Some Structural Characterizations of λ-Designs

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ABSTRACT. The twenty-five year old λ-design conjecture remains unsettled. Attempts to characterize these irregular, tight, 2-designs have produced a great number of parametric and dual structure characterizations of the so-called Type-I Designs. We establish some new structural characterizations and establish the conjecture in the smallest unsettled case ($\lambda = 14$) of the $2p$ family.

1 Introduction

Definition 1. A λ-design [5] is a combinatorial configuration consisting of a family of n subsets $\{S_1, S_2, \dots, S_n\}$ of an n-element set $X = \{x_1, x_2, \dots, x_n\}$ of size n such that

(i) $|S_i \cap S_j| = \lambda$ for $i \neq j$,

(ii) $|S_i| > \lambda > 0$ for $i, j = 1, 2, \dots n$.

(iii) For some i and j, $|S_i| \neq |S_j|$.

Note that property (ii) above rules out degeneracies and (iii) simply rules out the case of symmetric block designs. The only known examples of λ-designs are the so-called "Type-I Designs" constructed from symmetric block designs as follows. If $\{B_1, B_2, \dots, B_v\}$ are the blocks of a symmetric (v, k, λ)-design [6] with $k \neq 2\lambda$, then with Δ denoting symmetric difference

$$S_1 = B_1, S_2 = B_2 \Delta B_1, S_3 = B_3 \Delta B_1, \dots, S_v = B_v \Delta B_1$$

are the blocks of a λ-design with $\lambda' = k - \lambda$.

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If, in the symmetric design, $k = 2\lambda$, the above construction results in another symmetric design. In the contrary case, the λ'-design produced has very special structure. All block sizes are $2\lambda'$ save one.

The "λ-Design Conjecture" is the assertion that all λ-Designs are Type-I. It has been verified for prime values of λ by Singhi and Shrikhande [8] and for $\lambda \leq 10$ [1, 2, 4, 5, 7].

The fundamental structure result for λ-designs is most easily stated in terms of its (0,1) incidence matrix, A, which satisfies

$$A^t A = D + \lambda J \text{ with} \tag{1.1}$$

$$D = diag(k_1 - \lambda, k_2 - \lambda, \ldots, k_n - \lambda). \tag{1.2}$$

Here the columns of A correspond to the blocks of the design, A^t denotes the transpose of A, $diag$ denotes a diagonal matrix, the k_j are the block sizes, and J is the $n \times n$ matrix all of whose entries are 1.

Theorem 1.1 (Ryser-Woodall). *Let A be the $n \times n$ incidence matrix of a λ-design satisfying (1.1). Then A has precisely two row sums, r_1 and r_2, with $r_1 > r_2$, say. We have that $r_1 + r_2 = n + 1$, and putting $\rho = \frac{r_1 - 1}{r_2 - 1}$ and rearranging the rows of A if necessary*

$$AD^{-1}A^t = I + R, \tag{1.3}$$

where, if there are e_1 rows with sum r_1 and $e_2 = n - e_1$ rows with sum r_2, R is given in block form by

$$R = \begin{pmatrix} \rho J_{e_1} & J \\ J & (1/\rho) J_{e_2} \end{pmatrix}. \tag{1.4}$$

Of course I in (1.3) denotes the identity matrix and the subscripts on J indicate its order. We use **1** for a column vector of ones with a subscript for size, if necessary, and **0** for a zero vector or matrix.

In (1.3), A is arranged so that its first e_1 rows have the larger row sum, r_1 and A, in block form is given by

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{1.5}$$

Using k'_j for the jth column sum of A_1 and k^*_j for the jth column sum of A_2, multiplying (1.1) by **1** on the right and simplifying produces

$$k'_j(r_1 - 1) + k^*_j(r_2 - 1) = \lambda(n - 1). \tag{1.6}$$

This can also be written, using the definition of ρ in Theorem 1.1, as

$$k^*_j - \lambda = -\rho(k'_j - \lambda). \tag{1.7}$$

Note that (1.7) implies that if $k'_i = k'_j$ then $k_i = k_j$. It also shows that $k_j = 2\lambda$ if and only if $k'_j = k^*_j = \lambda$. In a Type-I design there are only two block sizes and one of these occurs for a unique block. A not too difficult result is the following.

