Ramsey Numbers For Cocircuits In Matroids

Fair Barbour Hurst
Talmage James Reid
Department of Mathematics
The University of Mississippi
University, MS 38677

ABSTRACT. We determine upper bounds on the number of elements in connected and 3-connected matroids with fixed rank and bounded cocircuit size. The existence of these upper bounds is a Ramsey property of matroids. We also determine size type function and extremal matroids in several classes of matroids with small cocircuits.

1 Introduction

Extremal matroid theory derives much of its motivation from extremal problems in geometry and graph theory. We draw our motivation here from extremal problems in planar geometries and the Ramsey theory of graphs. An excellent reference to extremal matroid theory is the survey paper of Kung [9]. In this paper Kung notes that the central problem of extremal matroid theory is, for a class of matroids $\mathcal{C}$, to determine the size function $h(r,\mathcal{C}) = \max\{|E(M)|: rM = r \text{ and } M \in \mathcal{C}\}$ and find the matroids of maximum size. We consider the closely related cardinality function $e(r,\mathcal{C}) = \max\{|E(M)|: rM \leq r \text{ and } M \in \mathcal{C}\}$ here. Exact values and bounds for the size functions of many classes of matroids which are subsets of some projective geometry have been computed (see, for example, [2,4,5,6,7,8,9,10,11,12,13,14,15,16]). It is clear that the size functions of these classes of matroids exist, although it is often difficult to compute them. It is not clear that the size function of the class of connected matroids with all cocircuits having fewer than a fixed number of elements exists as these matroids need not be representable over a finite field. We show here that the existence of this function is part of a fundamental Ramsey property of matroids implied by the existence of the matroid Ramsey numbers. The definition, a triangle inequality, and a binomial upper bound for the matroid Ramsey numbers are given in the next two results [18].

Definition 1. Let k and l be positive integers. Then $n(k,l)$ is the least positive integer n such that every connected matroid with n elements contains either a circuit with at least k elements or a cocircuit with at least l elements. □

Theorem 1. *Let k and l be integers exceeding one.*

(a) $n(k,l) < n(k-1,l) + n(k,l-1)$.

(b) $n(k,l) < \binom{k+l-4}{k-2}$ *if k and l exceed three.*

□

For each positive integer l, let $C^*(l)$ denote the class of all connected matroids with each cocircuit having fewer than l elements. Then $e(r, C^*(l))$ is the maximum number of elements a connected matroids M may have which has rank at most r and each cocircuit containing fewer than l elements. The next result establishes the existence of this cardinality function. The proofs of our main results are given in Section 2.

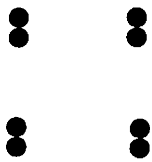
Theorem 2. $e(r, C^*(l)) < \min\{n(r+2,l), n(r+2,l-1)+r\}$ *if r and l exceed one. An upper bound for this minimum is* $\binom{r+l-3}{r} + r$ *when* $l \geq 5$.

An extremal matroid of a number $e(r, C^*(l))$ is a connected matroid M with $e(r, C^*(l))$ elements, rank at most r, and each cocircuit having fewer than l elements. We next define several classes of matroids which are shown in the next theorem to contain extremal matroids of specific numbers $e(r, C^*(l))$.

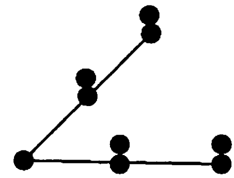
Let a, b, c, and d be positive integers. Then $L(a,b,c)$ denotes the matroid constructed by adding elements in parallel to a 3-element circuit so that parallel classes of sizes a, b, and c are obtained. The matroid $P(a,b,c,d)$ denotes the matroid constructed by adding elements in parallel to a 4-element circuit so that parallel classes of sizes a, b, c, and d are obtained. The matroid $P^+(a,a,a,a)$ is constructed by freely adding an element to the intersection of two distinct lines of $P(a,a,a,a)$. The uniform matroid with rank r on n elements is denoted by $U_{r,n}$. The rank-4 binary affine geometry is denoted by $AG(3,2)$. The graph H_8 and Euclidean representations for P_5 and some of the matroids mentioned are given in figure 1.

The next result gives all values of $e(r, C^*(l))$ and the extremal matroids for r at most three. The values $e(3, C^*(5)) = 8$ and $e(3, C^*(6)) = 9$ are used in a subsequent paper [3] in the proof of $n(5,6) = 11$. This is the largest known matroid Ramsey number. If $l \leq 2$, then $C^*(l)$ consists only of 1-element matroids. We ignore this case in the remaining theorems for this reason.

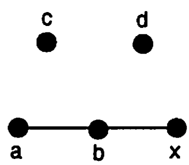
L(1,1,2)



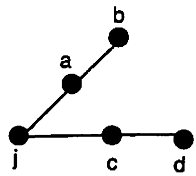
P(2,2,2,2)



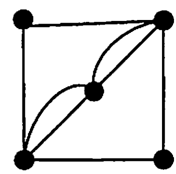
$P^+(2,2,2,2)$



P_5



$P^+(1,1,1,1)$



H_8

Figure 1

Theorem 3. *Let r and l be positive integers with $l \geq 3$.*

(a) $e(l, C^*(l)) = l - 1$.
The extremal matroid is $U_{1,l-1}$.

