Efficient Constructions of Antichain Cutsets

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Abstract

A subset S of an ordered set P is called a *cutset* if each maximal chain of P has nonempty intersection with S; if, in addition, S is also an antichain it is an *antichain cutset.* We consider new characterizations and generalizations of these and related concepts. The main generalization is to make our definitions in graph theoretic terms. For instance, a *cutset* is a subset S of the vertex set V of graph $G = (V, E)$ which meets each extremal path of G. Our principle results include (1) a characterization, by means of a closure property, of those antichains which are cutsets; (2) a characterization, by means of "forbidden paths" in the graph, of those graphs which can be expressed as the union of antichain cutsets; (3) a simpler proof of an existing result about N-free orders; and (4) efficient algorithms for many related problems, such as constructing antichain cutsets containing or excluding specified elements or forming a chain. We include a brief discussion of the use of antichain cutsets in a parsing problem for LR(k) languages.

1. Introduction

Several papers [Riv85b, Gin86, Hig86, Zah86, Riv87, Beh90] have investigated various characterizations and constructions related to antichains and cutsets. We consider several new characterizations and generalizations of these and related concepts. Our primary motivation is from a recent application [And90] of antichain cutsets in computer science. The main generalization, from the diagram approach [Riv85a, Riv89], is to make our definitions in graph theoretic terms. For instance, a *cutset* is a subset S of the vertex set V of graph $G = (V, E)$ which meets each extremal path of G. We conclude this section with a brief discussion of a new application of antichain cutsets.

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Syntax-directed translation [Lew68] is a useful technique which has found application in many formal language translation tasks (see [Pur80]) and for which additional uses are regularly described, e.g., [Ham92]. In addition to these applications, one finds program illustration [Hen89, Pur89], automatic compiler generation [And90], optimized data validation [Fin84], construction of LR-attributed [Jon80] grammars and attribute-influenced parsing [Wat79], and one-pass semantic analysis of programming languages [Kos84]. One factor in the variety of applications of syntax-directed translation is that null nonterminal insertion in LR(k) languages [Knu65] provides a simple, direct means to implement many interesting syntax-directed translation systems. (A *null nonterminal* is a symbol whose only translation string is the empty string.) The problem with such insertions is that, although null nonterminals do not change the language generated by the grammar, arbitrary insertion does not always preserve the parsing properties of the grammar. In particular, it is easy to construct an LR(k) grammar which is no longer LR(k) after altering just one of its productions by inserting a new null nonterminal.

It can be shown [And92] that positions in LR(k) productions which may or may not be successfully used for null nonterminal insertion are specified by antichain cutsets in graphs closely related to a graph representing the parsing automaton. Further, effective construction of such a syntax-directed translation system, which may involve many different null nonterminal symbols, can be reduced to a sequence of smaller problems relying on finding chains of antichain cutsets, some of which include specific vertices of the graphs and some of which exclude certain vertices. Although there are other applications of antichain cutsets, e.g., decomposition of 0-1 matrices into a direct sum of permutation matrices [Bru91], algorithmically better methods exist for those problems. We believe this to be an important example of a large-scale practical problem for which the use of antichain cutsets provides a natural characterization and efficient solution.

2. Antichain Cutsets

Much of the terminology surrounding cuts is common to both the theory of ordered sets and graphs, in particular networks. Unfortunately, this is often with slightly different meanings. To avoid confusion as much as possible, this section presents some of our terms in the context of relevant graph theory notation [Har69].

Unless stated otherwise, $G = (V, E)$ will denote a finite, directed graph with *vertex* set V and *edge* set E. As usual, E^+ denotes transitive closure while E^* denotes reflexive, transitive closure. If $(u, v) \in E^+$, say u *preceeds* (is a *predecessor* of) v. Vertex s is a *source* of G if and only if

$$V = \bigcup_{(s,v)\in E^*} \{v\}$$

and vertex t is a *sink* of G if and only if

$$V = \bigcup_{(v,t) \in E^*} \{v\}.$$

Note that a vertex which is a source and initial is the unique vertex which is either a source or initial in the graph. A similar statement holds for a final sink vertex, though in general a graph may have no vertex with these properties.

A *path* in graph G is a walk with distinct vertices, a *nontrivial* walk is of length at least one (counting edges) and a *cycle* is a nontrivial closed walk with distinct vertices except for the first and last. Define path p in graph $G = (V, E)$ to be *extremal* if it is a path from an initial vertex $s \in V$ to a final vertex $t \in V$. Throughout this section, finite directed graph $G = (V, E)$ will be assumed to have initial source node s and distinct final sink node t. Note that by adding or coalescing vertices, this assumption may be made without loss of generality in all that follows. $\mathcal{A} \subseteq V$ is an *antichain* in G if and only if v does not preceed v' for all nodes $v, v' \in \mathcal{A}$. Let $\mathfrak{A}(E)$ be the set of antichains in graph G.

Given graph $G = (V, E)$ and $S, S' \subseteq V$, define

$$S \sqsupseteq S' \iff \forall s \in S \; \exists s' \in S' \; (s, s') \in E^*.$$

Claim 2.1. *$\sqsupseteq$ is a partial order for the set of antichains $\mathfrak{A}(E)$ of graph $G = (V, E)$.*

As nodes s and t are distinct, each extremal path $r = (v_0, \ldots, v_{n+1})$ where $s = v_0$, $t = v_{n+1}$ and $(v_i, v_{i+1}) \in E$ is nontrivial. Let $V_r = \{v_1, \ldots, v_n\}$ be the (possibly empty) set of interior nodes of path r. Then $C \subseteq V$ is a *cutset* in G if and only if $V_r \cap C \neq \emptyset$ for all extremal paths r.

Let $\mathfrak{C}(E)$ be the set of cutsets in graph $G = (V, E)$ and cutset $C \in \mathfrak{C}(E)$ be a *proper* cutset if and only if no proper subset of C is a cutset in G, i.e., C is minimal with respect to $\subseteq$. Note that neither s nor t will be members of any cutset in G and this is the only distinction from a disconnecting set of nodes [For62].

Lemma 2.2. *Cutset C in $G = (V, E)$ is minimal if and only if for each $v \in C$ there exists extremal path r such that $V_r \cap C = \{v\}$.*

Proof. Since C is a cutset, for all $v \in C$ and for every extremal path r

$$\begin{aligned} V_r \cap C &= V_r \bigcap \left(\{v\} \cup (C \setminus \{v\})\right) \\ &= (V_r \cap \{v\}) \bigcup (V_r \cap (C \setminus \{v\})) \\ &\neq \emptyset. \end{aligned}$$

Now each $v \in C$ is the only element in C on some extremal path, otherwise $C \setminus \{v\}$ is still a cutset which contradicts the assumption that C is minimal. That is, for all $v \in C$ there exists extremal path r such that $V_r \cap (C \setminus \{v\}) = \emptyset$. So for all $v \in C$ there exists extremal path r such that

$$V_r \cap C = V_r \cap \{v\} \neq \emptyset.$$

Then for all $v \in C$

$$V_r \cap C = \{v\}$$

for some extremal path r. The converse is trivial. □

Antichain cutset $\mathcal{K} \in \mathfrak{A}(E) \cap \mathfrak{C}(E)$ is a proper antichain cutset as antichain cutsets are always minimal. Let $\mathfrak{AC}(E)$ be the set of antichain cutsets in graph $G = (V, E)$. The suffix $\mathfrak{C}$ of $\mathfrak{AC}$ is used to distinguish antichain cutsets, whose elements are vertices, from the antichain disconnectors to be introduced in section 3, whose elements are edges and are denoted $\mathfrak{AD}(E)$.

