Fractal Sequences and Interspersions

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1. Introduction.

Items from several sources are to be ordered along a line. Source 1 provides $u_2 - 1$ items before source 2 emerges with item $\#u_2$. After that, the items from source 2 are required to alternate with those from source 1. After $u_3 - 1$ items are ordered, source 3 emerges with item $\#u_3$, and thereafter, its items must be interspersed with the first two kinds. That is, between each item from each emergent source must be an item from each of the other two sources, as typified by the following ordering of 1's, 2's, and 3's:

$$1\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ 3\ 2\ 1\ 3\ 2\ 1\ 3 \qquad (1)$$

Here "1" occupies places numbered 1, 2, 3, 4, 6, 8, 11, 14; "2" occupies places 5, 7, 10, 13, and so on. Moreover, $u_2 = 5$ and $u_3 = 9$, as indicated by the following representation for the same ordering:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 6 & 8 & 11 & 14 \\ 5 & 7 & 10 & 13 & & & & \\ 9 & 12 & 15 & & & & & \end{array} \qquad (2)$$

Row 1 gives position numbers of items from source 1, row 2 those of items from source 2, and so on. The scheme now continues inductively: after $u_n - 1$ items are ordered, source n emerges with item $\#u_n$, and thereafter, the items from all n sources must be interspersed.

Such extensions of beginnings like (1) and (2) we call *fractal sequences* and *interspersions*, respectively. The term "fractal sequence" is chosen because the characteristic self-similarity property of fractals is manifest in the Upper and Lower Self-Similarity Theorems proved in Section 3 for fractal sequences. The term "interspersion" is introduced in [3]; the definition is given in Section 2.

Let N denote the set of positive integers. Except where stated otherwise, the sequences we consider consist solely of numbers in N, and the letters

$h, i, j, k, m, n, p, q, r$ designate elements of N, always arbitrary unless otherwise stated.

Definitions. A sequence $x = (x_n)$ is an *infinitive sequence* if for every i,

(F1) $x_n = i$ for infinitely many n; let $a(i, j)$ be the jth index n for which $x_n = i$.

As i and j range through N, the numbers $a(i, j)$ range through all of N. The array $A = \{a(i, j)\}$ is the *associated array* of x. An infinitive sequence x is a *fractal sequence* if two conditions hold:

(F2) if $i + 1 = x_n$, then there exists $m < n$ such that $i = x_m$;
(F3) if $h < i$ then for every j there is exactly one* k such that $a(i, j) < a(h, k) < a(i, j + 1)$.

According to (F2), the first occurrence of each $i \geq 2$ in x must be preceded at least once by each of the numbers $1, 2, \ldots, i - 1$, and according to (F3), between consecutive occurrences of i in x, each h less than i occurs exactly once. Next we combine (F1)-(F3) to obtain a stronger form of (F3).

Theorem 1. *Let x be a fractal sequence, A the associated array, and $r = a(i, j + 1) - a(i, j)$. Then*

(F3)′ *if $1 \leq h \leq r$, there is exactly one k such that* $a(i, j) < a(h, k) \leq a(i, j + 1)$.

Proof: Formally, the set $T = \{x_{a(i,j)+1},\ x_{a(i,j)+2}, \ldots,\ x_{a(i,j+1)}\}$ contains r elements. Were they not distinct, some p would appear twice in T, and by (F3), $p > i$. Write

$$i \quad p \quad p \quad i$$

to indicate that p occurs twice between the consecutive occurrences of i in the sequence x. (There may be other numbers in x separating these i's and p's.) By (F3), another occurrence of i must come between the two occurrences of p. This contradicts the consecutiveness of the two i's that are shown. Therefore,

*The quantifier "exactly one" means "one and only one." The price we pay for this well-established convenience is that one is not exactly one; e.g., if you hold up two fingers, the statement, "You are holding up one finger," is true, but the statement, "You are holding up exactly one finger," is false.

the r elements of T are distinct. Now suppose some number q in the set $S = \{1, 2, \ldots, r\}$ is missing from T.

Case 1: there exists p in T such that $p > q$. Here we write

$$q \quad i \quad p \quad i \quad q$$

to indicate that q must precede p, by (F2), and must precede the jth i since $q \notin T$, and also to indicate the first occurrence of q after the $(j+1)$st i. This arrangement violates (F3), since $i < q$.

