18] D.R. Stinson and L. Zhu, Towards the spectrum of Room squares with subsquares, *J. Comb. Theor Ser. A* 63 (1993), 129–142.
- [19] D.R. Stinson, L. Zhu and J.H. Dinitz, On the spectra of certain classes of Room frames, *Electronic Journal of Combinatorics* 1 (1994), #R7, pp. 21.
- [20] W.D. Wallis, One-factorizations of complete graphs, in *Contemporary Design Theory*, pp. 593–631, J.H. Dinitz and D.R. Stinson, eds., John Wiley and Sons, New York, 1992.
- [21] W.D. Wallis, A.P. Street and J.S. Wallis, *Combinatorics: Room squares, sun-free sets, Hadamard matrices Lect. Notes in Math.* 292, Springer-Verlag, Berlin, 1972.

Appendix: Nonisomorphic sets of orthogonal holey factorizations of type 2^5

The 64 sets of two orthogonal holey factorizations of type 2^5 are displayed below as Room frames. Beneath the index of each Room frame is the pair of indices of the constituent holey factorizations. For example, Room frame 1 is derived from holey factorizations 1 and 690. The first holey factorization of each pair, which yields the rows of the Room frame, is always in canonical (lexicographically lowest) form; the second holey factorization, yielding the columns, is not necessarily in canonical form.

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	27	18	09						36
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38					29			05	14
		57	46		13	02			
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		69	78					34		25	
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								58	49	16	07
26								19	08	37	
	27	18	09								36
	39	04	15		28						
38					29					05	14
		57	46			13	02				
47	56			03						12	

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26			19	08						37
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	39	04	15		28					
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							16	07	58	49	
26								19	08		37
	27	18	09								36
	39						28			04	15
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		46	57			13	02				
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26			19	08							37
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	49					58		16	07
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		18					09	27	36
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						58	49	16	07
26	37					19	08		
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	49	15			38				02
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				56	12	07	34		

46:
268
746

		79	68					35	24
		48	59	26	37				
				78	69	04	15		
				19	08			46	57
						39	28	17	06
27	36					18	09		
49	58							02	13
38	29	05	14						
56	47			03	12				
		16	07			25	34		

47:
269
746

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		48	59	26	37				
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						39	28	17	06
27	36					18	09		
49	58							02	13
38	29	05	14						
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56			07		12		34		

48:
282
746

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78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
49	58							03	12
		05	14	38	29				
		46	57			02	13		
25	34			07	16				

49:
282
705

		68	79	35				24	
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69			78			15	04		
47				08	19			56	
	39	06				28		17	
	27	18				09			36
	58	49		03				12	
38			14	29					05
	46	57			13	02			
25			07	16			34		

50:
283
746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
49	58							03	12
		05	14	38	29				
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25		46		07				13	

51:
283
705

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		59	26	48		37			
69			78			15	04		
47				08	19			56	
	39	06				28		17	
	27	18				09			36
	58	49		03				12	
38			14	29					05
	57		16		02	34			
25	46		07		13				

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284
746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
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49		05		38					12
		34		57		16	02		
25		46		07				13	

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285
746

		79	68					24	35
				26	37	48	59		
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	27	18				39			06
36			09				28	17	
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25		46		07				13	

54:
286
746

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78	69					15	04		
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	27	18				39			06
36			09				28	17	
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		34		57		16	02		
25		46		07				13	

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287
746

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				19	08			47	56
36		18			29				07
	27		09			38	16		
	58	14		39				02	
49		05		28					13
25		46			17	03			
	34		57	06			12		

56:
312
312

		79	68					35	24
69				37	26	59	48		
78				18		04			57
					19		05	46	
	27	08					39		16
	36		09			28			17
	49		15		38				02
	58	14		29					03
25			47	06			13		
34		56			07			12	

57:
312
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		79	68					35	24
	69			37	26	59	48		
78				18		04			57
					19		05	46	
	27	08					39		16
36			09			28			17
49				15		38			02
	58	14		29					03
25			47	06			13		
	34	56			07			12	