Theorem 1.2 [1]. *If $n-1$ blocks of a λ-design on n points have the same size the design is Type-I.*

We assume the reader is familiar with elementary results and notations for 2-designs: (v, b, r, k, λ) [6]. Recall that a 2-design obtained from a symmetric ($v = b$) design by removing a block and its incident points is called *residual*. Residual designs satisfy $r = k + \lambda$ and a design with the parameters of a residual design is called *quasi-residual*. Quasi-residual designs are characterized by the fact that $r = k + \lambda$. General 2-design parameters necessarily satisfy

$$r(k-1) = \lambda(v-1). \tag{1.8}$$

2 Results on Substructure

The dual structure of a Type-I design is quite uniform and characterizes them [3, 4]. These designs also have special substructure as well. The residual designs of both the symmetric design producing them and its complement are present as subdesigns. Indeed, in the Type-I case, the subdesign induced by the points contained in a single block is a residual design.

Theorem 2.1. *Assume the incidence matrix of a λ-design has the form*

$$A = \begin{pmatrix} 1 & N \\ 0 & M \end{pmatrix}, \tag{2.1}$$

where N is the incidence matrix of a 2-design, $\mathcal{D}$. Then the design is Type-I if and only if $\mathcal{D}$ is a quasi-residual 2-design.

Proof: Note that the block size, $\tilde{k}$, for $\mathcal{D}$ is λ and the replication number, $\tilde{r}$, is either $r_1 - 1$ or $r_2 - 1$. If the former holds then $k^*_1 = 0$ and $k_1 = \lambda + \lambda/\rho$. In the latter case $k'_1 = 0$ and $k_1 = \lambda + \lambda\rho$. We treat the former case as the cases are parallel with ρ and $1/\rho$ interchanged.

From (1.3) we obtain, putting the contribution from column one on the right hand side and with $E = diag(k_2 - \lambda, \ldots, k_n - \lambda)$:

$$NE^{-1}N^t = I + (\rho - \frac{1}{k_1 - \lambda})J = I + \rho(1 - \frac{1}{\lambda})J \tag{2.2}$$

From the remarks above about the parameters of D, we also have, using $\tilde{\lambda}$ for the number of blocks of D containing any pair of treatments:

$$NN^t = \lambda I + \tilde{\lambda} J \tag{2.3}$$

From the basic parameter relation (1.8) we have

$$(r_1 - 1)(\lambda - 1) = \tilde{\lambda}(k_1 - 1) = \tilde{\lambda}(\lambda + \frac{\lambda}{\rho} - 1). \quad (2.4)$$

Replacing $r_1 - 1$ by $\lambda + \tilde{\lambda}$ this becomes

$$\frac{\tilde{\lambda}}{\lambda} = \rho(1 - \frac{1}{\lambda}). \quad (2.5)$$

Hence comparing (2.3) and (2.2) we obtain

$$N(E^{-1} = \frac{1}{\lambda}I)N^t = 0. \quad (2.6)$$

However, in this case $k_i' \geq \lambda$ and (1.7) implies that $k_j \leq 2\lambda$. Thus $E^{-1} - \frac{1}{\lambda}I$ is a non-negative diagonal matrix. This implies that $E^{-1} = \frac{1}{\lambda}$ and we have that $k_j = 2\lambda$ for $j \geq 2$. Hence the λ-design is Type-I by Theorem 1.2.

As to the converse, of course if column one in (2.1) corresponds to the exceptional block size we are done. In general from (1.3), concentrating on two rows, say the first two, through N we obtain from the (1,2) position:

$$\frac{1}{K_1 - \lambda} + \frac{\tilde{\lambda}}{\lambda} = \rho \quad (2.7)$$

and, with $\tilde{r}$ as the row sum of N, from the (1,1) position:

$$\frac{1}{k_1 - \lambda} + \frac{\tilde{r}}{\lambda} = 1 + \rho. \quad (2.8)$$

Subtracting we have

$$\frac{\tilde{r}}{\lambda} - \frac{\tilde{\lambda}}{\lambda} = 1. \quad (2.9)$$

Thus $\tilde{r} - \tilde{\lambda} = \lambda = \tilde{k}$ and $\mathcal{D}$ is residual.

Clearly, if the incidence matrix A, normalized as in (2.1), has constant column sums in M then Theorem 1.2 applies to show that the underlying design is Type-I. Conversely, if $\mathcal{D}$ is Type-I, say produced from the symmetric design $\mathcal{S}$, then (2.1) implies, since all rows through N have the same sum, that $k_1' = 0$ or $k_1^* = 0$. Since all but one column sum in a Type-I design are 2λ with $k_j' = k_j^* = \lambda$, the first column corresponds to the exceptional block and M is the incidence matrix of the residual of $\mathcal{S}$.

