Counting triangulations of almost-convex polygons

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Abstract. We define an almost-convex polygon as a non convex polygon in which any two vertices see each other inside the polygon unless they are not adjacent and belong to a chain of consecutive concave vertices. Using inclusion-exclusion techniques, we find formulas for the number of triangulations of almost-convex polygons in terms of the number and position of the concave vertices. We translate these formulas into the language of generating functions and provide several simple asymptotic estimates. We also prove that certain balanced configurations yield the maximum number of triangulations.

Keywords. Triangulation, polygon, visibility, Catalan numbers, combinatorial geometry.

1. Introduction

A classical problem on combinatorial geometry, which goes back to Euler, asks for the number of ways of triangulating a convex polygon with n sides using $n-3$ internal diagonals. If we call this number t_n, the answer is given by (see [G] and Problem 7 in [D])

$$t_n = C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}, \qquad n \geq 3 \tag{1}$$

where C_n denotes the n-th Catalan number. For example, a quadrangle admits $C_2 = 2$ triangulations and a pentagon admits $C_3 = 5$. We accept that a segment is a convex polygon with two sides and that $t_2 = C_0 = 1$.

For the sake of completeness and future reference we briefly recall the proof of this classical result. Let P be a convex n-polygon and fix one side AB. In any triangulation of P, the side AB will be joined to some other vertex C of P forming a triangle ABC (see figure 1); in this way P is decomposed into a triangle and two other convex polygons of sizes k and $n-k+1$ (where k ranges from 2 to $n-1$), which can be triangulated in t_k and t_{n-k+1} ways, respectively. This gives the recurrence

$$t_n = t_2 t_{n-1} + \cdots + t_k t_{n-k+1} + \cdots + t_{n-1} t_2,$$

which, up to a shift, is the same recurrence satisfied by the Catalan numbers.

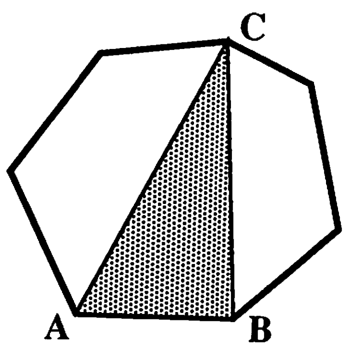

Fig. 1. Side AB can be joined to any other vertex C.

The question for convex polygons being settled, we ask whether we can find closed formulas for the number of triangulations of non-convex polygons. Triangulating a polygon is very sensitive to small changes in the internal visibility of the vertices, which makes the analysis of the general case rather difficult. Moreover, if we fix the number of concave (reflex) vertices then, as shown in the companion paper [HN], there is a wide range for the number of triangulations of an n-polygon. But we have been able to find closed formulas for a restricted class of polygons introduced below.

First we introduce some definitions which will be used in the sequel. Two vertices of a simple polygon are visible —a term widely used in the fields of Combinatorial and Computational Geometry— if the segment joining them is contained in the polygon. In particular, two consecutive vertices are always visible. A *reflex chain* in a polygon is a set of consecutive reflex vertices preceeded and followed by corresponding convex vertices. An *augmented reflex chain* is a reflex chain including the two extreme convex vertices. We define an *almost-convex* polygon as a polygon having the following maximum visibility property: two vertices are visible unless they belong to the same augmented reflex chain and are not consecutive. Our justification for the term almost-convex is that it represents a small departure from the convex model in which there is complete visibility: in other words, the only internal visibilities lost are those minimally imposed by the presence of reflex chains.

We denote by $P(n; k_1, k_2, \ldots, k_r)$ the class of almost-convex n-polygons having r reflex chains of lengths $k_1, k_2, \ldots, k_r$. If some of the k_i are repeated, we shorten the notation with an exponent so that, for example, $P(n; 1, 1, 2, 3, 3) = P(n; 1^2, 2, 3^2)$. Thus the two polygons shown in figure 2 are both in the class $P(11; 1, 2, 3)$. If all chains have the same length we write $P(n, k^r)$ instead of $P(n; k^r)$.

Our main result is Theorem 1: the number of triangulations of an almost-convex polygon depends only on the total number of vertices and on the number and lengths of its reflex chains, but not on their relative position. In other words, we can permute the reflex chains without changing the number of triangulations. We prove this using the principle of inclusion-exclusion and give a simple procedure for computing the number of triangulations in terms of the Catalan numbers. Before that, we handle in Section 2 a special case which illustrates the general technique we use for counting triangulations. Section 3 contains the main result together with an analysis of several particular cases. In Section 4 we prove that, fixing the total number of convex and reflex vertices, certain balanced configurations yield the maximum number of triangulations. In Section 5 we translate our results into the language of generating functions and obtain several asymptotic estimates. We conclude with some remarks and open problems.