Since $\sqsupseteq$ is a partial order for $\mathfrak{A}(E)$ it may be extended to a total order and part of the method of section 4 orders elements from maximum to minimum by repeated arbitrary selection from among the remaining maximal elements. There are, however, interesting subsets of $\mathfrak{A}(E)$, in fact subsets of $\mathfrak{AC}(E)$, for which $\sqsupseteq$ is already a total order. These are the sets of unilateral antichain cutsets, where two antichains are *unilateral* if no member of one of the antichains is a predecessor of any member of the other antichain.

Lemma 2.3. *Unilateral antichain cutsets are comparable under* $\sqsupseteq$.

In particular, Lemma 2.3 asserts that any set of pairwise unilateral antichain cutsets of an ordered set are linearly ordered. Unilateral antichain cutsets are used in section 4 when constructing linearly ordered antichain cutsets from unilateral antichains.

The next result shows that the elements of antichains partially ordered by $\sqsupseteq$ are contained only in trivial strong components, though the graph may have other nodes on cycles. This will be used in the next section to reduce the size of the graph for which antichains are being constructed by condensing cycles to strong components.

Lemma 2.4. *If* $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{A}(E)$ *and* $\mathcal{A}_1 \sqsupseteq \mathcal{A}_2$ *then* a_2 *does not preceed* a_1 *for all* $a_1 \in \mathcal{A}_1$ *and* $a_2 \in \mathcal{A}_2$.

3. Ideals I

The definitions of section 2: antichain, cutset, etc., may all be altered to apply to edges instead of vertices and doing so introduces the first of two implementations for finding an antichain of edges which meets each extremal path in G. Recall that unless stated otherwise, $G = (V, E)$ will be a finite, directed graph with initial source s and distinct final sink t. $\mathcal{A} \subseteq E$ is an *antichain* (of edges) in G if and only if $(u', v) \notin E^*$ for all edges $(u, u') \in \mathcal{A}$ and $(v, v') \in \mathcal{A}$. If $V' \subseteq V$, let $\overline{V'}$ denote $V \setminus V'$ and $\langle V', \overline{V'} \rangle$ denote $(V' \times \overline{V'}) \cap E$. Then $\langle V', \overline{V'} \rangle$ is defined to be a *disconnector* in G if and only if V' contains each initial node and no final node of G. Disconnector $\mathcal{D}$ in G is a *proper disconnector* if and only if no proper subset of $\mathcal{D}$ is a disconnector in G, i.e., $\mathcal{D}$ is minimal with respect to $\subseteq$. Let $\mathfrak{D}(E)$ be the set of disconnectors and $\mathfrak{AD}(E)$ denote the set of antichain disconnectors in graph $G = (V, E)$.

The first algorithm developed in this section will produce an antichain disconnector. Subsequently, this method will be adapted to produce an antichain cutset. The benefits of this approach are twofold. First, a uniform representation of antichain disconnectors and antichain cutsets may be given which leads directly to a simple method of generation for either. The second benefit is that it relates the conversion of an antichain disconnector into an antichain cutset and the removal of the excluded nodes characterized by Lemma 2.4.

For graph $G = (V, E)$, $\breve{E}$ will denote the inverse of edge relation E. $\mathcal{I} \subseteq V$ is an *ideal* in E for graph G if and only if for all i and j

$$\text{if } j \in \mathcal{I} \text{ and } (i, j) \in E \text{ then } i \in \mathcal{I}.$$

$\mathfrak{I}(E)$ will denote the set of all ideals in E. Thus $\mathfrak{I}(E)$ is exactly the set of closed sets of vertices under $\breve{E}$. Various consequences of the following corollary will be used repeatedly.

Corollary 3.1. *For the set of ideals $\mathfrak{I}(E)$ of an arbitrary graph $G = (V, E)$, the following are equivalent:*

(*i*) $\qquad \mathcal{I} \in \mathfrak{I}(E)$;

(*ii*) $\qquad j \in \mathcal{I} \Rightarrow \left(\bigcup_{(i,j) \in E^*} \{i\} \right) \subseteq \mathcal{I}$;

(*iii*) $\qquad i \notin \mathcal{I} \Rightarrow \left(\bigcup_{(i,j) \in E^*} \{j\} \right) \cap \mathcal{I} = \emptyset$.

Proof. Immediate from the definitions. □

Let $\mathfrak{I}(s,t,E)$ be the set of $\mathcal{I}$ such that $\mathcal{I} \in \mathfrak{I}(E)$ and $s \in \mathcal{I}$ and $t \notin \mathcal{I}$. Next a connection between ideals and antichain disconnectors is made.

Lemma 3.2. *For graph $G = (V, E)$,*

$$\langle \mathcal{I}, \overline{\mathcal{I}} \rangle \in \mathfrak{AD}(E) \text{ if and only if } \mathcal{I} \in \mathfrak{I}(s,t,E).$$

A direct consequence of Lemma 3.2 is the following.

Corollary 3.3. *For graph $G = (V, E)$, the map $f: \mathfrak{I}(s,t,E) \to \mathfrak{AD}(E)$ given by $f(\mathcal{I}) = \langle \mathcal{I}, \overline{\mathcal{I}} \rangle$ is a bijection.*

A ***strongly connected component*** or ***strong component*** of a graph $G = (V, E)$ is a subgraph $G' = (V', E')$ of G such that there is a path of E' edges between each ordered pair of vertices from V'. For $v, v' \in V$, let $v \sim v'$ if and only if v and v' are in the same maximal strong component of G. Since $\sim$ is an equivalence relation, define the quotient graph $G/\!\sim$ as usual. For $v \in V$, let $[v]$ denote the equivalence class of v under $\sim$.

