Some Distributions Related To A Random Walk In A Plane

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ABSTRACT. In this paper we consider a random walk in a plane in which a particle at any stage moves one unit in any one of the four directions, namely, north south, east and west with equal probability and derive the joint and marginal distributions of certain characteristics of this random walk by using combinatorial methods.

1 Introduction

Let us consider a random walk in a plane (i.e., a two-dimensional simple symmetric walk) in which a particle starting from the origin moves at any stage one unit in any one of the four directions, namely, north, south, east and west with equal probability. In this random walk, since every path of length d in the plane has the probability $(1/4)^d$, the determination of the distribution of any characteristic of the walk when the particle starting from $(0, 0)$ reaches a fixed point (a, b) after d steps needs the knowledge of the number of paths corresponding to the characteristic under consideration and the number of all paths of length d from $(0, 0)$ to (a, b). DeTemple and Robertson (1984) and DeTemple, Jones and Robertson (1988) have derived distributions of some characteristics of this random walk. Later Csáki, Mohanty and Saran (1990) and Saran and Rani (1994) have derived some distributions related to the boundaries $y - x = k_1$ and $y + x = k_2$. In this paper we consider the above mentioned two-dimensional symmetric random walk and derive the joint and marginal distributions of some characteristics related to the boundary $y-x = k$ ($k \geq 0$), namely, arrivals, positive arrivals, touchings, crossings, index of the ith arrival, length of the interval between

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the ith and the jth arrivals, thus generalizing and extending the earlier work in this direction.

2 Notations

We introduce the following symbols.

$\mathbf{E_{a,b;d}}$: a path of length d from $(0,0)$ to (a,b).

$\mathbf{N(a,b;d;k,r)}$: the number of $E_{a,b;d}$ paths reaching the line $y-x=k$ exactly r times, i.e., having exactly r arrivals.

$\mathbf{N^-(a,b;d;k,r)}$: the number of $E_{a,b;d}$ paths never crossing the line $y-x=k$ and reaching it from below exactly r times.

$\mathbf{N^+(a,b;d;k,r)}$: the number of $E_{a,b;d}$ paths reaching the line $y-x=k$ from above exactly r times (known as positive arrivals).

$\mathbf{N^*(a,b;d;k,r)}$: the number of $E_{a,b;d}$ paths crossing the line $y-x=k$ exactly r times.

$\mathbf{N(a,b;d;k,r;i,d_1)}$: the number of paths of type $N(a,b;d;k,r)$ where the ith arrival occurs in d_1 steps.

$\mathbf{N^-(a,b;d;k,r;i,d_1)}$: the number of paths of type $N^-(a,b;d;k,r)$ where the ith touch occurs in d_1 steps.

$\mathbf{N^+(a,b;d;k,r;i,d_1)}$: the number of paths of type $N^+(a,b;d;k,r)$ where the ith positive arrival occurs in d_1 steps.

$\mathbf{N^*(a,b;d;k,r;i,d_1)}$: the number of paths of type $N^*(a,b;d;k,r)$ where the ith crossing occurs in d_1 steps.

$\mathbf{N(a,b;d;k,r;i,d_1;j,d_2)}$: the number of paths of type $N(a,b;d;k,r;i,d_1)$ where the length of the interval between the ith and the jth arrival is d_2.

$\mathbf{N^-(a,b;d;k,r;i,d_1;j,d_2)}$: the number of paths of type $N^-(a,b;d;k,r;i,d_1)$ where the length of the interval between the ith and the jth touch is d_2.

$\mathbf{N^+(a,b;d;k,r;i,d_1;j,d_2)}$: the number of paths of type $N^+(a,b;d;k,r;i,d_1)$ where the length of the interval between the ith and the jth positive arrival is d_2.

$\mathbf{N^*(a,b;d;k,r;i,d_1;j,d_2)}$: the number of paths of type $N^*(a,b;d;k,r;i,d_1)$ where the length of the interval between the ith and the jth crossing is d_2.

3 Some auxiliary results

Some useful results needed in the sequel are quoted from Csáki, Mohanty and Saran (1990) and Saran and Rani (1994).