Case 2: $p \leq q$ for every p in T. In this case T consists of r positive integers less than q, and since $q \leq r$, the elements of T are not distinct. However, this was already proved impossible.

We conclude that $T = S$, and this proves (F3)′. ■

2. Interspersions.

Fractal sequences are closely related to interspersions – closely enough that many properties of fractal sequences are easily provable from properties already known to hold for interspersions. In this section we present the basic relationships, and in the next, we present proofs of self-similarity properties of fractal sequences.

Definition. An array $A = A(i, j)$ of positive integers is an *interspersion* if

- (I1) every positive integer occurs exactly once in A;
- (I2) every row of A is an increasing sequence;
- (I3) every column of A is an increasing sequence;
- (I4) if (u_j) and (v_j) are distinct rows of A, and if i and h are any indices for which $u_i < v_h < u_{i+1}$, then $u_{i+1} < v_{h+1} < u_{i+2}$.

Theorem 2. *If x is a fractal sequence, then the associated array A is an interspersion. Conversely, if A is an interspersion, then the sequence $x = (x_n)$ given by*

$$x_n = \text{the number } i \text{ such that } n = a(i, j) \qquad (3)$$

for some j is a fractal sequence.

Proof: First, suppose x is a fractal sequence. Clearly (F1) implies (I1) and (I2). By (F2), the first occurrence in x of i precedes that of $i + 1$. That is,

$a(i,1) < a(i+1,1)$, so that column 1 of A is increasing. Suppose for arbitrary j that column j is increasing. Suppose further that column $j+1$ is not increasing, so that

$$a(i+1,j+1) < a(i,j+1) \tag{4}$$

for some i. Now $a(i,j) < a(i+1,j)$ since column j is increasing, and $a(i+1,j) < a(i+1,j+1)$ since row $i+1$ is increasing. These inequalities and (4) give

$$a(i,j) < a(i+1,j+1) < a(i,j+1). \tag{5}$$

Next, $a(i+1,j) < a(i+1,j+1) < a(i,j+1)$, and also $a(i,j) < a(i+1,j)$, so that

$$a(i,j) < a(i+1,j) < a(i,j+1). \tag{6}$$

Inequalities (5) and (6) show two members of row $i+1$ lying between $a(i,j)$ and $a(i,j+1)$. This is impossible, by Theorem 1. Therefore, column $j+1$ is increasing, and by induction, (I3) holds.

Suppose next that

$$a(i,j) < a(h,k) < a(i,j+1). \tag{7}$$

That is, h occurs in x between the jth occurrence of i and the $(j+1)$st occurence of i. If h does not also occur between the $(j+1)$st and $(j+2)$nd occurrences of i, then x contains the arrangement

$$i \quad h \quad i \quad i \quad h, \tag{8}$$

the last h representing the first occurrence of h after the $(j+2)$nd i. But this arrangement violates (F3)$'$, so that the number of numbers between $a(i,j+1)$ and $a(i,j+2)$ is at least as great as the number of numbers between $a(i,j)$ and $a(i,j+1)$. That is,

$$a(i,j+1) - a(i,j) \leq a(i,j+2) - a(i,j+1) \tag{9}$$

Inequalities (7) now imply $1 \leq h \leq a(i,j+1) - a(i,j)$, by Theorem 1, so that

$$1 \leq h \leq a(i,j+2) - a(i,j+1),$$

by (9). By Theorem 2, there exists k' such that

$$a(i,j+1) < a(h,k') < a(i,j+2).$$

Clearly $k' \geq k+1$. If $k' \geq k+2$, then arrangement (8) would occur in x, a contradiction. Therefore $k' = k+1$, so that (I4) holds, and A is an interspersion.

For the converse, suppose A is an interspersion and x is given by (3). By (I2), there are infinitely many numbers n in row i, so that $x_n = i$ for infinitely many n. Since $a(i,j)$ in (3) means the jth number in row i, property (F1) holds.

By (I3), we have $a(i,1) < a(i+1,1)$, which is to say that the first occurrence of i in x precedes that of $i+1$, so that (F2) holds.