Lemma 3.4. *For directed graph $G = (V, E)$, let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the quotient graph $G/\!\sim$. Then the canonical homomorphism $h: G \to G/\!\sim$ induces a bijection $f: \mathfrak{AD}(E) \to \mathfrak{AD}(\tilde{E})$.*

Proof. Let $\langle Y, \overline{Y} \rangle \in \mathfrak{AD}(\tilde{E})$ and

$$X = \bigcup_{[y] \in Y} [y].$$

Clearly

$$\langle Y, \overline{Y} \rangle = \big\langle h(X), \overline{h(X)} \big\rangle.$$

But

$$\begin{aligned} X &= \bigcup_{[y] \in Y} [y] \\ &= \{\, x \mid x \sim y \text{ for } [y] \in Y \,\} \\ &= \{\, x \mid (x, y) \in E^* \text{ for } [y] \in Y \,\}. \end{aligned}$$

So $X \in \mathcal{I}(E)$ and by Lemma 3.2, $\langle X, \overline{X} \rangle \in \mathfrak{AD}(E)$. Thus f is a surjection from the antichain disconnectors of G to those of $\tilde{G}$.

Now

$$(w, x) \in \langle X, \overline{X} \rangle \in \mathfrak{AD}(E)$$

implies

$$[w] \subseteq X \text{ and } [x] \subseteq \overline{X}$$

since X must be an ideal. Then

$$(w, x) \in \langle X, \overline{X} \rangle \in \mathfrak{A}\mathfrak{D}(E)$$

if and only if

$$([w], [x]) \in \langle h(X), \overline{h(X)} \rangle \in \mathfrak{A}\mathfrak{D}(\tilde{E}).$$

So h is an injection since

$$\langle h(X_1), \overline{h(X_1)} \rangle = \langle h(X_2), \overline{h(X_2)} \rangle$$

implies

$$\langle X_1, \overline{X_1} \rangle = \langle X_2, \overline{X_2} \rangle.$$

□

To summarize, no edge in a nontrivial strong component of G may be in an antichain disconnector in G and only edges in nontrivial strong components are eliminated by condensing G. Therefore, at least some excluded vertices may be removed and the graph reduced in size by condensing strong components. $\tilde{E}$ now represents the edge relation of a directed acyclic graph and illustrates that the method can be used to find antichain disconnectors in a graph with cycles. Also, by converting any undirected edges into pairs of (oppositely) directed edges, the same result applies to undirected or mixed graphs. This is made precise next.

Corollary 3.5. *Given directed graph $G = (V, E)$ with source s and distinct sink t, let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the quotient graph $G/\sim$. If $\mathfrak{I}(s, t, \tilde{E}) \subseteq \mathfrak{I}(\tilde{E})$ is the subset of ideals such that $s \in \mathcal{I}$ and $t \notin \mathcal{I}$ for all $\mathcal{I} \in \mathfrak{I}(s, t, \tilde{E})$ then there is a bijection between $\mathfrak{I}(s, t, \tilde{E})$ and $\mathfrak{A}\mathfrak{D}(E)$ induced by the canonical homomorphism from G to $G/\sim$.*

Proof. By Corollary 3.3 there is a bijection $f: \mathfrak{I}(s, t, \tilde{E}) \to \mathfrak{A}\mathfrak{D}(\tilde{E})$. By Lemma 3.4 there is a bijection $g: \mathfrak{A}\mathfrak{D}(E) \to \mathfrak{A}\mathfrak{D}(\tilde{E})$. Then $h: \mathfrak{I}(s, t, \tilde{E}) \to \mathfrak{A}\mathfrak{D}(E)$ given by $h = g^{-1} \circ f$ is a bijection. □

Next, the method is adapted to finding, for graph $G = (V, E)$ where $s, t \in V$ and $(s, t) \notin E$, an antichain cutset $\mathcal{K} \subset V$ where $s, t \notin \mathcal{K}$. Before proceeding, note the condition $(s, t) \notin E$ is necessary because otherwise there exists no cutset $\mathcal{K}$ in G such that $s \notin \mathcal{K}$ and $t \notin \mathcal{K}$. The goal then is to use the method of Corollary 3.5 to find an antichain disconnector and ensure this determines an antichain cutset.

Given V, let $P = \{p\} \times V$ and $Q = \{q\} \times V$. For $v \in V$, often $(p, v) \in P$ and $(q, v) \in Q$ will be denoted p_v and q_v, respectively. Then for graph $G = (V, E)$ satisfying the restrictions on s and t above, define graph $G' = (V', E')$ such that $p_s, q_t \in V'$ and

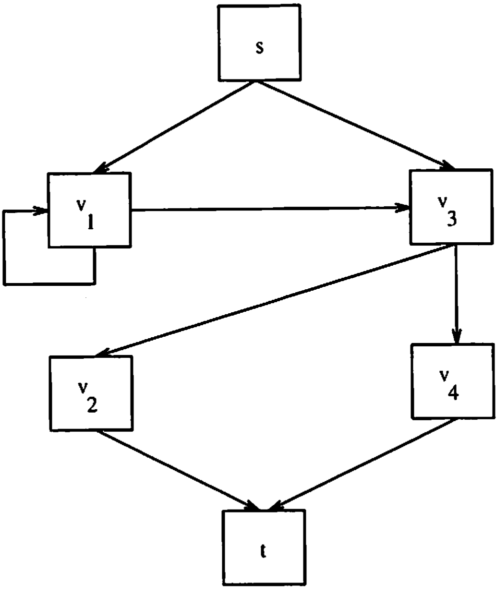

Figure 3.1. Example graph for Corollary 3.5.

$$v \in V \setminus \{s,t\} \iff p_v, q_v \in V' \text{ and } (q_v, p_v) \in E',$$

$$(i,j) \in E \iff (p_i, q_j) \in E' \text{ and } (q_j, p_i) \in E'.$$

The main effect of this transformation is to make the edges of G correspond to vertices in nontrivial strong components of G', and as such vertices are eliminated by condensing strong components in G', no member of any antichain disconnector in $\tilde{G}$ corresponds to an edge of G.

Let $\tilde{s} = [p_s]$ be the source of $\tilde{G}$ and $\tilde{t} = [q_t]$ be the sink of $\tilde{G}$. So the only edges in an antichain disconnector in $\tilde{G}$ are those introduced into G' to correspond to nodes of G. Thus, applying Corollary 3.5 to G' yields an antichain disconnector in $\tilde{G}$ which corresponds to an antichain cutset in G, as summarized next.

Theorem 3.6. *Let $G' = (V', E')$ be defined as above for $G = (V, E)$ where $s, t \in V$ and $(s,t) \notin E$. If $\langle U, \overline{U} \rangle \subset \tilde{E}$ is an antichain disconnector in $\tilde{G} = G'/\sim$ then*

$$\{\, v \mid q_v \in [u] \in U \text{ and } p_v \in [u'] \in \overline{U} \text{ for some } u, u' \in \tilde{V} \,\} \subset V$$

is an antichain cutset in G. If $T \subset V$ is an antichain cutset in G then

$$\{\, ([q_v], [p_v]) \mid v \in T \,\} \subset \tilde{E}$$

is an antichain disconnector in $\tilde{G}$.

