Domination Parameters Of Star Graph

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ABSTRACT. The n-star graph S_n is a simple graph whose vertex set is the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$ and two vertices α and β are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one i, $i \neq 1$. In the paper we determine the values of the domination number γ, the independent domination number γ_i, the perfect domination number γ_p and we obtain bounds for the total domination number γ_t and the connected domination number γ_c for S_n.

1 Introduction

By a graph we mean a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [7].

Akers and Krishnamurthy introduced the n-star graph S_n in [3]. The vertex set of S_n is the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$ and two vertices α and β are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one i , $i \neq 1$. The n-star graph has been proposed as an attractive alternative to the n-cube with many superior characteristics [2]. Day and Tripathi [8] have compared the topological properties of the n-star graph and the n-cube. In this paper we determine the values of the domination number γ, the independent domination number γ_i and the perfect domination number γ_p for the star graph S_n. We also obtain bounds for the total domination number γ_t and the connected domination number γ_c.

Let $G = (V, E)$ be a graph. A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S is called a *perfect dominating set* if every vertex in $V - S$ is adjacent to exactly one vertex in S [5]. A dominating set S is called an *independent dominating set* if no two vertices of S are adjacent. A subset S of V is called a *total dominating set* if every vertex in V is adjacent to some vertex in S. A dominating set S is called a *connected dominating set* if the subgraph induced by S is connected.

The domination number γ of G is defined to be the minimum cardinality of a dominating set in G. In a similar way, we define the *perfect domination number* γ_p, the *independent domination number* γ_i, the *total domination number* γ_t and the *connected domination number* γ_c.

A *domatic partition* of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximum number of classes of a domatic partition of G is called the *domatic number* of G and is denoted by $d(G)$ [6]. In a similar way we define the *perfect domatic number* $d_p(G)$, the *independent domatic number* $d_i(G)$ and the *total domatic number* $d_t(G)$.

A graph G is called *domatically full* if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of V where $\delta(G)$ is the minimum degree of a vertex of G [4]. A dominating set in a graph is called *indivisible* if it is not a union of two distinct dominating sets of G. The minimum number of classes of a partition of $V(G)$ into indivisible dominating sets is called the *adomatic number* of G and is denoted by $ad(G)$ [4].

We use the following theorem.

Theorem 1.1 [1]. *For any graph G of order p and maximum degree Δ, we have $\gamma \geq p/(\Delta + 1)$.*

2 Main Results

Theorem 2.1. $\gamma(S_n) = \gamma_i(S_n) = \gamma_p(S_n) = (n-1)!$ *for all n.*

Proof: Since S_n is $(n-1)$-regular it follows from Theorem 1.1 that $\gamma(S_n) \geq (n-1)!$. Also $S = \{\alpha \in V(S_n)/\alpha(1) = 1)$ is a dominating set of S_n which is independent and perfect and $|S| = (n-1)!$. Hence it follows that $\gamma(S_n) = \gamma_i(S_n) = \gamma_p(S_n) = (n-1)!$. □

Corollary 2.2. $d(S_n) = d_i(S_n) = d_p(S_n) = ad(S_n) = n$.

Proof: Let $A_i = \{\alpha \in V(S_n)/\alpha(1) = i\}$, $i = 1, 2, \ldots, n$. Clearly $V(S_n) = \cup_{i=1}^{n} A_i$ and each A_i is a minimal dominating set which is independent, indivisible and perfect. Hence the result follows. □

Corollary 2.3. *S_n is domatically full.* □

Lemma 2.4. *For any connected graph G of order p and maximum degree Δ, $\gamma_t \geq p/\Delta$.*

Proof: Let S be a γ_t set in G. Since each vertex in S dominates at most $\Delta - 1$ vertices in $V - S$, the result follows. □

Theorem 2.5. *$\gamma_t(S_n) = n!/(n-1)$ if n is even and $n!/(n-1) \le \gamma_t(S_n) \le \frac{(n-1)!(n-1)}{(n-2)}$ if n is odd.*

Proof:

Case (i) $n = 2m$.

Define $A_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i+1\}$ if i is odd and $A_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i-1\}$ if i is even. We claim that $A = \cup_{i=1}^{2m} A_i$ is a total dominating set of S_n. For odd i, each vertex of A_i has exactly one adjacent vertex in A_{i+1} so that $\langle A \rangle$ has no isolated vertices. Now let $\alpha \in V(S_n) - A$ and $\alpha(2) = i$. Let α' be the vertex obtained from α by interchanging $\alpha(1)$ and $i+1$ if i is odd and $\alpha(1)$ and $i-1$ if i is even. Clearly $\alpha' \in A$ and is adjacent to α. Hence A is a total dominating set of S_n. Also $|A| = n!/(n-1)$. Hence $\gamma_t(S_n) \le n!/(n-1)$. Also, by Lemma 2.4, $\gamma_t(S_n) \ge n!/(n-1)$ so that $\gamma_t(S_n) = n!/(n-1)$ when n is even.

Case (ii) $n = 2m+1$.

We define sets A_i and B_i for $i = 1, 2, \ldots, 2m$ as follows:

$A_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i+1\}$ if i is odd and
$A_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = i-1\}$ if i is even.
$B_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 2m+1, \alpha(3) = i+1\}$ if i is odd and
$B_i = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 2m+1, \alpha(3) = i-1\}$ if i is even.

Let $A = \cup_{i=1}^{2m}(A_i \cup B_i)$. Clearly $\langle A \rangle$ has no isolated vertices. Also $\cup_{i=1}^{2m} A_i$ dominates all vertices with $\alpha(2) = j$ $(j = 1, 2, \ldots, 2m)$ and $\cup_{i=1}^{2m} B_i$ dominates all vertices with $\alpha(2) = 2m+1$ so that A is a total dominating set of S_n and $|A| = \frac{(n-1)!(n-1)}{(n-2)}$.

Hence $\gamma_t(S_n) \le \frac{(n-1)!(n-1)}{(n-2)}$. Also by Lemma 2.4 $\gamma_t(S_n) \ge n!/(n-1)$ and the theorem follows. □

Corollary 2.6. *$d_t(S_n) = n-1$ if n is even.*

Proof: For $j = 2, 3, \ldots, n$ we define

$A_{ij} = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(j) = i+1\}$ if i is odd and
$A_{ij} = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(j) = i-1\}$ if i is even.

Let $A_j = \cup_{i=1}^{n} A_{ij}$. Then $\{A_2, A_3, \ldots, A_n\}$ is a partition of V into minimal total dominating sets and hence $d_t(S_n) = n-1$ if n is even. □

Theorem 2.6. $\frac{n!}{n-1} \le \gamma_c(S_n) \le 2(n-1)!$.

Proof: Define

$A_1^{(n)} = \{\alpha \in V(S_n)/\alpha(1) = 1\}$ and
$A_{i1}^{(n)} = \{\alpha \in V(S_n)/\alpha(1) = i, \alpha(2) = 1\}$ for $i = 2, 3, \ldots, n$.

Let $D_n = A_1^{(n)} \cup (\cup_{i=2}^{n} A_{i1}^{(n)})$. Clearly D_n is a dominating set of S_n. Let $X_n = \cup_{i=2}^{n} A_{i1}^{(n)}$. One can see that $\langle , X_n \rangle$ is a connected subgraph of S_n. Thus D_n is a connected dominating set of S_n and $|D_n| = 2(n-1)!$ so that $\gamma_c(S_n) \geq 2(n-1)!$. Also by Lemma 2.4, $\gamma_c(S_n) \geq \frac{n!}{n-1}$. □

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