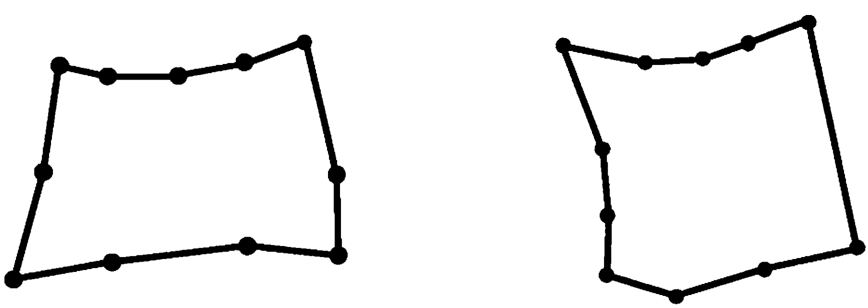

Fig. 2. Two polygons in the class $P(11; 1, 2, 3)$ with their reflex chains in different positions.

2. A special case: polygons with non adjacent reflex vertices

According to the notation introduced above, a $P(n, 1^k)$ polygon is one with n vertices, k of them reflex, no two of them adjacent and with maximum visibility, that is, the only internal visibilities lost are those between two convex vertices separated by a reflex one. This is possible only if $n \geq 2k$. Figure 3a gives an example of a $P(9, 1^2)$. We show that the number of ways of triangulating such a polygon depends only on n and k.

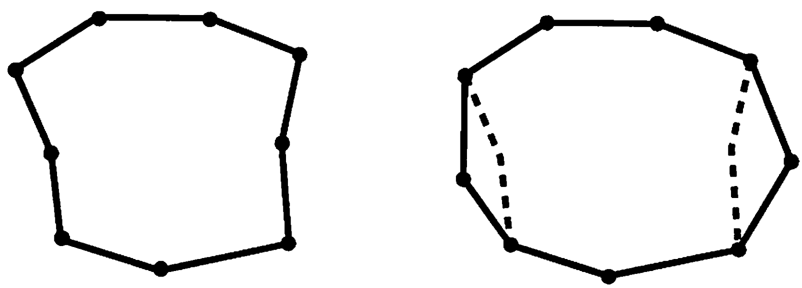

Fig. 3. A $P(9, 2)$ polygon and the corresponding convex nonagon.

Proposition 1. *The number of triangulations of a $P(n, 1^k)$ polygon, with $n \geq 2k$, is independent on the position of the k reflex vertices and is equal to*

$$t_n - kt_{n-1} + \binom{k}{2} t_{n-2} - \cdots + (-1)^j \binom{k}{j} t_{n-j} + \cdots + (-1)^k t_{n-k}, \quad (2)$$

where t_n is the number of triangulations of a convex n-gon.

PROOF. Take a convex n-polygon P^*, select k of its vertices not two of them consecutive, and push them towards the interior of P^* to get a polygon in $P(n, 1^k)$ (figure 3b). In this way we only lose k diagonals and it is clear that the number of triangulations of a $P(n, 1^k)$ equals the number of triangulations of P^* which do not use any of these diagonals. The formula now follows from the principle of inclusion-exclusion, since triangulating P^* using j of these diagonals amounts to triangulating a convex $(n - j)$-polygon. □

3. Almost-convex polygons

We now come to the general case and try to mimic the inclusion-exclusion technique of the previous section. So let $P = \{P_1, \ldots, P_n\}$ be a $P(n; k_1, \ldots, k_r)$ polygon. We consider again a convex polygon $P^* = \{P_1^*, \ldots, P_n^*\}$ together with the correspondence $P_i \longrightarrow P_i^*$. As before, the triangulations of P correspond uniquely to those of P^* which use none of the diagonals connecting non consecutive vertices of a corresponding augmented reflex chain in P. But we do not need to consider *all* these diagonals: we can limit ourselves to those in P^* connecting the two vertices immediately to the left and to the right of a vertex of P^* which is reflex in P, since a simple geometric argument shows that the triangulations we want to exclude will necessarily contain one of these special diagonals.

How many of these special diagonals are there? As many as reflex vertices are in P, that is, $k_1+k_2+\cdots+k_r$. But a triangulation of P^* cannot use any number of them arbitrarily: it cannot use two of them corresponding to a pair consecutive reflex vertices in P, for then the two diagonals would intersect. Thus we have run into the following combinatorial problem.