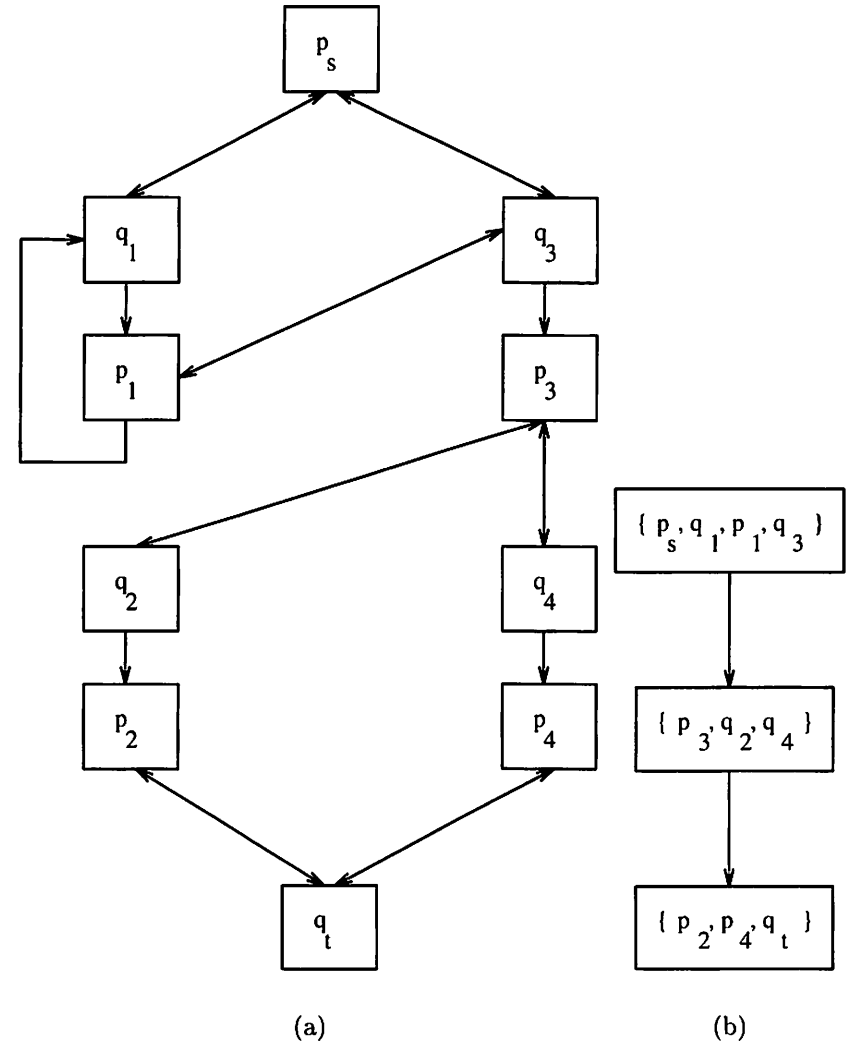

Figure 3.2. (a) modified graph G' and (b) condensed graph $\tilde{G}$ for graph G of Figure 3.1.

The method is illustrated by example. Figure 3.2(a) shows the sample graph of Figure 3.1 as modified by the construction prior to Theorem 3.6. This shows that the original vertices have been "split" and these pairs of nodes connected to pairs representing their successors in the original graph, making G'. In Figure 3.2(b), the strong components of G' shown in Figure 3.2(a) have been condensed to single vertices and relabled, making $\tilde{G}$. Suppose, applying Corollary 3.5, we select ideal $\{ [p_0] \} \cup \{ [q_0] \}$ and form antichain disconnector

$$\begin{aligned} &\langle \{ [p_0], [q_0] \}, \{ [p_3], [p_2], [p_5] \} \rangle \in \mathfrak{AD}(\tilde{E}) \\ &= \{ ([q_3], [p_3]) \} \subset \tilde{E} \end{aligned}$$

of $\tilde{G}$. Then by the construction in the proof of Lemma 3.4,

$$\begin{aligned} &\langle \{ p_0, q_1, p_1, q_3, q_0 \}, \{ p_3, q_2, q_4, p_2, p_4, q_5, p_5 \} \rangle \in \mathfrak{AD}(E') \\ &= \{ (q_3, p_3) \} \subset E' \end{aligned}$$

is an antichain disconnector of G'. Likewise, according to Theorem 3.6, $\{ v_3 \}$ is an antichain cutset for G.

As another example, if the construction prior to Theorem 3.6 is applied to the graph of Figure 6.3, any ideal which contains q_x also contains p_x. This immediately gives that x is in no antichain cutset of a graph with a subgraph isomorphic to Figure 6.3. We will examine this in more detail in sections 5 and 6.

4. Antichain Lattice

Having reduced, by Corollary 3.5 and Theorem 3.6, the problem of finding an antichain cutset to that of finding an ideal, next an algorithm for generating ideals is presented. Recall that an ideal $\mathcal{I} \in \mathfrak{I}(E)$ is closed under $\breve{E}$. By Corollary 3.1, it is clear that

$$\begin{aligned} j \in \mathcal{I} &\Rightarrow \bigcup_{(i,j) \in E^{\bullet}} \{i\} \subseteq \mathcal{I}, \\ i \in \overline{\mathcal{I}} &\Rightarrow \bigcup_{(i,j) \in E^{\bullet}} \{j\} \subseteq \overline{\mathcal{I}}. \end{aligned}$$

For $U \subseteq V$, the following program fragment generates an ideal $\mathcal{I} \in \mathfrak{I}(E)$ such that $U \cap \mathcal{I} = \emptyset$. If a suitable representation for E is chosen, the algorithm runs in time $O(|V| + |E|)$ and additional space $O(|V|)$. Since condensing the graph has similar complexity [Tar72], taken with Theorem 3.6 this represents a significant improvement over the $O(|V|^6)$ existing method [Riv87].

Algorithm 4.1.

```
begin
  I := V;
  for u ∈ U do
    I := I \ ⋃_{(u,u')∈E+} {u'}
  end;
  output(I);
end.
```

Note that algorithm 4.1 produces a maximal ideal not containing the given set U, i.e., only U and its successors are omitted. This means that if an antichain cutset $\mathcal{K}$ containing a specific antichain $\mathcal{A}$ exists, algorithm 4.1 may be used with $U = \mathcal{A}$ to produce the ideal $\mathcal{I}$ used to generate $\mathcal{K}$. This property of algorithm 4.1 will be used in algorithm 4.3.

Finally, return to a problem mentioned earlier: construct a sequence of antichain cutsets forming a chain given a partial specification of members of the chain. More specifically, given directed graph $G = (V, E)$ with $s, t \in V$ and antichains $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq V$, find a chain of antichain cutsets $\mathcal{K}_1 \sqsupseteq \cdots \sqsupseteq \mathcal{K}_n$ such that $\mathcal{A}_i \sqsupseteq \mathcal{K}_i$ for $1 \leq i \leq n$. See section 2 for the definition of relation $\sqsupseteq$.

