On Packing Designs with Block Size 5 and Indexes 3 and 6

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ABSTRACT. Let V be a finite set of order v. A (v, κ, λ) packing design of index λ and block size κ is a collection of κ-element subsets, called blocks, such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks, $\sigma(v, \kappa, \lambda)$, in a packing design. It is well known that $\sigma(v, \kappa, \lambda) \leq [\frac{v}{\kappa}[\frac{v-1}{\kappa-1}\lambda]] = \psi(v, \kappa, \lambda)$, where $[x]$ is the largest integer satisfying $x \geq [x]$. It is shown here that $\sigma(v, 5, 6) = \psi(v, 5, 6)$ for all positive integers $v \geq 5$ with the possible exceptions of $v = 43$ and that $\sigma(v, 5, 3) = \psi(v, 5, 3)$ for all positive integers $v \equiv 1, 5, 9, 17 \pmod{20}$ and $\sigma(v, 5, 3) = \psi(v, 5, 3) - 1$ for all positive integers $v \equiv 13$ mod (20) with the possible exception of $v = 17, 29, 33, 49$.

1 Introduction

A (v, κ, λ) packing design (or respectively covering design) of order v, block size κ and index λ is a collection β of κ-element subsets, called blocks, of a v-set V such that every 2-subset of V occurs in at most (at least) λ blocks.

Let $\sigma(v, \kappa, \lambda)$ denote the maximum number of blocks in a (v, κ, λ) packing design; and $\alpha(v, \kappa, \lambda)$ denote the minimum number of blocks in a (v, κ, λ) covering design. A (v, κ, λ) packing design with $|\beta| = \sigma(v, \kappa, \lambda)$ will be called a maximum packing design. Similarly, (v, κ, λ) covering design with $|\beta| = \alpha(v, \kappa, \lambda)$ is called a minimum covering design. It is well known [21] that

$$\sigma(v, \kappa, \lambda) \le \left\lfloor \frac{v}{\kappa} \left\lfloor \frac{v-1}{\kappa - 1} \lambda \right\rfloor \right\rfloor = \psi(v, \kappa, \lambda)$$

and

$$\alpha(v, \kappa, \lambda) \ge \left\lceil \frac{v}{\kappa} \left\lceil \frac{v-1}{\kappa - 1} \right\rceil \right\rceil = \phi(v, \kappa, \lambda)$$

where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \le x$ and $\lceil x \rceil$ is the smallest integer satisfying $x \le \lceil x \rceil$. When $\sigma(v, \kappa, \lambda) = \psi(v, \kappa, \lambda)$ the (v, κ, λ) packing design is called optimal packing design. Similarly when $\alpha(v, \kappa, \lambda) = \phi(v, \kappa, \lambda)$ the (v, κ, λ) covering design is called minimal covering design. Packing designs with $\lambda = 3$ are called tripacking.

Several researchers have been involved in determining the packing number $\sigma(v, \kappa, \lambda)$ known up to date (see bibliography). Our interest here is in the case $\kappa = 5$ and $\lambda = 3, 6$. Our goal is to prove the following.

Theorem 1.1.

(1) $\sigma(v, 5, 6) = \psi(v, 5, 6)$ for all positive integers $v \ge 5$ with the possible exception of $v = 43$

(2) $\sigma(v, 5, 3) \equiv \psi(v, 5, 3) - 1$ for all $v \equiv 13$ mod (20) and $\sigma(v, 5, 3) = \psi(v, 5, 3)$ for all $v \equiv 1, 5, 9, 13$ (mod 20) with the possible exceptions of $v = 17, 29, 33, 49$.

2 Recursive Constructions

In order to describe our recursive constructions we require several other types of combinatorial designs. A balanced incomplete block design, $B[v, \kappa, \lambda]$, is a (v, κ, λ) packing design where every 2-subset of points is contained in precisely λ blocks. If a $B[v, \kappa, \lambda]$ exists then it is clear that $\sigma(v, \kappa, \lambda) = \lambda v(v-1)/\kappa(\kappa - 1) = \psi(v, \kappa, \lambda)$ and Hanani [16] has proved the following existence theorem for $B[v, 5, \lambda]$.

Theorem 2.1. *Necessary and sufficient conditions for the existence of a $B[v, 5, \lambda]$ are that $\lambda(v-1) \equiv 0$ (mod 4) and $\lambda v(v-1) \equiv 0$ (mod 20) and $(v, \lambda) \ne (15, 2)$.*

Corollary. *(1) $\sigma(v,5,3) = \psi(v,5,3)$ for all positive integers v where $v \equiv 1$ or 5 (mod 20). (2) $\sigma(v,5,6) = \psi(v,5,6)$ for all positive integers v where $v \equiv 1$ or 5 (mod 10).*

A (v, κ, λ) packing design with a hole of size h is a triple (V, H, β) where V is a v-set, H is a subset of V of cardinality h, and β is a collection of κ - element subsets, called blocks, of V such that

1) no 2-subset of H appears in any block;

2) every other 2-subset of V appears in at most λ blocks;

3) $|\beta| = \psi(v,\kappa,\lambda) - \psi(h,\kappa,\lambda)$.