Problem: There are r strings of lengths $k_1, k_2, \ldots, k_r$ and we want to select j cells among the r strings without two of them being consecutive in one string (see figure 4 for an illustration). In how many ways can this be done?

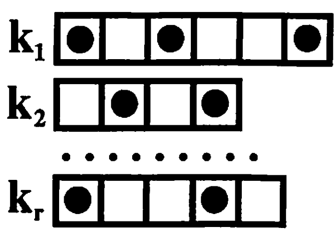

Fig. 4. Selecting non consecutive cells.

If we denote this number by

$$\left\| \begin{matrix} k_1, \ldots, k_r \\ j \end{matrix} \right\|, \tag{3}$$

then, by inclusion-exclusion, the following holds.

Theorem 1. *The number of triangulations of a $P(n; k_1, \ldots, k_r)$ polygon is equal to*

$$t_n - \left\| \begin{matrix} k_1, \ldots, k_r \\ 1 \end{matrix} \right\| t_{n-1} + \cdots + (-1)^j \left\| \begin{matrix} k_1, \ldots, k_r \\ j \end{matrix} \right\| t_{n-j} + \cdots, \tag{4}$$

where the sum extends until the term $\left\| \begin{matrix} k_1, \ldots, k_r \\ j \end{matrix} \right\| t_{n-j}$ becomes zero. In particular, this number depends only on $n, k_1, \ldots, k_r$ and not on the relative position of the reflex chains. □

We denote the expression in (4) by

$$t(n;k_1,\ldots,k_r)=\sum_j(-1)^j\left\|{k_1,\ldots,k_r \atop j}\right\|t_{n-j}$$

and use the exponent notation whenever some of the k_i are repeated (we also write a comma instead of a semicolon when $r=1$). For example, the number in (2) is $t(n,1^k)$. Clearly, Proposition 1 can be deduced from Theorem 1, since

$$\left\|{1,\overset{k}{\ldots},1 \atop j}\right\|=\binom{k}{j}.$$

A particular case that deserves further comment is that of *spiral* polygons. They are defined in Computational Geometry as those composed of a chain of convex vertices followed by a chain of reflex vertices and have been studied because of their relative simplicity (see [EC]). Spiral polygons with k reflex vertices and maximum internal visibility are those in the class $P(n,k)$. The number $\left\|{k \atop i}\right\|$ corresponds to the selection of i cells, no two of them consecutive, in a string of length k. The solution to this problem is well known:

$$\left\|{k \atop j}\right\|=\binom{k-j+1}{j}.$$

Thus we have

$$t(n,k)=\sum_{j\geq 0}(-1)^j\binom{k-j+1}{j}t_{n-j}.$$

In order to calculate $t(n,k_1,\ldots,k_r)$ for arbitrary n and $k_1,\ldots,k_r$ we need a general procedure for computing the numbers $\left\|{k_1,\ldots,k_r \atop j}\right\|$. This can be done by means of the following recursive formula.

Proposition 2. *The numbers defined in (3) satisfy the recurrence equation*

$$\left\|{k_1,\ldots,k_r \atop j}\right\|=\sum_{i=0}^{j}\left\|{k_1 \atop i}\right\|\cdot\left\|{k_2,\ldots,k_r \atop j-i}\right\|.$$

PROOF. Select i cells from the first string and the remaining $j-i$ from the remaining strings. □

In section 5 we will see how to translate these formulas in terms of certain polynomials and generating functions.

4. A maximum property

Let us fix n and k and ask the following question: which distribution of k concave vertices into reflex chains of an n-gon will produce the maximum number of triangulations? The answer is that the chains must be *balanced*, all having the same length (due to rounding effects, some of them will have one vertex more than the others).

Theorem 2. *Among all almost-convex polygons with n vertices, k of them being reflex, the maximum number of triangulations is reached by the* balanced *one: the k reflex vertices are evenly distributed in $n-k$ reflex chains of length $\lfloor \frac{k}{n-k} \rfloor$ (which can be zero) or $\lceil \frac{k}{n-k} \rceil$ between any two convex vertices. Moreover, this maximum is strict.*

The theorem will follow at once from the following lemma, which asserts that transfering vertices from one reflex chain to a shorter one increases the number of triangulations.