An application for such a chain of antichain cutsets was described at the end of section 1. In that application as in others, when the antichains $\mathcal{A}_i$ are unilateral it is desirable to produce antichain cutsets $\mathcal{K}_i$ which are unilateral. When this is possible, an injective mapping from the antichains then induces an injective mapping from the antichain cutsets.

As sections 3 and 5 discuss finding an antichain cutset, the principles by which a chain of antichain cutsets may be found are needed at this point. To this end, consider the structure of ideals and antichains in more detail. Let $\mathfrak{A}(E)$ be the set of antichains in G, $\mathfrak{AC}(E)$ the set of antichain cutsets in G.

Claim 2.1 was that $(\mathfrak{A}(E), \sqsupseteq)$ is a partial order. In fact, Dilworth [Dil60] showed it is a distributive lattice, which we get from claim 5.1 and the fact that $(\mathfrak{I}(E), \subseteq)$ is a distributive lattice. It was shown that $(\mathfrak{AC}(E), \sqsupseteq)$ is a linear order in Lemma 2.3, so the problem of finding a chain of antichain cutsets containing a given set of antichains is really one of finding a special extension of the given partial order to a linear order. That is, extend the partial order $(\{\mathcal{A}_1, \cdots, \mathcal{A}_n\}, \sqsupseteq)$ of specified antichains to a linear order $(\{\mathcal{K}_1, \cdots, \mathcal{K}_n\}, \sqsupseteq)$ of antichain cutsets such that $\mathcal{A}_i \sqsupseteq \mathcal{K}_i$ for $1 \leq i \leq n$.

First, a definition will be useful. Given $S' \subseteq S$ and binary relation R on S, define

$$max_R(S') = \{ s \in S' \mid \text{there exists no } s' \in S' \text{ such that } s \, R \, s' \}.$$

Then the following lemma suggests a particular order in which a chain could be discovered.

Lemma 4.2. *For* $\mathcal{I} \in \mathfrak{I}(E)$*, the following are equivalent:*

(*i*) $\mathcal{I} \in max_{\subseteq}(\mathfrak{I}(E))$;

(*ii*) $max_{E}(\mathcal{I}) \in max_{\sqsupseteq}(\mathfrak{A}(E))$;

(*iii*) $(s \in \mathcal{I} \wedge t \notin \mathcal{I}) \Rightarrow \langle \mathcal{I}, \overline{\mathcal{I}} \rangle = max_{\sqsupseteq}(\mathfrak{A}\mathfrak{D}(E))$.

Proof. (*i*) ⇒ (*ii*) by Theorem 5.3. (*ii*) ⇒ (*iii*) by Lemma 3.2 and Theorem 5.3. (*iii*) ⇒ (*i*) by Lemma 3.2. □

For each $0 \leq i \leq n$, algorithm 4.3 constructs a graph G_i to find the maximum (in $\sqsupseteq$) antichain cutset $\mathcal{K}_i$ containing some maximum remaining $\mathcal{A}_k$. It does so by collapsing every ineligible vertex either to the initial or final vertex where they may not be used by algorithm 4.1. As was observed, algorithm 4.1 finds a maximal ideal in each G_i, so the chain of antichain cutsets is produced from maximum to minimum element. Note that the nodes adjacent to s are minimum in $\mathfrak{A}\mathfrak{C}(E)$ and those adjacent to t are maximum in $\mathfrak{A}\mathfrak{C}(E)$. Algorithm 4.3 is given in a form that does not require the $\mathcal{A}_i$ to be distinct and could be simplified if they were. For $V' \subseteq V$, the *restriction* of edge set E to edges only involving vertices of V' is denoted $E \restriction V'$.

Algorithm 4.3.

Given directed graph $G = (V, E)$ with source $s \in V$, sink $t \in V$, and $(s,t) \notin E$ and antichains $\mathcal{A}_1, \ldots, \mathcal{A}_n$, such that $\{s,t\} \cap \mathcal{A}_i = \emptyset$ for $1 \leq i \leq n$, algorithm 4.3 finds a chain of antichain cutsets $\mathcal{K}_1 \sqsupseteq \cdots \sqsupseteq \mathcal{K}_n$ such that $\mathcal{A}_i \sqsupseteq \mathcal{K}_i$ for $1 \leq i \leq n$.

begin
  $\mathcal{A} := \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$;
  $i := n + 1$; $\mathcal{K}_{n+1} := \{ v \mid (v,t) \in E \}$;
  `while` $\mathcal{A} \neq \emptyset$ `do`
    $i := i - 1$;
    find k such that $\mathcal{A}_k \not\sqsupseteq \mathcal{A}_j$ for all $\mathcal{A}_j \in \mathcal{A}$;
    remove $\mathcal{A}_k$ from $\mathcal{A}$;
    `comment` make G_i;
    $V_i' := \{s_i\} \cup \big(V \setminus \{ v' \mid (v', v) \in E^+ \text{ for some } v \in \mathcal{A}_k \}\big)$;
    $E_i' := \big(\{s_i\} \times \mathcal{A}_k\big) \cup \big(E \restriction V_i'\big)$;
    $V_i := \{t_i\} \cup \big(V_i' \setminus \{ v' \mid (v, v') \in E^+ \text{ for some } v \in \mathcal{K}_{i+1} \}\big)$;
    $E_i := \big(\mathcal{K}_{i+1} \times \{t_i\}\big) \cup \big(E_i' \restriction V_i\big)$;
    find maximum antichain cutset $\mathcal{K} \sqsubseteq \mathcal{A}_k$ in $G_i = (V_i, E_i)$;
    `output`($\mathcal{K}_i := \mathcal{K}$);
  end;
end.

By Lemma 2.3, the $\mathcal{K}_i$ form a chain of antichain cutsets because

$$\mathcal{K}_i \bigcap \bigcup_{\substack{v \in \mathcal{K}_{i-1} \\ (v',v) \in E^+}} \{v'\} = \emptyset.$$

Suitably altered, the algorithm of [Ste86] can be used to generate all cutsets in each of the above applications, rather than just the one cutset generated by using algorithm 4.1.

5. Ideals II

There is, however, an alternative to the approach embodied in Corollary 3.5 in the case that an antichain cutset is desired. Corollary 3.3 established a bijection between certain ideals and antichain disconnectors. For antichain cutsets a more general bijection between antichains and ideals will be established. Throughout recall that s denotes a source node and t a sink node.

For $\mathfrak{A}(E)$ the set of antichains in graph $G = (V, E)$ and $\mathcal{A} \in \mathfrak{A}(E)$, define

$$\hat{\mathcal{A}} = \{ v' \mid (v', v) \in E^* \text{ for some } v \in \mathcal{A} \}.$$

Clearly $\hat{\mathcal{A}} \in \mathfrak{I}(E)$. Let partial order $(\mathfrak{A}(E), \sqsupseteq)$ be as is defined in section 2.

Claim 5.1. *For all* $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{A}(E)$

$$\hat{\mathcal{A}}_1 \subseteq \hat{\mathcal{A}}_2 \textit{ if and only if } \mathcal{A}_1 \sqsupseteq \mathcal{A}_2.$$

The next result is required before showing that ideals and antichains are in 1-1 correspondence.