It is clear that if there exists a (v, κ, λ) packing design with a hole of size h and $\sigma(h,\kappa,\lambda) = \psi(h,\kappa,\lambda)$ then $\sigma(v,\kappa,\lambda) = \psi(v,\kappa,\lambda)$.

Let κ, λ and v be positive integers and M be a set of positive integers. A group divisible design $GD[\kappa,\lambda,M,v]$ is a triple (V,β,γ) where V is a set of points with $|V| = v$, and $\gamma = \{G_1,\ldots,G_n\}$ is a partition of V into n sets called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;

2) $|G_i| \in M$ for all $G_i \in \gamma$;

3) every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

If $M = \{m\}$ then the group divisible design is denoted by $GD[\kappa,\lambda,m,v]$.

A $GD[\kappa,\lambda,m,\kappa m]$ is called a transversal design and denoted by $T[\kappa,\lambda,m]$. It is well known that a $T[\kappa,1,m]$ is equivalent to κ -2 mutually orthogonal Latin squares of side m.

In the sequel we shall use the following existence theorem for transversal designs. The proof of this result may be found in [1], [12], [13], [16], [20], [21].

Theorem 2.2. *There exists a $T[6,1,m]$ for all positive integers m with the exception of $m \in \{2,3,4,6\}$ and the possible exception of $m \in \{10,14,18,22,26,30,34,38,42,44\}$.*

Theorem 2.3. *If there exists a $GD[6,6,5,5n]$ and a $(20+h,5,6)$ packing design with a hole of size h then there exists a $(20(n-1)+4u+h,5,6)$ packing design with a hole of size $4u+h$.*

Proof: Take a $GD[6,6,5,5n]$ and delete $5-u$ points from the last group. Inflate this design by a factor of 4. On the blocks of size 5 and 6 construct a

$GD[5,1,4,20]$ and a $GD[5,1,4,24]$ respectively. Add h points to the groups and on the first $n-1$ groups construct a $(20+h,5,6)$ packing design with a hole of size h, and take the h points with the last group to be the hole of size $4u+h$.

It is clear to apply the above theorem we require the existence of a $GD[6,6,5,5n]$. Our authority for that is the following lemma of Hanani [16].

Lemma 2.1. *There exists a $GD[6,6,5,35]$ and a $GD[6,6,5,45]$.*

If in the definition of $GD[\kappa,\lambda,m,v]$ (similarly $T[\kappa,\lambda,m]$) condition 3 is changed to be read as (3) every 2-subset $\{x,y\}$ of V such that x and y are neither in the same group (column) nor in the same row is contained in exactly λ blocks of β. Then the resultant design is called a modified group divisible design (modified transversal design) and is denoted by $MGD[\kappa,\lambda,m,v](MT[\kappa,\lambda,m])$. Notice that this means that a block can contain at most one element from any given row.

A resolvable design is a design of which the blocks can be partitioned into parallel classes. We write $RB, RMGD$ with the appropriate parameters. It is clear that a $RMGD[5,1,5,5m]$ is the same as $RT[5,1,m]$ with one parallel class of blocks singled out, and since $RT[5,1,m]$ is equivalent to $T[6,1,m]$ we have the following.

Theorem 2.4. *There exists a $RMGD[5,1,5,5m]$ for all positive integers m with the exception of $m \in \{2,3,4,6\}$ and the possible exception of $m \in \{10,14,18,22,26,30,34,38,42,44\}$.*

The next two theorems are in the form most useful to us.

Theorem 2.5 [2]. *If there exists a $RMGD[5,1,5,5m]$ and a $GD[5,\lambda,\{4,s^*\},4m+s]$, where $*$ means there is exactly one group of size s, and there exists a $(20+h,5,\lambda)$ packing design with a hole of size h then there exists a $(20m+4u+h+s,5,\lambda)$ packing design with a hole of size $4u+h+s$ where $0 \leq u \leq m-1$.*

It is clear that the application of the above theorem requires the existence of a $GD[5,1,\{4,s^*\},4m+s]$. We observe that we may choose $s=0$ if $m \equiv 1 \pmod 5$; $s=4$ if $m \equiv 0$ or $4 \pmod 5$, and $s=\frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$ (see [2]). We may also apply the following [15].

Theorem 2.6. *There exists a $GD[5,1,\{4,8^*\},4m+8]$ for all positive integers $m \equiv 0$ or $2 \pmod 5$, $m \geq 7$, with the possible exception of $m=10$.*

Theorem 2.7. *If there exists (1) a $RMGD[5,1,5,5m]$ (2) there exists a $GD[5,6,2,2m]$ or a $GD[5,6,2,2(m+1)]$ (3) there exists a $(10+h,5,6)$ packing design with a hole of size h. Then there exists a $(10m+2u+e+h,5,6)$ packing design with a hole of size $2u+e+h$ where $0 \leq u \leq m-1$*

and $e = 0$ *if a* $GD[5,6,2,2m]$ *exists and* $e = 2$ *if a* $GD[5,6,2,2(m+1)]$ *exists.*

Proof: The proof of this theorem is the same as theorem 2.5 of [6].