(i) For $a > b$

$$N^-(a,b;d;0,0) = \frac{a-b}{d}\binom{d}{\frac{d-a-b}{2}}\binom{d}{\frac{d-a+b}{2}}$$

where $d \geq a+b$ and $d-a-b \equiv 0 \pmod 2$.

(ii)

$$N^-(a,a;d;0,1) = \frac{1}{d-1}\binom{d-1}{\frac{d-2}{2}}\binom{d}{\frac{d-2a}{2}}$$

= the number of paths of length d from $(0,0)$ to (a,a) lying entirely below the line $y = x$ except at the end point.

(iii) For $a \geq b-k$, $k \geq 0$ and $r \geq 1$

$$N^-(a,b;d;k,r) = \frac{a-b+2k+r-1}{d-r+1}\binom{d-r+1}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}}.$$

When $k = 0$, the starting point is counted as a touch.

(iv) For $a \geq b-k$, $k \geq 0$ and $r \geq 1$

$$N(a,b;d;k,r) = 2^{r-1}N^-(a,b;d;k,r).$$

When $k = 0$, the starting point is counted as an arrival.

(v) For $k > 0$ and $r \geq 1$

$$N^*(x,k+x;d;k,r) = \frac{k+1+2r}{d+1}\binom{d+1}{\frac{d-k-2r}{2}}\binom{d}{\frac{d-k-2x}{2}}.$$

(vi) For $a > b$ and $r \geq 1$

$$N^*(a,b;d;0,r) = \frac{a-b+1+2r}{d+1}\binom{d+1}{\frac{d-a+b-2r}{2}}\binom{d}{\frac{d-a-b}{2}}.$$

(vii) For $r \geq 1$

$$N^*(a,a;d;0,r) = \frac{4(r+1)}{d}\binom{d}{\frac{d-2(r+1)}{2}}\binom{d}{\frac{d-2a}{2}}.$$

(viii) For $a \geq b-k$, $k \geq 0$ and $r \geq 0$

$$N^+(a,b;d;k,r) = \frac{a-b+2k+2r+1}{d+1}\binom{d+1}{\frac{d-a+b-2r-2k}{2}}\binom{d}{\frac{d-a-b}{2}}.$$

When $k=0$, the starting point is counted as a positive arrival.

4 Joint distributions

Theorem 1.

(a) For $a \geq b-k$, $k > 0$, $r \geq 1$, $d_1 - k \geq 2(i-1)$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 - a + b - k \geq 2(r-j)$

$$N^-(a,b;d;k,r;i,d_1;j,d_2) = \frac{(k+i-1)(j-i)(r-j+a-b+k)}{(d_1-i+1)(d_2-j+i)(d-d_1-d_2-r+j)} \binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}}\binom{d_2-j+i}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2-r+j}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}}\binom{d}{\frac{d-a-b}{2}} \qquad (1)$$

(b) For $k=0$, $a \geq b$, $r \geq 1$, $d_1 \geq 2i$, $d_2 \geq 2(j-i)$ and $d-d_1-d_2-a+b \geq 2(r-j)$

$$N^-(a,b;d;k,r;i,d_1;j,d_2) = \frac{i(j-i)(r-j+a-b)}{(d_1-i)(d_2-j+i)(d-d_1-d_2-r+j)} \binom{d_1-i}{\frac{d_1-2i}{2}}\binom{d_2-j+i}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2-r+j}{\frac{d-d_1-d_2-a+b-2r+2j}{2}}\binom{d}{\frac{d-a-b}{2}}. \qquad (2)$$

Proof: Let the path, as envisaged in (1), touch the line $y-x=k$ for the ith time at the point $(x_1, k+x_1)$ in d_1 steps and, for the jth time at the point $(x_2, k+x_2)$ in d_1+d_2 steps, respectively, where $d_1-k \equiv 0 \pmod 2$ and $d_2 \equiv 0 \pmod 2$. Then the required number of paths is given by