Corollary 5.2. *For all* $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{A}(E)$

$$\hat{\mathcal{A}}_1 = \hat{\mathcal{A}}_2 \iff \mathcal{A}_1 = \mathcal{A}_2.$$

Proof.

$$\begin{aligned} & \hat{\mathcal{A}}_1 = \hat{\mathcal{A}}_2 \\ \iff & \hat{\mathcal{A}}_1 \subseteq \hat{\mathcal{A}}_2 \text{ and } \hat{\mathcal{A}}_2 \subseteq \hat{\mathcal{A}}_1 \end{aligned}$$

and by claim 5.1

$$\begin{aligned} \iff & \mathcal{A}_1 \sqsupseteq \mathcal{A}_2 \text{ and } \mathcal{A}_2 \sqsupseteq \mathcal{A}_1 \\ \iff & \mathcal{A}_1 = \mathcal{A}_2. \end{aligned}$$

□

Since for any $V' \subseteq V$, $max_E(V')$ is an antichain in graph $G = (V, E)$, the following may be proved.

Theorem 5.3. *The mapping $f: \mathfrak{A}(E) \to \mathfrak{I}(E)$ given by $f(\mathcal{A}) = \hat{\mathcal{A}}$ is a bijection.*

Proof. Note that $\mathcal{I} = \widehat{max_E(\mathcal{I})}$ for every $\mathcal{I} \in \mathfrak{I}(E)$. The inverse is also particularly simple as $\mathcal{A} = max_E(\hat{\mathcal{A}})$ for every $\mathcal{A} \in \mathfrak{A}(E)$. This surjection together with the injection of Corollary 5.2 establishes the result. □

Theorem 5.3 identifies an antichain for graph $G = (V, E)$ with an ideal of vertices in E and shows that, to find an antichain cutset, no antichain will be overlooked by restricting our attention to those generated by the following method using ideals. Given graph $G = (V, E)$ and nodes $j, k \in V$, define *compatibility* relation Ξ by

$$j \,\Xi\, k \iff (i,j) \in E \text{ and } (i,k) \in E \text{ for some } i \in V.$$

The next lemma gives a condition in which the set of maximal elements of an ideal in a graph, which by Theorem 5.3 is equivalent to an antichain in the graph, will also be a cutset in the graph. This condition may be compared to the finite case of an existing result. Let N stand for the four element ordered set $\{a, b, c, d\}$ in which $a < c$, $d < b$, and $d \prec c$ (that is, if $d < y \leq c$ then $y = c$). Say an ordered set is *N-free* if it contains no subset isomorphic to N. Then for an ordered set in which all chains are finite, every maximal antichain is an antichain cutset if and only if the ordered set is N-free [Gri69].

Lemma 5.4. *For graph $G = (V, E)$ and $\mathcal{I} \in \mathfrak{I}(E)$ with $s \in \mathcal{I}$ and $t \notin \mathcal{I}$,*

$$max_E(\mathcal{I}) \text{ is a cutset in } G \text{ if and only if } \mathcal{I} \text{ is closed under } \Xi.$$

Proof. Assume $max_E(\mathcal{I})$ is a cutset in G but $\mathcal{I}$ is not closed under Ξ. Then there exist $i \in \mathcal{I}$, $j \in \mathcal{I}$, and $k \notin \mathcal{I}$ such that $(i,j) \in E$, $(i,k) \in E$ and

$$\Big(\bigcup_{(k,k') \in E^*} \{k'\} \Big) \bigcap max_E(\mathcal{I}) = \emptyset.$$

This is because $s \in \mathcal{I}$, $t \notin \mathcal{I}$ and since $k \notin \mathcal{I}$ then $k' \notin \mathcal{I}$ so $k' \notin max_E(\mathcal{I})$. But since $i \in \mathcal{I}$, $j \in \mathcal{I}$, and $(i,j) \in E$, $i \notin max_E(\mathcal{I})$. Now by the definition of max_E,

$$(i', i) \in E^* \Rightarrow i' \notin max_E(\mathcal{I}).$$

Then there is a walk from s to i, edge (i,k), and walk from k to t which contains no element of $max_E(\mathcal{I})$. But then $max_E(\mathcal{I})$ is not a cutset, a contradiction.

Otherwise, let $\mathcal{I}$ be closed under Ξ but $max_E(\mathcal{I})$ not be a cutset. Then there exists a walk from s to t which contains no element of $max_E(\mathcal{I})$. Since $s \in \mathcal{I}$ let i be a vertex of that walk such that $i \in \mathcal{I}$ and there is edge $(i, k) \in E$ where $k \notin \mathcal{I}$, which must hold for some k since $t \notin \mathcal{I}$. Since $i \notin max_E(\mathcal{I})$, there exists edge $(i, j) \in E$ where $j \in \mathcal{I}$. But $(i, k) \in E$ and $(i, j) \in E$ where $j \in \mathcal{I}$ and $k \notin \mathcal{I}$ means $\mathcal{I}$ is not closed under Ξ, a contradiction. □

The second method to find an antichain cutset is given below.

Algorithm 5.5

For graph $G = (V, E)$ with $s, t \in V$ and $(s, t) \notin E$:
1) add an undirected edge to E between each pair of nodes that have a common parent. This ensures that ideals will be closed under Ξ;
2) find ideal $\mathcal{I} \in \mathfrak{I}(\tilde{E})$ such that $\tilde{s} \in \mathcal{I}$ and $\tilde{t} \notin \mathcal{I}$ for $\tilde{G} = (\tilde{V}, \tilde{E}) = G/\sim$;
3) then $max_E(\mathcal{I}) \in \mathfrak{AC}(E)$.

Taking advantage of transitivity in step 1), we need only add edges between a node and its left sibling in a breadth-first traversal. Note that step 2) need not be restricted to the maximal connected subgraph of $\tilde{G}$ with source $\tilde{s}$ and sink $\tilde{t}$, though the resulting ideal will be smaller in general if isolated vertices are removed from $\tilde{G}$. The advantage of algorithm 5.5 over that implied by Corollary 3.5 is that more cycles are introduced and eliminated by 5.5, so that an ideal is constructed in a smaller graph. The disadvantage, as a general method of constructing antichains which separate initial from final nodes, is that it cannot find disconnectors.

Algorithm 5.5 gives another $O(|V| + |E|)$ method for finding an antichain cutset containing or excluding certain vertices. Finally, algorithm 5.5 gives a slight improvement when used with the method described in [Riv87] to determine, for a subset K of an ordered set P, whether or not K is a cutset of P. However, it should be noted that for finite P, the case discussed, breadth-first search provides a $O(|E|)$ alternative. For most problems however, our methods are only as fast as existing methods [Möh89].