We close this section with the following notation that will be used later. A block $< \kappa, \kappa+m, \kappa+n, \kappa+j, f(\kappa) >$ mod(v) where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd will be denoted by $< 0, m, n, j > \cup\{a, b\}$. Similarly a block $< (0,\kappa)(0,\kappa+m)(1,\kappa+m)(1,\kappa+j)f(\kappa) >$ mod$(-, v)$ where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd will be denoted by $< (0,0)(0,m)(1,n)(1,j) > \cup\{a,b\}$.

3 The Structure of Packing and Covering Designs

Let (V, β) be a (v, κ, λ) packing design, for each 2-subset $e = \{x, y\}$ of V define $m(e)$ to be the number of blocks in β which contain e. Note that by the definition of a packing design we have $m(e) \leq \lambda$ for all e.

The complement of (V, β), denoted by $C(V, \beta)$ is defined to be the graph with vertex set V and edges e occurring with multiplicity $\lambda - m(e)$ for all e. The number of edges (counting multiplicities) in $C(V, \beta)$ is given by $\lambda\binom{v}{2} - \mid \beta \mid \binom{\kappa}{2}$. The degree of the vertex x in $C(V, \beta)$ is $\lambda(v-1) - r_x(\kappa-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V, β) covering design denoted by $E(V, \beta)$, to be the graph with vertex set V and edges e occurring with multiplicity $m(e) - \lambda$ for all e. The number of edges in $E(V, \beta)$ is given by $|\beta|\binom{\kappa}{2} - \lambda\binom{v}{2}$; and the degree of each vertex is $r_x(\kappa - 1) - \lambda(v - 1)$ where r_x is as before.

Lemma 3.1. *Let* (V, β) *be a* $(v, 5, 4)$ *covering design with* $|\beta| = \phi(v, 5, 4)$ *then the degree of each vertex of* $E(V, \beta)$ *is divisible by 4 and the number of edges in the graph is 0, 6, 8 when* v *(mod 10)* $\in \{1,5\}, \{2,4\}, \{3\}$ *respectively.*

The only graph with 6 edges and each vertex of degree divisible by 4 is the graph consisting of $v-3$ isolated vertices and 3 vertices, each 2 of which are connected by two edges. Therefore when $v \equiv 2$ or 4 (mod 5) a $(v, 5, 4)$ minimal covering design contains a triple $\{a, b, c\}$ each of its pairs appears in precisely 6 blocks and all other pairs appears in 4 blocks.

Lemma 3.2. *Let* v *be odd number and* (V, β) *a* $(v, 5, 2)$ *minimal covering design then the degree of each vertex of* $E(V, \beta)$ *is divisible by 4, and the number of edges in the graph is 0, 4 or 8 when* v *(mod 10)* $\in \{1,5\}\{3\}$ *or* $\{7,9\}, v \neq 9, 15$.

The only graph with 4 edges and every vertex of degree divisible by 4 is the graph with four parallel edges connecting two vertices and $v-2$ isolated vertices. Therefore, when $v \equiv 3$ (mod 10) a $(v, 5, 2)$ minimal covering design contains a pair of points which appears in 6 blocks while each other pair appears in precisely two blocks.

4 Packing of index 3 and order $v \equiv 1 \pmod 4$

In this section we need to consider only the cases $v \equiv 9, 13, 17 \pmod{20}$.

In view of the discussion in Section 3, if $(v, 5, 3)$ is a packing design and $v \equiv 13 \pmod{20}$ and $|\beta| = \psi(v, 5, 3)$ then the degree of $C(V, \beta)$ is divisible by 4 and the total number of edges in $C(V, \beta) = 4$. This means that the graph $C(V, \beta)$ has only 4 parallel edges between two vertices, which is impossible since $\lambda = 3$. Thus:

Lemma 4.1. *For $v \equiv 13 \pmod{20}$, $\sigma(v, 5, 3) \leq \psi(v, 5, 3) - 1$.*

We need two small packings:

Lemma 4.2. $\sigma(9, 5, 3) = \psi(9, 5, 3)$

Proof: Let $X = Z_9$. Then the required blocks are 02367 02478 03456 03578 04568 12348 12368 12467 13457 15678

Lemma 4.3. $\sigma(13, 5, 3) = \psi(13, 5, 3) - 1$

Proof: Let $X = Z_{10} \cup \{A, B, C\}$. Then the required blocks are
02159 0124B 013467 0238A 0356A 057AB 079BC 12678 1359B 137AC 148AB 1568C 169AC 23679 238BC 2457C 245AC 269AB 34589 346BC 4789A 5678B

In [15] it was shown that

Theorem 4.4: [15].

1) *If $v \equiv 13 \pmod{20}$, $v \geq 53$ then there exist a $(v, 5, 1)$ packing design with a hole of size 13.*

2) *If $v \equiv 9$ or $17 \pmod{20}$, $v \geq 37$ and $v \neq 49$, then there exist $(v, 5, 1)$ packing design with a hole of size 9.*

Thus take 3 copies of the packing that exist for $v \equiv 13 \pmod{20}$ and replace the hole (or the block of size 13) by the packing obtained from Lemma 4.3. Similarly for the case of $v \equiv 9$ or $17 \pmod{20}$ we replace the hole (or the blocks of size 9) by the packing obtained from Lemma 4.2.