By Lemma 1 of [**3**], each of the numbers $a(i,j)+1, a(i,j)+2, \ldots, a(i,j+1)$ lies in exactly one row of A numbered from 1 up to the number $r = a(i,j+1) - a(i,j)$. By Lemma 2 of [**3**], r is the number of terms of the first column of A which are $\leq a(i,j+1)$. Therefore, $i \leq r$, so that if $h < i$, then exactly one k satisfies $a(i,j) < a(h,k) < a(i,j+1)$. ■

In the wake of Theorem 2, we may speak of the *associated interspersion* of any fractal sequence, and the *associated fractal sequence* of any interspersion.

3. Self-similarity of a fractal sequence.

Definition. Suppose $x = (x_n)$ is an infinitive sequence. The *upper-trimmed subsequence* of x is the sequence $\Lambda(x)$ obtained from x by deleting the first occurrence of n, for each n. More precisely, if the positive integers $\{a(i,j)\}$ are written in increasing order, then $\Lambda(x)$ is the sequence (λ_k), where λ_k is the kth number $x_{a(i,j)}$ such that $i \geq 1$ and $j \geq 2$.

Example. First, we display the first few terms of an interspersion known as the Wythoff array (see [5]):

1	2	3	5	8	13	21	34	55	89	144
4	7	11	18	29	47	76	123	199	322	521
6	10	16	26	42	68	110	178	288	466	754
9	15	24	39	63	102	165	267	432	699	1131
12	20	32	52	84	136	220	356	576	932	1508
14	23	37	60	97	157	254	411	665	1076	1741
17	28	45	73	118	191	309	500	809	1309	2118
19	31	50	81	131	212	343	555	898	1453	2351
22	36	58	94	152	246	398	644	1042	1686	2728

It is now easy to tell the first several terms of the associated fractal sequence; the first occurrence of each n is marked:

$$\overline{1}\ 1\ 1\ \overline{2}\ 1\ \overline{3}\ 2\ 1\ \overline{4}\ 3\ 2\ \overline{5}\ 1\ \overline{6}\ 4\ 3\ \overline{7}\ 2\ \overline{8}\ 5\ 1\ \overline{9}\ 6\ 4\ \overline{10}\ 3 \qquad (10)$$

The marked terms are deleted, and the remaining terms comprise the upper-trimmed subsequence:

$$1\ 1\ 1\ 2\ 1\ 3\ 2\ 1\ 4\ 3\ 2\ 5\ 1\ 6\ 4\ 3 \ldots$$

Theorem 3 (Upper Self-Similarity Theorem). *If x is a fractal sequence, then $\Lambda(x) = x$.*

Proof: Suppose x is a fractal sequence, with associated interspersion A. The removal of the first occurrence of n in x, for every n, corresponds to the removal of the first column of A, leaving an array $\widehat{A}$ with terms given by $\widehat{a}(i,j) = a(i,j+1)$. Let $\widetilde{A}$ be the array given by $\widetilde{a}\,(i,j) =$ the number of terms of $\widehat{A}$ which are $\leq a(i,j+1)$. According to Theorem 1.1 of [4], we have $\widetilde{A} = A$. Clearly, the fractal sequence associated with $\widetilde{A}$ is $\Lambda(x)$, so that $\Lambda(x) = x$. ■

The sequence $1,3,2,1,4,3,2,1,5,4,3,2,1,\ldots$ is an infinitive sequence satisfying $\Lambda(x) = x$, but it is not a fractal sequence. However, it is easy to state and prove a partial converse for Theorem 3.

Theorem 4. *If x is an infinitive sequence satisfying* (F2), *and $\Lambda(x) = x$, then x is a fractal sequence.*

Proof: By hypothesis, x satisfies (F1) and (F2). Suppose x fails to satisfy (F3). Then for some p and q, there are two occurrences of q in x with no p between them. Let i be the least q for which such a p exists, and let h be the least p for this choice of q.

Case 1: $h < i$. Schematically, x contains one of two arrangements:

Arrangement 1: $h \quad i \quad i$

Here, the initial h represents the first occurrence of h in x; the first i represents the first occurrence of i in x; the second i represents the next i in x, and there is no h between these two i's. For this arrangement, $\Lambda(x)$ would have its first i before its first h, so $\Lambda(x)$ could not equal x.

Arrangement 2: $h \quad i \quad \ldots h \quad i \quad h \quad i \quad i$

Here the h's represent 1st, $(j-1)$st, and jth occurrences, and the i's represent 1st, $(j-1)$st, jth, and $(j+1)$st occurrences. After deleting the first occur-rence of each n, we see that $\Lambda(x)$ must contain the arrangement

$$\ldots h \quad i \quad h \quad i \quad i,$$

indicating that there is no h between the $(j-1)$st i and the jth i. Since this is not true about x, we have $\Lambda(x) \neq x$.