Corollary 2.2. *The incidence matrix A in (2.1) with N the incidence matrix of a 2-design corresponds to a Type-I λ-design if and only if M is also the incidence matrix of a 2-design.*

The next result should be valid without the restriction on c but we have been unable to establish this. We recall that a 2-design, $\mathcal{D}$, is a c-multiple if the blocks can be partitioned into c classes each class of blocks forming the same 2-design on the points of $\mathcal{D}$.

Theorem 2.3. *Let the $n \times n$ incidence matrix, A, of a λ-design, $\mathcal{D}$ be in the form (2.1) where N is the incidence matrix of a 2-design which is a c-multiple where $c > 1$. Then $\mathcal{D}$ is Type-I.*

Proof: Assume that N represents a c-multiple of a $2-(v, b, r, k, \lambda)$-design. Clearly all the points of the 2-design represented by N have the same replication, $cr + 1$, in $\mathcal{D}$. We first argue that we can assume that they have the smaller replication number r_2 . Should $r_1 = cr + 1$ form the matrix

$$A' = \begin{pmatrix} 1 & J - N \\ 0 & M \end{pmatrix}. \tag{2.10}$$

This matrix represents the so-called point complement of $\mathcal{D}$ with respect to block one and is either a symmetric block design (in which case $\mathcal{D}$ is Type-I and we are finished) or again a λ-design (with a new value for λ, of course). Now $J - N$ also represents a c multiple 2-design and the rows through $J - N$ have sum $n - 1 - (r_1 - 1) + 1 = n + 1 - r_1 = r_2$ from The Ryser-Woodall Theorem.

Hence we consider the case that $cr + 1 = r_2$, $c > 1$. Since the c-multiple has repeated blocks intersecting in $k = \lambda$ points, the corresponding blocks in the residual represented by the matrix M are disjoint. Since some of the rows of M have sum r_1 and the blocks in N containing a corresponding point are distinct we must have $b \geq r_1$. But then $2b > r_1 + r_2 = n + 1 = cb + 2$, denying that $c > 1$.

3 Remarks on the case $\lambda = 14$

The settling of the λ-design conjecture for specific values of λ beyond the first few, which did result in the Singhi-Shrikhande result for the prime case, have not proven particularly elucidating. In view of Woodall's result that $\rho \leq \lambda$ which bounds n as a function of λ (roughly λ^3, see [1]) it is not surprising that specific cases can be handled, though, for example, $\lambda = 12$ remains unresolved. Relying heavily on several technical lemmas established by Seress [7] for the case that $\lambda/2$ is a prime, the second author has established the following.

Theorem 3.1 (Tzong-Pyng Tsaur). *A λ-design with $\lambda = 14$ is Type-I.*

Remarks on the proof. If one puts $\rho = x/y$ with x and y relatively prime, a basic result of Singhi and Shrikhande is

Theorem 3.2 [8]. *In any λ-design $x < y + \lambda < 2\lambda$ holds. Furthermore if $gcd(\lambda, x - y) = 1$ the design is Type-I.*

In view of this, the present case ($\lambda = 14$) reduces to $gcd(\lambda, x - y) = 2$ or $gcd(\lambda, x - y) = 7$. Moreover the following result due to Seress further reduces the possibilities.

Theorem 3.3 [7]. *In a λ-design with $\lambda/2 = p$, p a prime, if either (i) x and y are both odd with $y \geq 1$ or (ii) $y = 1$ and $x \geq p$ then the design is Type-I.*

Should $gcd(14, x - y) = 7$ this reduces the possibilities for ρ to one of the numbers in the set $\{13/6, 12/5, 11/4, 10/3, 9/2\}$. For each case one uses (1.7) to restrict the possible column types. Counting arguments and further results from [7] are used to complete the argument.

In the case $gcd(14, x - y) = 2$, ρ is even more tightly constrained to be either 3,5, or 7 which yield the parameter possibilities in the following table.

n	e_1	e_2	r_1	r_2
91	17	74	76	16
163	14	149	136	28
235	11	224	196	40
307	8	299	256	52
379	5	374	316	64

Relatively direct, if tedious, counting arguments rule out these five possibilities. We omit further detail.

4 Summary

It should be the case that if the points on a block of a λ-design form a 2-design on the remaining blocks restricted to those points the design is a Type-I design. We have been unable to establish that without further assumptions. Moreover the general case of $\lambda = 2p$, p a prime, remains unsettled. It follows from some unpublished work of Xiang-dong Hou that the conjecture is valid for $\lambda = 2p$ provided no block size exceeds $4p$. Finally, it is curious to note that, perhaps due to the peculiar number theoretic arguments, the case of $\lambda = 12$ has not been completed.

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