Thus we have

Corollary 4.5.

1) *If $v \equiv 13 \pmod{20}$ then $\sigma(v, 5, 3) = \psi(v, 5, 3) - 1$ with the possible exception of $v = 33$.*

2) *If $v \equiv 9$ Or $17 \pmod{20}$ then $\sigma(v, 5, 3) = \psi(v, 5, 3)$ with the possible exception of $v = 17, 29, 49$.*

5 Packing of index 6

In this section we distinguish the following cases.

5.1 Packing of order $v \equiv$ 0, 2, 4 or 6 (mod 10)

It is easy to see that $(\sigma, v, 5, 6) = \psi(v, 5, 6)$ for all positive integers $v \equiv 0$, 2 or 6 (mod 10) by simply taking a $(v, 5, 2)$ and a $(v, 5, 4)$ optimal packing design. Furthermore since a (22,5,2) and (22,5,4) packing designs with a hole of size 2 exist [4], [10] it follows that there exists a (22,5,6) packing design with a hole of size 2. For $v \equiv 4$ (mod 10) we treat the values under 100 individually. For this purpose the following lemma is very useful.

Lemma 5.1.1. *Let v be a positive integer, $v \equiv 4$ (mod 10). If there exists a $(v, 5, 2)$ packing design with a hole of size 4 and there exists a $(v, 5, 4)$ minimal covering design then $\sigma(v, 5, 6) = \psi(v, 5, 6)$.*

Proof: The blocks of a $(v, 5, 6)$ optimal packing design may be constructed as follows.

1) take a $(v, 5, 4)$ minimal covering design. This design has a triple, say, $\{a, b, c\}$ such that each pair of this triple appears 6 times and all other pairs appear precisely 4 times.

2) take a $(v, 5, 2)$ packing design with a hole of size 4, say, $\{a, b, c, d\}$. Now it is easy to check that these 2 steps yield a $(v, 5, 6)$ optimal packing design for $v \equiv 4$ (mod 10).

Lemma 5.1.2. *There exists a $(v, 5, 2)$ packing design with a hole of size 4 for $v = 34, 54, 74, 94$.*

Proof: For $v = 34, 54, 74$ see [4]. For $v = 94$ apply Theorem 2.7 with $h = 2$ and $m = 9$. See [6] for the existence of a $GD[5, 2, 6, 20]$, and for a (12,5,6) packing design with a hole of size 2 take a (12,5,2) and (12,5,4) packing design with a hole of size 2 [4], [10].

Corollary. $\sigma(v, 5, 6) = \psi(v, 5, 6)$ *for* $v = 34, 54, 74, 94$.

Proof: For $v = 34, 54, 74, 94$ there exists a $(v, 5, 4)$ minimal covering design such that the pairs of a triple, say, $\{a, b, c\}$ occurs in 6 blocks. On the other hand, for $v = 34, 54, 74, 94$ there exists a $(v, 5, 2)$ packing design with a hole of size 4. Now apply Lemma 5.1.1 to give the result.

Lemma 5.1.3. $\sigma(v, 5, 6) = \psi(v, 5, 6)$ *for* $v = 14$ *and for all positive integers* $v \equiv 4$ *(mod 20).*

Proof: For $v = 14$ let $X = Z_{10}\{A, B, C, D\}$. Then the required blocks are 0128C 0123A 0135D 01356 016AB 01789 02369 02478 0249A 0279D 0345D

038BC 0458A 049BD 0567C 068AB 06ABD 09BCD 1237B 1268D 127BD 129AC 1346C 1358B 14589 146BD 147CD 1479A 14ABC 157AD 1689C 2349B 2346D 245AC 24568 25679 257BC 258AB 25BCD 28ACD 347BC 3478D 359CD 359AB 3679A 367AC 378AC 389AD 4569C 4678B 567AD 5789B 689CD

For $v \equiv 4 \pmod{20}$ proceed as follows.

1. take a $B[v+1, 5, 1]$ design and assume we have the block $\{1\ 2\ 3\ v\ v+1\}$, where $\{1, 2, 3\}$ are arbitrary numbers. In this block change $v+1$ to 5 and in all other blocks change $v+1$ to v.

2. take a $(v, 5, 4)$ optimal packing design, [10]. In this design there is one pair, say, $(5, v)$ that does not appear at all and every other pair appears exactly 4 times. Furthermore assume we have the following two blocks

   $\{4\ 6\ 7\ 8\ v\}$ $\{4\ 7\ 8\ 5\ 1\}$

   where $\{4, 6, 7, 8\}$ are arbitrary numbers. In the first block change v to 1 and in the second block change 1 to v.

3. take a $(v-1, 5, 1)$ optimal packing design, [3]. This design has precisely v missing pairs so without loss of generality we may assume that (2,5) (3,5) and (1,6) are missing pairs in this design.

Now it is easily checked that the above three steps yield a $(v, 5, 6)$ optimal packing design for all $v \equiv 4 \pmod{20}$.