Arrangement 3: $h \quad h \quad \ldots h \quad i \quad i$

Here, j h's before the first i, in x, yield $j-1$ h's in $\Lambda(x)$ before the first i, so that $\Lambda(x) \neq x$.

Arrangement 4:

$$h \quad h \quad \ldots h \quad i \quad h \quad i \quad h \quad \ldots i \quad h \quad i \quad i$$

Here x has initially j h's, then k pairs i h, and then two i's not separated by an h. Consequently, $\Lambda(x)$ has initially $j-1$ h's followed by a jth h, followed by only $k-1$ pairs i h, followed by two i's that are not separated by an h. Again, $\Lambda(x) \neq x$.

Case 2: $h > i$. The proof here is similar to that for Case 1 and is omitted.

Since both cases lead to contradictions, we conclude that (F3) holds, so that x is a fractal sequence. ∎

Definition. Suppose $x = (x_n)$ is an infinitive sequence. The *lower-trimmed sequence* of x is the sequence $V(x)$ obtained by subtracting 1 from each x_n and then removing all 0's. Explicitly, for each n, let $\ell(n)$ be the least number ℓ such that the number of j satisfying $j \leq \ell$ and $x_j > 1$ is n; then $V(x)$ is the sequence (v_n) given by $v_n = x_{\ell(n)} - 1$.

Example. Let w be the Wythoff fractal sequence of (10). Subtracting 1 and removing 0's leaves $V(x)$, beginning with

1 2 1 3 2 1 4 5 3 2 6 1 7 4 8 5 3 9 2 10 6 1 11 7 4 12 13 8 5

Theorem 5 (Lower Self-Similarity Theorem). *If x is a fractal sequence, then $V(x)$ is a fractal sequence.*

Proof: Suppose x is a fractal sequence and $i \geq 1$. By (F1), the number $i+1$ occurs infinitely many times in x, so that i occurs infinitely many times in $V(x)$. Clearly (F2) for x implies (F2) for $V(x)$. Now by (F3), if $2 \leq h < i+1$, then between each pair of consecutive occurrences of $i+1$ in x there is exactly one occurrence of h. Thus, between each pair of consecutive occurrences of i in $V(x)$, there is exactly one occurrence of $h-1$. ∎

Theorem 5 can be stated in terms of an interspersion, as follows: *if the first row of an interspersion is deleted and each remaining term a is replaced by $a-t$, where t is the number of deleted terms which are $< a$, then the resulting array is an interspersion.* In Theorem 7, we show a large class of fractal sequences x for which $V(x)$ is not only a fractal sequence, but is the sequence x itself. Each such sequence, of course, has an associated interspersion which remains invariant under the first-row deletion operation just described.

4. Signature sequences.

Definition. For any irrational number θ, let $S(\theta)=\{c+d\theta\colon\ c \in N,\ d \in N\}$, and let $c_n(\theta)+d_n(\theta)\theta$ be the sequence obtained by arranging the elements of $S(\theta)$ in increasing order. A sequence x is a *signature sequence* if there exists a positive irrational number θ such that $x=(c_n(\theta))$. In this case, x is the *signature of θ*.

Theorem 6. *Let θ be a positive irrational number. The signature of θ is a fractal sequence.*

Proof: The signature (c_n) of θ is defined from the ordering of the set $S(\theta)$:

$$c_1+d_1\theta \ < \ c_2+d_2\theta \ < \ c_3+d_3\theta \ < \cdots \tag{11}$$

Obviously (F1) and (F2) hold. Let $a(i,j)$ be the jth index n for which $c_n=i$. Suppose $h<i$, and for arbitrary j write $c+d\theta$ for $c_{a(i,j)}+d_{a(i,j)}\theta$. Let $k=[\frac{c-h}{\theta}]+d+1$. Then k is the only integer satisfying

$$\frac{c-h}{\theta}+d<k<\frac{c-h}{\theta}+d+1,$$

or equivalently,

$$c+d\theta<h+k\theta<c+(d+1)\theta.$$

Thus, there is exactly one k such that $a(i,j)<a(h,k)<a(i,j+1)$. ∎

Theorem 7. *If x is a signature sequence, then $V(x) = x$.*

Proof: Suppose x is the signature sequence of a positive irrational number θ. Arrange the set $S(\theta)$ in the natural order (11). When 1 is subtracted from each term and all terms of the form $0 + d\theta$ are removed, the remaining sequence is still in natural order. Moreover, all the terms are of the form $c_n - 1 + d_n\theta$, for $c_n \geq 2$, $d_n \geq 1$, or equivalently, $c_n + d_n\theta$, for $c_n \geq 1$, $d_n \geq 1$, as in (11). Therefore, $V(x) = x$. ∎