Finally, this section concludes with a modification of Lemma 5.4. Let $(P, <)$ be a poset. $I \subseteq P$ is an *ideal* if for each $x \in I$ and $y \in P$, if $y < x$ then $y \in I$. Ideal I is *regular* if sup C exists in I for each nonempty chain $C \subseteq I$. Let ideal I be *diffuse* if each maximal chain $C \subseteq P$ meets I. Similarly, ideal I is closed under Ξ provided for all $y \in I$ and $x, z \in P$, if $x < y$ and $x < z$ then $z \in I$. With the usual definitions of antichain and cutset for ordered sets, the characterization of antichain cutsets can be extended.

Theorem 5.6. *Let $(P, >)$ be an ordered set in which each ideal is regular. Then for diffuse ideal $I \subset P$, $K = \{\sup C \mid$ maximal chain $C \subseteq I\}$ is a* *cutset in P if and only if I is closed under Ξ.*

Proof. Assume K is a cutset in G but I is not closed under Ξ. By transitivity, let $z \notin I$ be such that $x < z$ and $x < y$ for some $y \in I$, $x \in P$. Since I is an ideal $x \in I$ but $x < y$ so $x \notin K$. Since I is diffuse and z is not in ideal I, let maximal chain C go from x to z without meeting K. Then K is not a cutset, a contradiction.

Otherwise, let I be closed under Ξ but K not be a cutset. Then there exists a maximal chain C which meets I but does not intersect K. Let $x \in I$ be an element of C. Since x is not maximal on C, for otherwise $x \in K$, let $z \notin I$ be such that $x < z$. But since I is regular, there is $y \in K$ from some chain through x such that $x < y$. Then $y \in I$ and $z \notin I$ so I is not closed under Ξ. □

Note that in Theorem 5.4, the case for finite graphs, the condition on diffuse ideals is insured by requiring the existence of source s. A slightly weaker condition could be used, similar to [Gri69], by requiring each maximal antichain meet each extremal path but we will not pursue further variations.

6. Exclusions

Methods were presented in sections 3 and 5 for finding antichain cutsets in finite directed graphs. Section 4 presented a method for finding an antichain cutset containing or excluding certain specified nodes of the graph. Though Lemma 2.4 shows that no vertex in a nontrivial strong component is contained in an antichain cutset, certainly there are acyclic graphs in which some node which is neither initial nor final is not contained in any antichain cutset. In fact, it is not difficult to find a finite example of such a directed acyclic graph. Before proceeding therefore, it is appropriate to characterize those nodes which do not appear in any antichain cutset. Consider the construction of graph $G' = (V', E')$ from $G = (V, E)$ given prior to Theorem 3.6. In this context, node $x \in V$ is not contained in any antichain cutset in $G = (V, E)$ if and only if every ideal in E' which contains $q_x \in V'$ also contains $p_x \in V'$. Recall that $\breve{R}$ denotes the inverse of relation R. Then since ideals in relation R are sets closed under $\breve{R}$, it is clear that node $x \in V$ is not contained in any antichain cutset in G if and only if $(p_x, p_x) \in E'^{+}$.

Next, a characterization of finite sets which are the union of antichain cutsets is given.

Theorem 6.1. *Node $x \in V$ is not contained in any antichain cutset in graph $G = (V, E)$ if and only if $(x, x) \in (E^{+}\breve{E})^{*}E^{+}$.*

Proof. For $x \in V$, without loss of generality consider the walk r in E corresponding to a cycle r' from p_x to p_x in E'. A *cycle* is a nontrivial

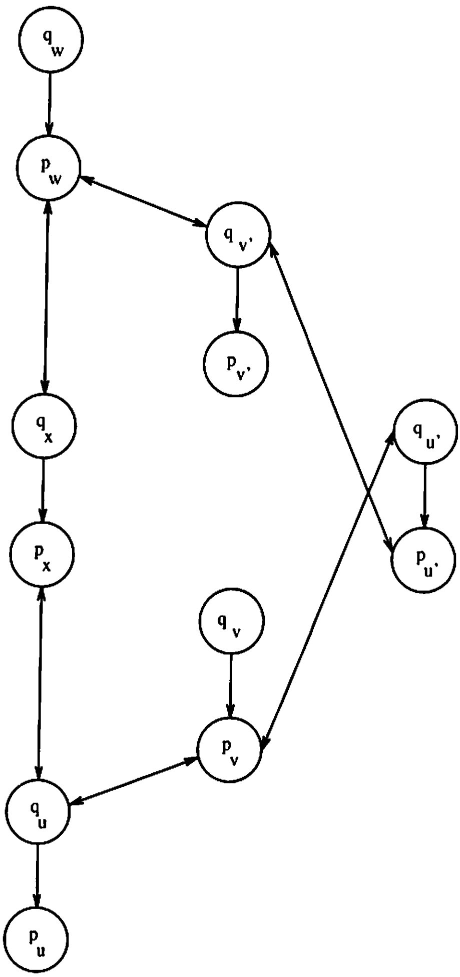

Figure 6.1. Graph for Theorem 6.1 proof.

closed walk with distinct vertices except for the ends.
(i) Cycle r' begins

$$p_{u_1}, q_{u_2}, p_{u_2}, \ldots, p_{u_{i-1}}, q_{u_i}$$

for some $i > 1$ where $u_1 = x$ for the initial segment of r'. The corresponding segment of r is

$$u_1, u_2, \ldots, u_{i-1}, u_i$$

and $(u_1, u_i) \in E^+$.
(ii) If r' continues to p_{u_i} then r is unchanged and case (i) is repeated.
(iii) Assume then that r' continues p_{v_1} where $u_i \neq v_1$ so that r continues

with $(u_i, v_1) \in \breve{E}$ and again (i) is repeated. Thus, zero or more times, (i) is followed by (ii) in r' and $(x, w_1) \in (E^+\breve{E})^*$. The final segment of r' is a sequence described by (i) and ends

$$p_{w_1}, q_{w_2}, p_{w_2}, \ldots, q_x, p_x$$

so $(w_1, x) \in E^+$. Altogether then, $(p_x, p_x) \in E'^+$ and $(x, x) \in (E^+\breve{E})^* E^+$ (see Figure 6.1). □

For transitive graphs, property $(x, x) \in (E^+\breve{E})^* E^+$ of Theorem 6.1 reduces to the much simpler $(x, x) \in E\breve{E}E$.