Theorem 5.1.1. *$\sigma(v, 5, 6) = \psi(v, 5, 6)$ for all positive integers v where $v \equiv 2$, 4 or 6 (mod 10).*

Proof: For $v \equiv 2$ or 6 (mod 10) a $(v, 5, 6)$ optimal packing design is obtained by taking a $(v, 5, 4)$ and $(v, 5, 2)$ optimal packing designs. For $v \equiv 4 \pmod{20}$ the result follows from Lemma 5.1.3. For $v = 14, 34, 54, 74, 94$ the result follows from Lemma 5.1.3 and corollary to Lemma 5.1.2. For $v \geq 114, v \neq 134$, simple calculations show that v can be written in the form $v = 20m + 4u + s + h$ where m, u, h and s are chosen so that

1. there exists a $RMGD[5, 1, 5, 5m]$
2. $4u + h + s \equiv 14 \pmod{20}$, $14 \leq 4u + h + s \leq 94$
3. $0 \leq u \leq m - 1, s \equiv 0 \pmod 4$ and $m \not\equiv 8 \pmod{10}$
4. $h = 2$.

Now apply theorem 2.5 to give the result.

For $v = 134$ apply theorem 2.3 with $n = 7, h = 2$ and $u = 3$ gives us a (134,5,6) packing design with a hole of size 14. But $\sigma(14, 5, 6) = \psi(14, 5, 6)$ hence $\sigma(134, 5, 6) = \psi(134, 5, 6)$.

5.2 Packing of order $v \equiv 8 \pmod{10}$

In this section the values below 100 are treated individually.

Lemma 5.2.1. *$\sigma(v,5,6) = \psi(v,5,6)$ for $v = 8, 18, 28, 38, 48, 58, 68, 78, 88, 98$.*

Proof: For $v = 8, 18, 28, 38, 48, 68, 78, 88$ the required constructions are given in the following table. In general, the construction in this table and other tables to come is as follows. Let $X = Z_2 \times Z_{(v-n)/2} \cup H_n$ or $X = Z_{v-n} \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbit of the tabulated base block mod $(v-n)/2$ or mod $(v-n)$ respectively unless it is otherwise specified. For $v = 98$ apply theorem 2.7 with $m = 9, h = 2$ and $u = 2$. For $v = 58$ take a $T[6,1,5]$. Delete a point from last group and inflate the design by a factor of 2 and index 6. Finally, on each group construct a $(v,5,6)$ optimal packing design where $v = 8, 10$.

v	Point Set	Base Blocks
8	Z_8	< 0 1 2 3 4 > < 0 1 3 4 6 >
18	Z_{18}	< 0 1 2 3 4 > < 0 1 4 8 13 > < 0 1 6 10 12 > < 0 2 6 10 13 > < 0 2 7 10 13 >
28	Z_{28}	< 0 1 2 3 8 >, twice < 0 2 6 14 19 > < 0 3 10 14 19 > < 0 3 10 16 20 > < 0 2 6 15 19 > < 0 3 9 17 21 > < 0 3 10 15 20 >
38	Z_{38}	< 0 1 2 4 13 > < 0 3 8 18 24 > < 0 4 9 19 26 > < 0 6 12 19 30 > < 0 1 3 10 17 > < 0 4 13 19 24 > < 0 1 4 12 17 > < 0 1 2 4 8 > < 0 2 12 20 27 > < 0 3 12 23 28 > < 0 5 14 22 28 >
48	$Z_{40} \cup H_8$	$\{0\ 8\ 16\ 24\ 32\} + i, i \in \{0, 1, \ldots, 7\}$, twice < 0 1 2 4 10 > < 0 3 13 17 28 > < 0 5 11 22 27 > < 0 5 12 19 31 > < 0 3 9 13 h_1 > < 0 1 17 19 h_2 > < 0 4 10 22 h_3 > < 0 6 8 20 h_4 > < 0 1 2 9 h_5 > < 0 3 7 18 h_6 > < 0 5 17 26 h_7 > < 0 7 17 30 h_8 >. Let $H_8 = \{a, b, c, d\} \times Z_2$ add the following base blocks, < 0 1 3 8 a_0 > < 0 3 14 27 b_0 > < 0 4 19 25 c_0 > < 0 5 16 25 d_0 >
68	$Z_{60} \cup H_8$	On $Z_{60} \cup H_7$ construct a (67,5,2) packing design with a hole of size 7, [5], where the hole is H_7, and take the following blocks $\{0\ 12\ 24\ 36\ 48\} + i, i \in \{0, 1, \ldots, 11\}$. Three times. < 0 10 30 40 h_8 > half orbit. < 0 1 3 7 41 > < 0 5 18 28 44 > < 0 8 17 35 46 > < 0 1 3 7 44 > < 0 6 14 34 45 > < 0 8 18 33 46 > < 0 1 3 16 h_1 > < 0 4 9 27 h_2 > < 0 5 22 29 h_3 > < 0 9 20 39 h_4 > < 0 1 3 8 h_5 > < 0 4 19 29 h_6 > < 0 9 26 37 h_7 > < 0 12 25 39 h_8 > .
78	$Z_{70} \cup H_8$	On $Z_{70} \cup H_3$ construct a (73,5,2) packing design with a hole of size 3, say, H_3, and take the following blocks. $\{0\ 14\ 28\ 42\ 56\} + i, i \in \{0, 1, \ldots 13\}$, twice < 0 17 35 52 h_4 > half orbit < 0 5 11 21 41 > < 0 2 6 24 32 > < 0 3 10 39 54 > < 0 9 20 32 57 > < 0 1 9 13 28 > < 0 2 18 38 48 > < 0 6 23 37 49 > < 0 1 3 8 h_1 > < 0 4 9 29 h_2 > < 0 7 26 43 h_3 > < 0 11 27 42 h_4 > < 0 12 25 49 h_5 > < 0 1 2 5 h_6 > < 0 3 9 22 h_7 > < 0 7 30 38 h_8 > Let $H_8 = \{a, b, c, d\} \times Z_2$ and add the blocks < 0 10 21 45 c_0 > < 0 14 29 47 d_0 >
88	$Z_{80} \cup H_8$	On $Z_{80} \cup H_7$ construct a (87,5,2) packing design with a hole of size 7, say, H_7, [19] and take the following blocks $\{0\ 16\ 32\ 48\ 64\} + i, i \in \{0, 1, \ldots, 15\}$, Three times. < 0 12 40 52 h_8 > half orbit. < 0 1 3 7 24 > < 0 8 16 43 54 > < 0 12 27 41 60 > < 0 2 6 11 24 > < 0 7 15 28 50 > < 0 10 27 41 61 > < 0 1 5 23 47 > < 0 2 12 29 38 > < 0 6 20 44 55 > < 0 1 3 15 23 > < 0 3 31 50 h_1 > < 0 4 25 41 h_2 > < 0 1 6 9 h_3 > < 0 5 15 40 h_4 > < 0 7 26 59 h_5 > < 0 7 37 46 h_6 > < 0 11 29 42 h_7 > < 0 13 30 57 h_8 >