Lemma. *For $p \geq 2$ and arbitrary $k, k_1, \ldots, k_r$ it holds that*

$$t(n; k, k+p, k_1, \ldots, k_r) < t(n; k+1, k+p-1, k_1, \ldots, k_r).$$

PROOF. The proof is by induction on p. First of all, due to Theorem 1, we can choose any representative P in the class $P(n, k, k+p, k_1, \ldots, k_r)$, and we do it in such a way that the two reflex chains of lengths k and $k+p$ are separated by only one reflex vertex. Do the same for P' in the class $P(n, k+1, k+p-1, k_2, \ldots, k_r)$. The situation is depicted in figure 5, where $\{A, V_1, \ldots, V_k, B\}$ and $\{B, W_1, \ldots, W_{k+p}, C\}$ are two consecutive augmented reflex chains of P, and $\{A, V_1, \ldots, V_{k+1}, B\}$ and $\{B, W_1, \ldots, W_{k+p-1}, C\}$ belong to P'.

Now take every triangle with base $W_{k+p}C$ and third vertex X in P. This disects P into L and R. Do the same in P' with basis $W_{k+p-1}C$ and vertex X' obtaining L' and R'. For every permissible choice of X we can take $X' = X$. Then polygons R and R' have the same number of triangulations and, by induction hypothesis, L' has more triangulations than L if $p > 2$.

When $p = 2$, L and L' have exactly the same number of triangulations, since the two resulting chains are in both cases of lengths k and $k+1$ and Theorem 1 applies again. But in this case we have for X' one more choice than for X, namely $X' = V_{k+1}$, and this choice provides at least one more triangulation for P' than for P. □

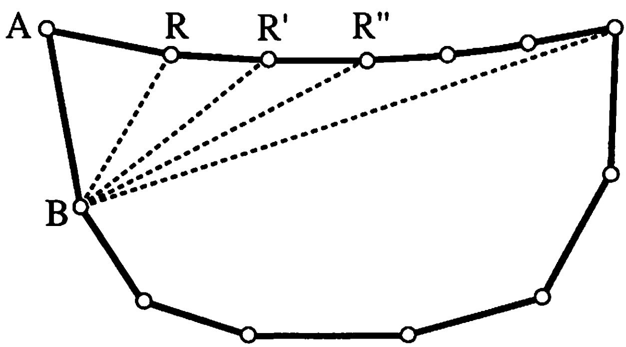

Fig. 5. Dissection of P and P'.

5. Generating functions and asymptotic analysis

Let us recall that the generating function for the Catalan numbers is $\frac{1-\sqrt{1-4z}}{2z}$. This, together with (1), implies that the generating function of the t_n is

$$T(z) = \sum_{n\geq 0} t_n z^n = \frac{z}{2}(1 - \sqrt{1-4z}).$$

We now define a sequence p_k of polynomials by means of

$$p_k(z) = \sum_{j\geq 0} (-1)^j \binom{k-j+1}{j} z^j \tag{5}$$

and note that the j-th coefficient of p_k is precisely $(-1)^j \left\| {k \atop j} \right\|$. From Proposition 2 it follows that

$$p_{k_1}(z)\cdots p_{k_r}(z) = \sum_{j\geq 0} (-1)^j \left\| {k_1,\ldots,k_r \atop j} \right\| z^j.$$

If we let $T_{k_1,\ldots,k_r}(z) = \sum_{n\geq 0} t(n;k_1,\ldots,k_r)z^n$ then Theorem 1 gives an explicit expression for this generating function:

$$T_{k_1,\ldots,k_r}(z) = p_{k_1}(z)\cdots p_{k_r}(z)T(z). \tag{6}$$

An application of Darboux's lemma to the former expression allows us to compute an asymptotic estimate for the numbers $t(n;k_1,\ldots,k_n)$.

Theorem 3. *Fix positive integers $k_1, k_2, \ldots, k_r$. Then, as $n \to \infty$,*

$$t(n;k_1,\ldots,k_n) \sim \frac{(1+\frac{k_1}{2})\cdots(1+\frac{k_r}{2})}{2^{k_1+\cdots+k_r}} t_n.$$

PROOF. The function $T(z) = \sum_n t_n z^n$ has an algebraic singularity at $z = 1/4$; when multiplied by a polynomial $p(z)$, Darboux's lemma (see [W]) tells us that the asympotic behaviour of the coefficients in $p(z)T(z)$ is $p(1/4)t_n$. If we want to apply this to (6) we only need to compute the value of $p_k(1/4)$ for every k. This is most easily done by means of the recurrence

$$\begin{aligned} &p_0 = 1; \qquad p_1 = 1 - z \\ &p_k = p_{k-1} - z\,p_{k-2}, \qquad \text{for } k \geq 2, \end{aligned} \tag{7}$$

satisfied by the polynomials p_k and which is proved from the definition (5) and the addition formula for the binomial coefficients. If we let $\alpha_k = p_k(1/4)$, then

$$\alpha_{k+1} = \alpha_k - \frac{1}{4}\alpha_{k-1}$$

with initial conditions $\alpha_0 = 1$ and $\alpha_1 = 3/4$, and the unique solution is $\alpha_k = (1 + \frac{k}{2})\frac{1}{2^k}$. □