$$\begin{aligned}
&N^-(a,b;d;k,r;i,d_1;j,d_2) \\
&\quad = \sum_{x_1}\sum_{x_2} N^-(x_1,k+x_1;d_1;k,i) \\
&\qquad N^-(x_2-x_1,x_2-x_1;d_2;0,j-i) \\
&\qquad N^-(a-x_2,b-k-x_2;d-d_1-d_2;0,r-j) \\
&\quad = \sum_{x_1=-(d_1+k)/2}^{(d_1-k)/2} \sum_{x_2=-(d_2-2x_1)/2}^{(d_2+2x_1)/2} \frac{k+i-1}{d_1-i+1} \\
&\qquad \binom{d_1}{\frac{d_1-k-2x_1}{2}} \binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}} \frac{j-i}{d_2-j+i} \\
&\qquad \binom{d_2-j+i}{\frac{d_2-2j+2i}{2}} \binom{d_2}{\frac{d_2-2x_2+2x_1}{2}} \frac{r-j+a-b+k}{d-d_1-d_2-r+j} \\
&\qquad \binom{d-d_1-d_2-r+j}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}} \binom{d-d_1-d_2}{\frac{d-d_1-d_2-a-b+k+2x_2}{2}},
\end{aligned}$$

by (iii) of Section 3. Since both d_2 and d_1-k are multiples of 2, therefore, on substituting $u=(d_2+2x_1)/2$ and $v=(d_1-k)/2$ and using repeatedly the Chu-Vandermonde identity

$$\sum_{x=0}^{n} \binom{a}{x}\binom{b}{n-x} = \binom{a+b}{n}, \tag{3}$$

it leads to (1). Proceeding in a similar manner, as for $k>0$, one can easily establish (2).

Deductions:

A. Summing (1) over d_2 from $2j-2i$ to $d-d_1-a+b-k-2r+2j$ and using the convolution identity in Mohanty ((1979), p. 25), we get (1) in Saran and Rani (1994).

B. Summing (2) over d_2 from $2j-2i$ to $d-d_1-a+b-2r+2j$, we get (2) in Saran and Rani (1994).

Theorem 2.

(a) For $a > b - k$, $k > 0$, $r \geq 1$, $d_1 - k \geq 2(i-1)$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 - a + b - k \geq 2(r-j)$

$$N^*(a,b;d;k,r;i,d_1;j,d_2) = \frac{(k+2i-1)(2j-2i)(a-b+k+1+2r-2j)}{(d_1+1)d_2(d-d_1-d_2+1)} \binom{d_1+1}{\frac{d_1-k-2i+2}{2}} \binom{d_2}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2+1}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}} \binom{d}{\frac{d-a-b}{2}} \tag{4}$$

(b) For $k > 0$, $r \geq 1$, $d_1 - k \geq 2(i-1)$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 \geq 2(r-j+1)$

$$N^*(a,a+k;d;k,r;i,d_1;j,d_2) = \frac{(k+2i-1)(2j-2i)(2r-2j+2)}{(d_1+1)d_2(d-d_1-d_2)} \binom{d_1+1}{\frac{d_1-k-2i+2}{2}} \binom{d_2}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2}{\frac{d-d_1-d_2-2r+2j-2}{2}} \binom{d}{\frac{d-2a-k}{2}} \tag{5}$$

(c) For $a > b$, $r \geq 1$, $d_1 \geq 2i$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 - a + b \geq 2(r-j)$

$$N^*(a,b;d;0,r;i,d_1;j,d_2) = \frac{2i(2j-2i)(a-b+1+2r-2j)}{d_1 d_2(d-d_1-d_2+1)} \binom{d_1}{\frac{d_1-2i}{2}} \binom{d_2}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2+1}{\frac{d-d_1-d_2-a+b-2r+2j}{2}} \binom{d}{\frac{d-a-b}{2}} \tag{6}$$

(d) For $r \geq 1$, $d_1 \geq 2i$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 \geq 2(r-j+1)$

$$N^*(a,a;d;0,r;i,d_1;j,d_2) = \frac{2i(2j-2i)(2r-2j+2)}{d_1 d_2(d-d_1-d_2)} \binom{d_1}{\frac{d_1-2i}{2}} \binom{d_2}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2}{\frac{d-d_1-d_2-2r+2j-2}{2}} \binom{d}{\frac{d-2a}{2}}. \tag{7}$$