Theorem 5.2.1. *For all positive integers $v \equiv 8 \pmod{10}$, $\sigma(v,5,6) = \psi(v,5,6)$.*

Proof: For $8 \leq v \leq 98$ the result follows from Lemma 5.2.1. For $v \geq 108, v \neq 128$, 138, 178 simple calculations show that v can be written in the form $v = 20m + 4u + s + h$ where m, u, h and s are chosen so that:

1. there exists a $RMGD[5,1,5,5m]$

2. $4u+h+s \equiv 8 \pmod{10}$, $8 \leq 4u+s+h \leq 98$

3. $0 \leq u \leq m-1, s \equiv 0 \pmod 4$ and $m \not\equiv 8 \pmod{10}$

4. $h=2$.

Now apply Theorem 2.5 to give the result.

For $v = 128$ apply Theorem 2.3 with $n = 7, h = 0$, and $u = 2$. For $v = 138, 178$ apply Theorem 2.3 with $h = 2, u = 4, n = 7$ and 9 respectively gives us a (138,5,6) and (178,5,6) packing design with a hole of size 18. But $\sigma(18,5,6) = \psi(18,5,6)$ hence $\sigma(138,5,6) = \psi(138,5,6)$ and $\sigma(178,5,6) = \psi(178,5,6)$.

5.3 Packing of order $v \equiv 7$ or 9 (mod 10)

It is easy to see that if there exists a $(v,5,2)$ packing design with a hole of size h and we take three copies of this design then the resultant design is a $(v,5,6)$ packing design with a hole of size h.

Lemma 5.3.1. *(1) For all positive integer $v \geq 29, v \equiv 7$ or 9 (mod 10) $v \neq 147$ there exist a $(v,5,2)$ packing design with a hole of size h where $h = 7$ or 9. (2) There exist a (147,5,2) packing design with a hole of size 27.*

Proof: For all such v, $v \neq 137$, 139, 147 the result is given in [5]. For $v = 137$, 139, 147 see [2].

Lemma 5.3.2. *(1) $\sigma(v,5,6) = \psi(v,5,6)$ for $v = 7,9,17,19,27$. (2) There exists a (23,5,6) packing design with a hole of size 3.*

Proof: For a (23,5,6) packing design with a hole of size 3 take 3 copies of a (23,5,2) packing design with a hole of size 3. For $v = 7,9,19,27$ the constructions are given in the next table. For $v = 17$ let $X = Z_{10} \cup \{A,B,C,D,E,F,G\}$. Then the blocks are