As particular cases of the theorem we get the following two estimates

$$\begin{aligned} &(a) \quad t(n, 1^k) \sim (3/4)^k t_n, \\ &(b) \quad t(n, k) \sim (1 + \tfrac{k}{2})(1/2)^k t_n. \end{aligned}$$

which can be rephrased as follows. (a) Every time we transform a convex vertex into a reflex one in a convex polygon while maintaining maximum visibility (this implies that the reflex vertices will not be adjacent) the number of triangulations decreases, asymptotically, by a factor of 3/4.

(b) If we insert a reflex chain of length k in a convex polygon while maintaining maximum visibility —and produce an almost-convex spiral polygon— the number of triangulations decreases, asymptotically, by a factor of $(1 + k/2)2^{-k}$.

We end this section with a remark. Theorem 1 and equations (7) imply the following recurrence relation for the numbers $t(n, k)$:

$$t(n, k) = t(n - 1, k - 1) + t(n, k + 1), \qquad \text{for } n \geq k + 4\ .$$

A pure geometric proof of this fact can be given as follows. Let P be a polygon in the class $P(n, k)$ and consider the diagonal joining the first reflex vertex R with the second convex vertex B at its left (see figure 6). Triangulations of $P(n, k)$ which use the diagonal BR are counted by the number $t(n - 1, k - 1)$. If a triangulation does not contain BR then it contains none of the diagonals $BR', BR'', \ldots$ joining B to the reflex vertices.

The situation is then as if A were a reflex vertex and B the first convex vertex and these triangulations are counted by $t(n, k+1)$ (note that $n \geq k+4$ guarantees that there are at least four convex vertices).

The recurence above can be used to prove by induction a closed formula for the numbers $t(n, k)$, namely

$$t(n, k) = \frac{k+2}{n-1}\binom{2n-k-5}{n-k-3}.$$

We finally remark that the numbers $t(n, k)$ also appear in [S] and in in [BV] in the context of enumerating certain lattice paths.

6. Concluding remarks and open problems

The class of polygons dealt with in this paper is suitable for combinatorial analysis because being almost-convex is a very strong condition for a polygon. We believe a natural class of polygons in which one should try to enumerate triangulations is that of spiral polygons: they are simple in structure but quite complicated with respect to the internal visibility of the vertices. The general tight bounds obtained in [HN] apply to spiral polygons; the problem then is to pick additional parameters from which to obtain exact formulas.

Returning to almost-convex polygons, it is not difficult to prove that $t(a+b+3; a, b) = \binom{a+b}{a}$. It would be nice to find similar simple formulas for the very symmetric configurations in $P(3k+3; k, k, k)$ or, more generally, for $P(rk+r, k^r)$.

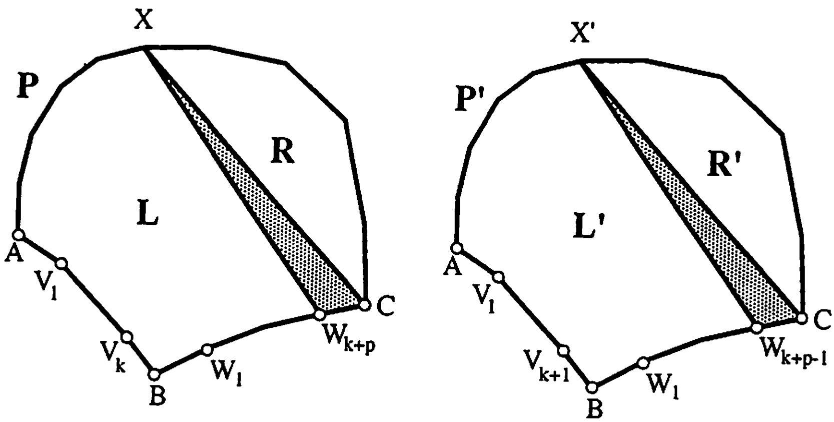

Fig. 6. Proof of the recurrence $t(n,k) = t(n-1,k-1) + t(n,k+1)$.

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