58:
319
319

			68	79					35	24
69					26	37	48	59		
78					18			04		57
						09	15		46	
	27	19						38		06
	36			08			29			17
	58		14	39						02
	49	05			28					13
25		47				16	03			
34			56	07					12	

59:
319
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			68	79					35	24
					26	37	48	59		
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78						09	15		46	
	27	19						38		06
36			08			29				17
	58		14	39						02
49		05			28					13
25		47				16	03			
	34		56	07					12	

60:
342
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69					37	26	58	49			
78					18		04				57
						19		05	46		
	27	08						39			16
	36			09				28		17	
	59	14				38					02
	48		15	29							03
25			47	06						13	
34		56			07				12		

61:
342
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69						04	18			57	
78						19	05	46			
		27	08	16				39			
		36	17	09			28				
		59			38					02	14
		48			29					15	03
25			47	06	13						
34		56			12	07					

62:
343
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69					37	26	58	49			
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						19			05	46	
		36	08					29			17
		27		09					38		16
		48		15	39						02
		59	14		28						03
25			47	06				13			
34		56			07				12		

			79	36		58	24
		68			37	49	25
	69			18		04	57
78				09	15	46	
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	35	47		16	02		
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		68			27	49	35
69				18		04	57
	78			09	15	46	
64:	37	19			28		06
746		26		08		39	17
746		59		14	38		02
	48	05	29				13
	25	47		16	03		
	34		56	07		12	

The 28 sets of three orthogonal hole factorizations of type 2^5 are listed below in the following way. First, the index of the triple is given as $\#i$, for $1 \leq i \leq 28$. Beneath the index of the triple is the parenthesized index of the Room frame that yields the first two factorizations of the triple. The third hole factorization is fully listed with the holes bracketed. Beneath the index of the Room frame are the indices of the three factorizations; as with the Room frames, only the first factorization is necessarily in canonical form. Thus, for example, triple number 6 consists of Room frame 8 (which in turn consists of canonical factorization 1 together with a factorization isomorphic to canonical factorization 746) and the fully listed factorization which is isomorphic to canonical factorization 746.

	[01]	28	36	49	57
	[01]	29	37	46	58
	[23]	05	19	47	68
#1:	[23]	08	14	56	79
(1)	[45]	02	17	38	69
1	[45]	06	12	39	78
690	[67]	03	18	24	59
690	[67]	09	13	25	48
	[89]	04	16	27	35
	[89]	07	15	26	34

	[01]	28	36	49	57
	[01]	29	37	46	58
	[23]	05	19	47	68
#2:	[23]	08	14	56	79
(2)	[45]	02	16	39	78
1	[45]	07	12	38	69
690	[67]	03	18	24	59
690	[67]	09	13	25	48
	[89]	04	17	26	35
	[89]	06	15	27	34

#3:
(3)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 18 47 69
[23] 09 14 56 78
[45] 02 17 39 68
[45] 06 12 38 79
[67] 03 19 25 48
[67] 08 13 24 59
[89] 04 16 27 35
[89] 07 15 26 34

#5:
(5)
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690
690

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[23] 04 18 56 79
[23] 09 15 47 68
[45] 02 17 38 69
[45] 06 12 39 78
[67] 03 19 25 48
[67] 08 13 24 59
[89] 05 16 27 34
[89] 07 14 26 35

#7:
(9)
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746
746

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[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 17 38 69
[45] 06 13 29 78
[67] 03 18 24 59
[67] 09 12 35 48
[89] 04 16 25 37
[89] 07 15 26 34

#9:
(11)
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746

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[01] 28 36 47 59
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[45] 03 16 29 78
[45] 07 12 38 69
[67] 02 19 34 58
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[89] 04 17 26 35
[89] 06 15 24 37

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[45] 02 16 38 79
[45] 07 12 39 68
[67] 03 19 25 48
[67] 08 13 24 59
[89] 04 17 26 35
[89] 06 15 27 34

#15:
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#17:
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[89] 07 15 26 34

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[89] 05 16 24 37
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#18:
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