012BG 0123C 0134C 0169B 017BE 018AD 0245G 02478 02579 026AC 035DG 0369E 037AC 039BF 0456E 0456C 04EFG 059AF 0678C 078DF 089DG 08ABF 0ADEG 0BDEF 1239F 1236G 128DF 129AG 1358A 13CDE 1457F 1467B 1479E 1489E 149CD 1568F 156DG 157AD 15BCE 168AF 17BEG 1ACFG 2348B 235EF 235BC 247CD 247DF 24ABD 2568F 257AE 267AE 269DG 26ABE 289DE 28CEG 29BCF 345EF 346AG 34689 346AD 358DE 3678B 378BG 379CD 379FG 37AFG 3ABDE 45BDG 489AB 48ACG 49AEF 4BCFG 569BD 578BG 579AC 589CG 59ABC 679EG 67CDF 6BCDF 6CEFG

v	Point Set	Base Blocks
7	$Z_4 \cup H_3$	$< 0\ 1\ 2\ h_1\ h_2 >$ $< 0\ 1\ 2\ h_1\ h_3 >$ $< 0\ 1\ 2\ h_2\ h_3 >$
9	$Z_2 \times Z_3 \cup H_3$	$< (0,0)(0,1)(1,1)h_1h_2 >< (0,0)(1,0)(1,1)h_1h_2 >< (0,0)(0,1)(1,0)h_1h_3 >$ $< (0,0)(1,1)(1,2)h_1h_3 >< (0,0)(0,1)(1,2)h_2h_3 >< (0,0)(1,0)(1,2)h_2h_3 >$ $< (0,0)(0,1)(0,2)(1,0)(1,1) >$
19	$Z_2 \times Z_8 \cup H_3$	On $\{0\} \times Z_8 \cup H_3$ construct a B[11,5,2]. On $\{1\} \times Z_8 \cup H_3$ construct a B[11,5,2] and take the following blocks $< (0,0)(0,1)(0,4)(1,0)(1,1) >$ $< (0,0)(0,1)(0,3)(1,0)(1,1) >$ $< (0,0)(0,2)(1,0)(1,1)(1,3) >$ $< (0,0)(0,2)(1,0)(1,3)(1,5) >$ $< (0,0)(0,4)(1,2)(1,6)h_1 >$ $< (0,0)(0,1)(1,2)(1,4)h_1 >$ $< (0,0)(0,1)(1,4)(1,5)h_2 >$ $< (0,0)(0,2)(1,4)(1,7)h_2 >$ $< (0,0)(0,3)(1,2)(1,6)h_3 >$ $< (0,0)(0,3)(1,5)(1,7)h_3 >$
27	$Z_2 \times Z_{12} \cup H_3$	$\{(0,0)(0,4)(0,8)h_1h_3\} + (0,i), i \in Z_4$ $\{(0,0)(0,4)(0,8)h_2h_3\} + (0,i), i \in Z_4$ $\{(0,0)(0,4)(0,8)h_1h_2\} + (0,i), i \in Z_4$ $< (0,0)(0,1)(0,2)(0,7)(1,0) >$ $< (0,0)(0,1)(0,5)(1,0)(1,1) >$ $< (0,0)(0,1)(1,0)(1,2)(1,4) >$ $< (0,0)(0,2)(0,3)(1,5)(1,9) >$ $< (0,0)(0,3)(1,0)(1,1)(1,2) >$ $< (0,0)(0,2)(0,10)(1,4)(1,7) >$ $< (0,0)(0,4)(1,4)(1,8)(1,11) >$ $< (0,0)(0,2)(0,5)(1,3)(1,8) >$ $< (0,0)(0,5)(1,2)(1,6)(1,8) >$ $< (0,0)(0,3)(0,6)(0,9)h_1 > +(-,i), i \in Z_3$, twice $< (0,0)(0,1)(1,5)(1,10)h_1 >$ $< (1,0)(1,1)(1,5)(1,7)h_1 >$ $< (0,0)(0,2)(1,7)(1,10)h_2 >$ $< (0,0)(0,3)(1,6)(1,9)h_2 >$ $< (0,0)(0,5)(1,1)(1,7)h_3 >$ $< (0,0)(0,6)(1,4)(1,11)h_3 >$ Let $\{h_2, h_3\} = \{a\} \times Z_2$, and add $< (1,0)(1,1)(1,3)(1,4)a_0 >$.

Theorem 5.3.1. $\sigma(v,5,6) = \psi(v,5,6)$ *for all positive integers* $v, v \equiv 7$ *or 9 (mod 10).*

Proof: For $v = 7, 9, 17, 19, 27$ the result follows from Lemma 5.3.2. For $v \geq 29$ we have shown, Lemma 5.3.1, that there exists a $(v,5,2)$ packing design with a hole of size 7, 9 or 27 and hence a $(v,5,6)$ packing design with a hole of size 7, 9 or 27. But $\sigma(v,5,6) = \psi(v,5,6)$ for $v = 7, 9$ or 27 hence $\sigma(v,5,6) = \psi(v,5,6)$ for $v \geq 29$.

5.4 Packing of order $v \equiv 3 \pmod{10}$

In this section we combine different designs to obtain our result.

Theorem 5.4.1. $\sigma(v,5,6) = \psi(v,5,6)$ *for all positive integers* v, $v \equiv 3$ *(mod 10) with the possible exceptions of* $v = 43$.

Proof: For $v \neq 13, 43, 53, 63, 73, 83$ the construction is as follows.

1. take a $(v,5,2)$ minimal covering design [18]. Such design exists for all $v \equiv 3 \pmod{10}$, $v \neq 13$, with the possible exception of $v = 43, 53, 63, 73, 83$. Furthermore this design has one pair, say $\{a,b\}$ that appears 6 times while each other pair appears exactly twice.