Theorem 8. *Suppose x is the signature sequence of θ. Let $a(1, j)$ denote the position of the jth occurrence of 1 in x. Then*

$$a(1, j+1) - a(1, j) = 1 + [j\theta]. \tag{12}$$

Proof: On one hand, the number of terms of x positioned between the jth 1 and the $(j+1)$st 1 is clearly $a(1, j+1) - a(1, j) - 1$. Now let us count the number of $a + b\theta$ satisfying

$$1 + j\theta < a + b\theta < 1 + (j+1)\theta.$$

Equivalently,

$$j - \frac{a-1}{\theta} < b < j + 1 - \frac{a-1}{\theta},$$

so that $b = [j + 1 - \frac{a-1}{\theta}]$, and the values of $a > 1$ for which $b \in N$ are $2, 3, \ldots, [j\theta + 1]$. So, there are $[j\theta]$ numbers $a + b\theta$ of the kind being counted, and (12) follows. ∎

Theorem 9. *Suppose x is the signature sequence of θ. Let A be the associated interspersion. Then*

$$a(1, j) = j + [\theta] + [2\theta] + \cdots + [(j-1)\theta] \tag{13}$$

and

$$a(i, 1) = i + [\frac{1}{\theta}] + [\frac{2}{\theta}] + \cdots + [\frac{i-1}{\theta}]. \tag{14}$$

Moreover, $\theta = 2 \lim_{j \to \infty} \frac{a(1,j)}{j^2} = \frac{1}{2} \lim_{i \to \infty} \frac{i^2}{a(i,1)}$.

Proof: Since $a(1,j)$ is the position of the jth occurrence of 1 in x, (13) follows inductively from (12), since $a(1,1) = 1$. The interspersion A' associated with $1/\theta$ is the transpose of A, since the order of the numbers in (11) remains unchanged when they are all divided by θ. Thus, $a'(i,j) = a(j,i)$, and in particular, column 1 of A is row 1 of A'. By Theorem 8,

$$a'(1,j+1) - a'(1,j) = 1 + \left[\tfrac{j}{\theta}\right],$$

so that $a(i+1,1) - a(i,1) = 1 + \left[\frac{i}{\theta}\right]$, and (14) follows by induction.

Equation (13) yields

$$a(1,j) = j + \theta - \epsilon_1 + 2\theta - \epsilon_2 + \cdots + (j-1)\theta - \epsilon_{j-1}$$

$$= j + \frac{(j^2-j)\theta}{2} - Q,$$

where $Q = \sum_{k=1}^{j-1} \epsilon_k$, $0 < \epsilon_k < 1$, and $k = 1, 2, \ldots, j-1$.

From this easily follows $\theta = 2 \lim\limits_{j\to\infty} \frac{a(1,j)}{j^2}$. A proof that $\theta = \frac{1}{2} \lim\limits_{i\to\infty} \frac{i^2}{a(i,1)}$ is obtained in the same way from (14). ■

5. Beatty sequences and Graham's test.

How can one tell from a given fractal sequence whether it is a signature sequence? The condition $V(x) = x$ is not enough. For example, the sequence

$$1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, 6, 5, 4, 3, 2, 1, \ \ldots$$

satisfies $V(x) = x$ but is not a signature sequence. To see this, note that A is given by

$$a(i,j) = j + \frac{(i+j-1)(i+j-2)}{2},$$

so that $\lim\limits_{i\to\infty} \frac{i^2}{a(i,1)} = 2$, and by Theorem 9, x is not a signature sequence.