Proof: Let the path as envisaged in (4) cross the line $y - x = k$ for the ith time at the point $(x_1, k + x_1)$ in d_1 steps, and for the jth time at the point $(x_2, k + x_2)$ in $d_1 + d_2$ steps, respectively. Then the required number of paths is given by

$$
\begin{aligned}
&N^*(a, b; d; k, r; i, d_1; j, d_2) \\
&\quad = \sum_{x_1}\sum_{x_2} N^*(x_1, k + x_1; d_1; k, i - 1) \\
&\qquad N^*(x_2 - x_1, x_2 - x_1; d_2; 0, j - i - 1) \\
&\qquad N^*(a - x_2, b - k - x_2; d - d_1 - d_2; 0, r - j) \\
&\quad = \sum_{x_1 = -(d_1+k)/2}^{(d_1-k)/2} \; \sum_{x_2 = -(d_2 - 2x_1)/2}^{(d_2+2x_1)/2} \frac{k + 1 + 2i - 2}{d_1 + 1} \\
&\qquad \binom{d_1+1}{\frac{d_1-k-2i+2}{2}} \binom{d_1}{\frac{d_1-k-2x_1}{2}} \frac{2j - 2i - 2 + 2}{d_2} \\
&\qquad \binom{d_2}{\frac{d_2-2j+2i-2+2}{2}} \binom{d_2}{\frac{d_2-2x_2+2x_1}{2}} \frac{a - b + k + 1 + 2r - 2j}{d - d_1 - d_2 + 1} \\
&\qquad \binom{d-d_1-d_2+1}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}} \binom{d-d_1-d_2}{\frac{d-d_1-d_2-a-b+k+2x_2}{2}},
\end{aligned}
$$

by (v), (vi) and (vii) of Section 3. Simplifying the above expression in a similar manner as in Theorem 1, it leads to (4).

Likewise (5) to (7) can be established.

Deduction:

Summing (4) to (7) each over d_2, we get, respectively, results (3) to (6) of Saran and Rani (1994).

Similarly the following two results can easily be obtained by using similar arguments:

Theorem 3.

(a) For $a \geq b - k$, $k > 0$, $r \geq 1$, $d_1 - k \geq 2(i - 1)$, $d_2 \geq 2(j - i)$

and $d - d_1 - d_2 - a + b - k \geq 2(r - j)$

$$N(a,b;d;k,r;i,d_1;j,d_2) = 2^{r-1}\frac{(k+i-1)(j-i)(a-b+k+r-j)}{(d_1-i+1)(d_2-j+i)(d-d_1-d_2-r+j)} \binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}}\binom{d_2-j+i}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2-r+j}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}}\binom{d}{\frac{d-a-b}{2}}. \quad (8)$$

(b) For $a \geq b$, $r \geq 1$, $d_1 \geq 2i$, $d_2 \geq 2(j-i)$ and $d-d_1-d_2-a+b \geq 2(r-j)$

$$N(a,b;d;0,r;i,d_1;j,d_2) = 2^{r}\frac{i(j-i)(a-b+r-j)}{(d_1-i)(d_2-j+i)(d-d_1-d_2-r+j)} \binom{d_1-i}{\frac{d_1-2i}{2}}\binom{d_2-j+i}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2-r+j}{\frac{d-d_1-d_2-a+b-2r+2j}{2}}\binom{d}{\frac{d-a-b}{2}}. \quad (9)$$

Theorem 4. *For $a \geq b-k$, $k \geq 0$, $r \geq 1$, $d_1 - k \geq 2i$, $d_2 \geq 2(j-i)$ and $d - d_1 - d_2 - a + b - k \geq 2(r-j)$*

$$N^{+}(a,b,d;k,r;i,d_1;j,d_2) = \frac{(k+2i)(a-b+k+2r-2j+1)(2j-2i)}{d_1 d_2(d-d_1-d_2+1)} \binom{d_1}{\frac{d_1-k-2i}{2}}\binom{d_2}{\frac{d_2-2j+2i}{2}} \binom{d-d_1-d_2+1}{\frac{d-d_1-d_2-a+b-k-2r+2j}{2}}\binom{d}{\frac{d-a-b}{2}}. \quad (10)$$

When $k = 0$, the starting point is counted as a positive arrival.

Deductions:

A. Summing (8) and (9) each over d_2, we get, respectively, results equivalent to (7) and (8) of Saran and Rani (1994).

B. Summing (10) over d_2, we get a result which is in agreement with (16) in Saran and Rani (1994).

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