2. take two copies of a $(v,5,2)$ optimal packing design [4], [6]. This design has a triple, say, $\{a,b,c\}$ that its pairs do not appear in any block while each other pair appears exactly twice.

Now it is easily checked that the above two steps yield an optimal packing design for $v \equiv 3 \pmod{10}$, $v \neq 13, 43, 53, 63, 73, 83$.

For $v = 13$ let $X = \{1, 2, \dots, 13\}$ then the blocks are

{1 2 3 5 11} {1 2 3 11 12} {1 2 6 9 13} {1 2 8 10 12} {1 2 8 11 13}
{1 2 9 10 11} {1 3 4 9 12} {1 3 5 8 13} {1 3 7 9 13} {1 3 7 8 9}
{1 4 5 9 10} {1 4 5 7 12} {1 4 8 10 13} {1 5 6 7 10} {1 5 6 7 12}
{1 6 8 10 11} {1 7 11 12 13} {2 3 4 6 13} {2 3 4 7 13} {2 3 5 9 10}
{2 3 7 8 10} {2 4 5 6 12} {2 4 5 8 9} {2 5 7 9 13} {2 5 10 12 13}
{2 6 7 8 11} {2 7 8 10 12} {2 7 9 11 12} {3 4 5 8 11} {3 4 6 10 11}
{3 4 7 8 12} {3 5 6 10 12} {3 5 7 10 11} {3 6 8 9 12} {3 6 9 10 13}
{3 6 11 12 13} {4 5 8 11 13} {4 6 7 10 13} {4 6 7 9 11} {4 6 8 9 12}
{4 7 9 10 11} {4 10 11 12 13} {5 6 7 8 13} {5 6 8 9 11} {5 9 11 12 13}
{8 9 10 12 13}

For $v = 53, 63, 73, 83$ we show that there exists a $(v, 5, 6)$ packing design with a hole of size 13 and since $\sigma(13, 5, 6) = \psi(13, 5, 6)$ it follows that $\sigma(v, 5, 6) = \psi(v, 5, 6)$ for $v = 53, 63, 73, 83$.

For $v = 53$ take $RB[40, 4, 1]$, [17]. There are 13 parallel classes, to each parallel class add a new point. The resultant design is a (53,5,1) packing design with a hole of size 13. Take six copies of this design gives us a (53,5,6) packing design with a hole of size 13.

For $v = 73$ take six copies of a (73,5,1) packing design with a hole of size 13, [15].

For $v = 63$, 83 see the next table. In this table we use a $B[61, 5, 2]$ with a hole of size 11. Such design may be constructed by taking a $T[6, 1, 5]$, inflate this design by a factor of 2. On the blocks construct a $GD[5, 2, 2, 12]$, [16]. Add a point to the groups and on the first five groups construct a $B[11, 5, 2]$ and take the last group with the point to be the hole.

v	Point Set	Base Blocks
63	$Z_{50} \cup H_{13}$	On $Z_{50} \cup H_{11}$, construct a B[61,5,2] with a hole of size 11, say H_{11} and take the following blocks. $\{0\ 10\ 20\ 30\ 40\} + i, i \in \{0, 1, \dots, 9\}$, twice < 0 2 8 24 36 > < 0 4 6 18 h_1 > < 0 5 13 37 h_2 > < 0 3 7 27 h_3 > < 0 1 2 5 h_4 > < 0 3 8 25 h_5 > < 0 6 19 33 h_6 > < 0 7 19 34 h_7 > < 0 9 18 33 h_8 > < 0 10 21 39 h_9 > < 0 1 2 7 h_{10} > < 0 3 11 20 h_{11} > < 0 4 19 31 h_{12} > < 0 7 16 29 h_{13} > Let $\{h_{12}, h_{13}\} = \{a\} \times Z_2$, then add the block < 0 10 21 35 a_0 >
83	$Z_{70} \cup H_{13}$	On $Z_{70} \cup \{h_1\}$ construct a B[71,5,2] and take the following blocks. $\{0\ 14\ 28\ 42\ 56\} + i, i \in \{0, 1, \dots, 13\}$, twice < 0 2 6 32 44 > < 0 8 18 30 54 > < 0 4 20 38 h_1 > < 0 3 25 33 h_2 > < 0 5 11 31 h_3 > < 0 1 3 8 h_4 > < 0 4 13 23 h_5 > < 0 6 19 55 h_6 > < 0 10 25 43 h_7 > < 0 11 27 50 h_8 > < 0 12 29 53 h_9 > < 0 1 3 10 h_{10} > < 0 4 9 29 h_{11} > < 0 6 21 39 h_{12} > < 0 11 28 47 h_{13} > Let $\{h_2, h_3, \dots, h_{13}\} = \{a, b, c, d, e, f\} \times Z_2$ and add the blocks < 0 13 27 48 a_0 > < 0 1 2 9 b_0 > < 0 3 15 26 c_0 > < 0 5 30 43 d_0 > < 0 7 29 46 e_0 > < 0 14 33 49 f_0 > .

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