The condition $V(x) = x$ nevertheless leads one to try to "self-generate" (cf. [6, page 10]) fractal sequences that satisfy $V(x) = x$ in general and are signature sequences in particular. Consider, for example, extending the initial run

$$1, 2, 1$$

in accord with the restriction $V(x) = x$. We use an arrow ($\rightarrow$) to indicate an extension. Initially,

$$1, 2, 1 \rightarrow 1, 2, 1, 3 \rightarrow 1, 2, 1, 3, 2$$

this last run being necessary in order that the first three terms of $V(x)$ be $1, 2, 1$. Now, there are two possible numbers to choose from to follow the 2 in the fifth position, as indicated here:

ext. 1: 1 2 1 3 2 $\rightarrow$ 1 2 1 3 2 1 $\rightarrow$ 1 2 1 3 2 1 4 3 2 (then 1 5 or 5 1)

ext. 2: 1 2 1 3 2 $\rightarrow$ 1 2 1 3 2 4 $\rightarrow$ 1 2 1 3 2 4 1 3 5 2 4 (then 1 6 or 6 1)

Let us examine possible extensions of extension 1:

ext. 1.1: 1 2 1 3 2 1 4 3 2 $\rightarrow$ 1 2 1 3 2 1 4 3 2 1 5 4 3 2 (then 1 6 or 6 1)

ext. 1.2: 121321432 $\rightarrow$ 121321432$\widehat{51}$43625$\widehat{1}$43625$\widehat{1}$473625 (then 18 or 81)

These examples show that when constructing x so that $V(x) = x$ (for the portion of the sequence that has been constructed), a choice is sometimes necessary when placing the number 1. The opportunity to place 1 occurs each time a set of the form $\{1, 2, \ldots, h\}$ has been assigned consecutive places in the construction, and then the choice, when there is one, is between 1 or $h + 1$. (As indicated in extension 1.2 by the mark ($\widehat{\ }$), at some of the locations in question, the position of 1 is determined – that is, there is no choice.)

In order to solve the problem of placing 1's so that we achieve not only $V(x) = x$ but also have a signature sequence, we use Graham's test [2] for Beatty sequences. If θ is a positive irrational number, then the sequence

$$[\theta],\ [2\theta],\ [3\theta], \ldots$$

is the *Beatty sequence of* θ. (These sequences have attracted much interest since their appearance in 1926 in connection with Beatty's problem: if

$$\frac{1}{\theta} + \frac{1}{\theta'} = 1,$$

then the two Beatty sequences $[n\theta]$ and $[n\theta']$ partition N, or in other parlance, they form a complementary system.) Following [2] and [6, pp. 29-30],

Graham's test. *For any finite sequence* $s = (s_1, s_2, \ldots s_n)$*, let*

$$\mathcal{S}_{i,k} = \{s_i + s_{k-i} : 1 \leq i < k\}$$

and decree s ***nearly linear*** *if*

$$\max \mathcal{S}_{i,k} \leq s_k < 1 + \min \mathcal{S}_{i,k}$$

for $k = 2, 3, \ldots, n$. *Then s is nearly linear if and only if s is the initial segment of a Beatty sequence. In other words, to test whether* $(s_1, s_2, \ldots s_n)$ *is an initial segment of a Beatty sequence, look at the sums* $s_1 + s_{n-1}$, $s_2 + s_{n-2}, \ldots$, $s_{n-1} + s_1$. *If all these sums have the same value, v say, then* s_n *must equal v or* $v + 1$*; but if they take on the two values v and* $v + 1$*, and no others, then* s_n *must equal* $v + 1$. *If anything else happens, s is not part of a Beatty sequence.*

In view of Theorem 8, it is easy to see that Graham's test solves our problem: let s_j be the number of terms of the sequence x under construction that are placed after the jth 1 and before the $(j+1)$st 1; as long as the numbers s_j pass Graham's test, the finite sequence thus far constructed is the first part of a Beatty sequence (actually, infinitely many Beatty sequences), but not otherwise.

References

[1] S. Beatty, Problem 3173, *Amer. Math. Monthly* 33 (1926) 159; 34 (1927), 159.

[2] R. L. Graham, S. Lin, and C-S. Lin, "Spectra of Numbers," *Math. Mag.* 51 (1978) 174-176.

[3] C. Kimberling, "Interspersions and Dispersions," *Proc. Amer. Math. Soc.* 117 (1993) 313-321.

[4] C. Kimberling, "The First Column of an Interspersion," *Fibonacci Quart.* 32 (1994) 301-314,

[5] C. Kimberling, "Stolarsky Interspersions," to appear in *Ars Combinatoria.*

[6] N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York, 1973.