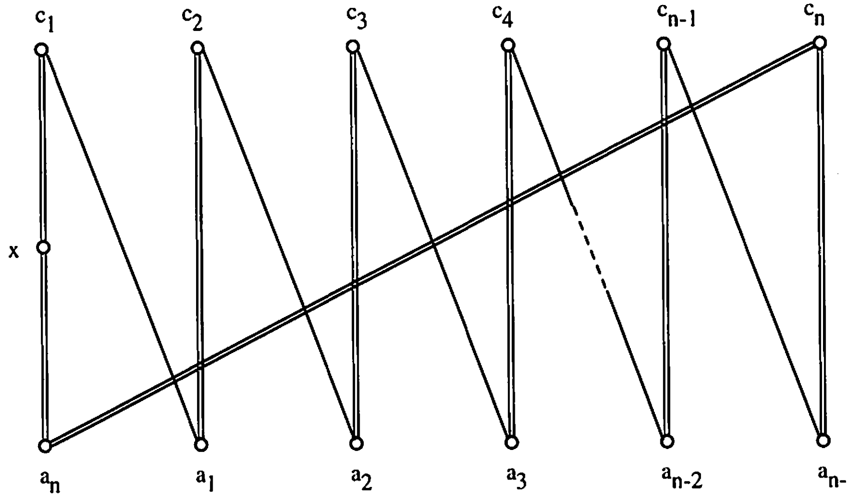

Figure 6.2. Graph of generalized alternating-cover cycle.

Recall that for partially ordered set $\mathfrak{S}$, a *chain* in $\mathfrak{S}$ is a totally ordered subset $c \subseteq \mathfrak{S}$ and path p in graph $G = (V, E)$ is *extremal* if p is a path from an initial vertex $s \in V$ to a final vertex $t \in V$. In the case of an arbitrary graph, Theorem 6.1 characterizes nodes not contained in any antichain which meets each extremal path. For another illustration of Theorem 6.1, consider its application to a graph with some additional structure. If G is the graph of a finite set V with partial order relation E and c is an extremal path in G, then the set of vertices of c form a maximal chain. When G is such a graph, Theorem 6.1 characterizes nodes which are not contained in any antichain which meets each maximal chain. In other words, when applied to the graph of a partial order, Theorem 6.1 characterizes nodes excluded from antichain cutsets in the order theoretic sense.

Consider set $\mathfrak{S}$ with partial order relation $\geq$. As usual, $u > v$ if and only if $u \geq v$ and $u \neq v$. Also, let $u \succ v$ if and only if $u > v$ and there exists no w such that $u > w > v$ and define $\prec$ similarly. For $n > 1$, define subset $\{x, a_1, c_1, \ldots, a_n, c_n\} \subseteq V$ to be a *generalized alternating-cover cycle* (for x) [Riv85b] if and only if

$$c_1 > x > a_n,$$
$$c_2 > a_1, \ldots, c_{n-1} > a_{n-2}, c_n > a_{n-1}$$

and

$$c_1 \succ a_1, c_2 \succ a_2, \ldots, c_n \succ a_n.$$

If these are the only comparabilities among elements $\{x, a_1, c_1, \ldots, a_n, c_n\}$, the graph of such a set is shown in Figure 6.2.

Lemma 6.2. *If $G = (V, E)$ is the graph of a finite partial order $\mathfrak{S} = (V, \geq)$ and $S \subseteq V$ is a generalized alternating-cover cycle,*

$$x \in S \text{ if and only if } (x, x) \in (E^+ \breve{E})^* E^+.$$

Proof.

$$\begin{aligned} x \in S &\iff x > a_n \prec c_n > \cdots \prec c_2 > a_1 \prec c_1 > x \\ &\iff (x, a_n) \in E^+ \wedge (a_n, c_n) \in \breve{E} \wedge \cdots \\ &\qquad \wedge (c_2, a_1) \in E^+ \wedge (a_1, c_1) \in \breve{E} \wedge (c_1, x) \in E^+ \\ &\iff (x, x) \in (E^+ \breve{E})^n E^+. \end{aligned}$$

□

Then the following has been obtained.

Corollary 6.3 [Riv85b]. *In a finite ordered set, an element is contained in an antichain cutset if and only if it is not contained in a generalized alternating-cover cycle.*

Proof. Theorem 6.1 and Lemma 6.2. □

The $O(|V| + |E|)$ complexity of Theorem 6.1 provides an efficient alternative to checking for the existence of generalized alternating-cover cycle.

As a final application of Theorem 6.1, let an ordered set be *upper levelled* [Riv89] if it has a diagram in which, for each element, all upper covers are on a horizontal level. A tree is a simple example of such a structure. On the other hand, if the diagram is not upper levelled, then some element a_1 has two upper covers x and c_2 which are forced to be at different levels. Figure 6.3 shows a special case of a diagram which is not upper levelled.

Since c_1 and c_2 are required by a_2 to be at the same level, and c_2 is a cover of a_1, call element x a *witness* to a diagram which is not upper levelled. That is, x comes between a_1 and what should be its cover c_1.

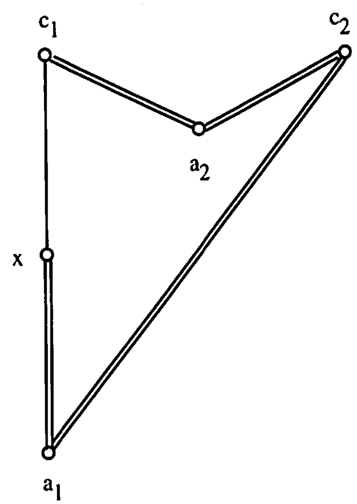

Figure 6.3. x witnesses a simple diagram which is not upper levelled.

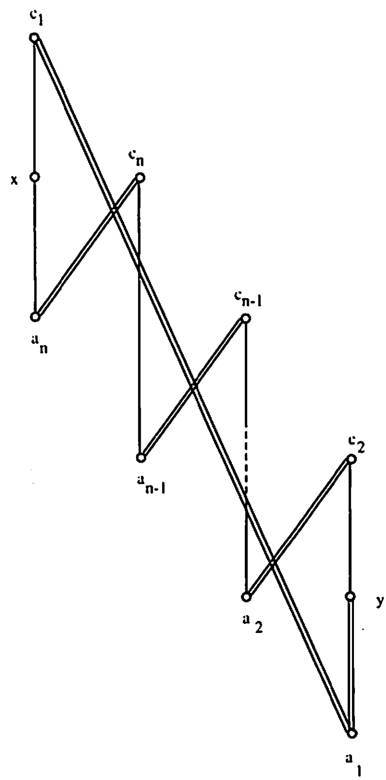

Figure 6.4. x witnesses a diagram which is not upper levelled.

Corollary 6.4. *If $G = (V, E)$ is the diagram of a finite partial order $\mathfrak{G} = (V, \geq)$, $x \in V$ is a witness that G is not upper levelled if and only if $(x, x) \in (E^+\breve{E})^*E^+$.*

Proof. Corollary 2 [Pel87] and Lemma 6.2. □

Theorem 6.1 can also provide a connection between two apparently unrelated structural results about ordered sets. A finite ordered set is upper levelled if and only if it contains no alternating cover cycle (Corollary 2 [Pel87]). Rival [Riv89] notes that this echoes a rewording of Corollary 6.3: a finite ordered set is the union of antichain cutsets if and only if it contains no alternating cover cycle [Riv85b]. Theorem 6.1 indicates that it is the presence of a distinguished element x on a special semi-walk which characterizes both these results.

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