(b) $e(2, C^*(l)) = \frac{3}{2}l - \frac{3}{2}$ *if* $l = 2n + 1$ *for some positive integer* n.
The extremal matroid is $L(n, n, n)$.

(c) $e(2, C^*(l)) = \frac{3}{2}l - 2$ *if* $l = 2n$ *for some positive integer* n.
The extremal matroid is $L(n-1, n-1, n)$.

(d) $e(3, C^*(l)) = 2l - 2$ *if* $l = 2n + 1$ *for some positive integer* n.
The extremal matroid is $P(n, n, n, n)$.

(e) $e(3, C^*(l)) = 2l - 3$ *if* $l = 2n$ *for some positive integer* n.
The extremal matroids are $P(n-1, n-1, n-1, n)$ *and* $P^+(n-1, n-1, n-1, n-1)$ *if* $l \geq 6$. *If* $l = 4$, *then the extremal matroids are the connected rank-3 matroids with five elements.*

The final main result implies that 3-connected matroids with small cocircuits have fewer elements than connected matroids with small cocircuits.

Theorem 4. *Let M be a 3-connected rank-r matroid with all cocircuits having fewer than l elements for integers $r \geq 5$ and $l \geq 3$. Then* $|E(M)| \leq e(r-2, C^*(l)) + 2$.

Section 1 concludes with some terminology and results used here. The matroid terminology used mostly follows Oxley [17]. Let M be a matroid. The ground set of M is denoted by $E(M)$. Let $X \subseteq E(M)$. Then $M \setminus X$ and M/X denote the deletion and contraction of X from M, respectively. The restriction of M to X is denoted by $M|X$. The closure and rank of X in M are denoted by $cl(X)$ and rX, respectively. Three-element circuits of M are called *triangles*.

If k is a positive integer, then a bipartition (A, B) of $E(M)$ is a *k-separation* of M if $|A| \geq k$, $|B| \geq k$, and $rA + rB - rM \leq k - 1$ [19]. For an integer $n \geq 2$, M is *n-connected* if and only if M has no k-separations for any $k < n$. A 2-connected matroid is said to be connected. A matroid M is connected if and only if its dual M^* is connected (see [17, (4.2.8)]).

The next three results on matroid connectivity are used here. The first result is due to Tutte [19]. The second result follows from Tutte's result by induction. The last result is due to Akkari and Oxley [1].

Theorem 5. *If M is a connected matroid and $e \in E(M)$, then $M \setminus e$ or M/e is connected.* □

Corollary 1. *If X is a subset of a connected matroid M, then there exists a connected minor N of M with $E(N) = X$.* □

Theorem 6. *Let M be a matroid having at least four elements. Then M is 3-connected and $M/e, f$ is disconnected for all pairs $\{e, f\}$ of distinct elements if and only if every pair of distinct elements of M is in a triangle.* □

2 The proofs

We first establish an upper bound for the size functions of the connected matroids with small cocircuits.

The proof of Theorem 2: Let M be a member of $C^*(l)$ with rank at most r. If $|E(M)| \geq n(r+2, l)$, then either M has a circuit with at least $r+2$ elements or M has a cocircuit with at least l elements. The former does not occur as $rM \leq r$, while the latter does not occur as $M \in C^*(l)$. Thus $|E(M)| < n(r+2, l)$.

Suppose $rM^* \geq n(r+2, l-1)$. Let B^* be a cobasis of M. Then there exists a connected minor N of M with ground set B^* by Corollary 1. Hence $|E(N)| = |B^*| \geq n(r+2, l-1)$ implies that N has either a circuit with at least $r+2$ elements or a cocircuit with at least $l-1$ elements. The former does not occur as $rN \leq rM \leq r$. Hence N has a cocircuit D with at least $l-1$ elements. The set D is contained in some cocircuit of M (see [17, (3.1.11)]). Thus D is a cocircuit of M as M has no cocircuits with l or more elements. However, D is contained in the cobasis B^* of M; a contradiction. Thus $rM^* < n(r+2, l-1)$. It follows that $|E(M)| =$

$rM + rM^* < r + n(r+2, l-1)$. Thus $|E(M)| < \min\{n(r+2,l), n(r+2, l-1) + r\}$ for all connected matroids M in $C^*(l)$ with rank at most r. Hence $e(r, C^*(l)) < \min\{n(r+2,l), n(r+2,l-1)+r\}$. If $l \geq 5$, then $\min\{n(r+2,l), n(r+2,l-1)+r\} \leq n(r+2,l-1)+r < \binom{r+l-3}{r} + r$ by Theorem 1(b). □

The next result of Reid [18] is used in the proof of Theorem 3. An extremal matroid of a number $n(k,l)$ is a connected matroid M with $n(k,l)-1$ elements such that each circuit of M has fewer than k elements and each cocircuit of M has fewer than l elements.

Theorem 7. *Let l be an integer exceeding two.*

(a) $n(3,l) = l$.
The extremal matroid is $U_{1,l-1}$.

(b) $n(4,l) = \frac{3}{2}l - \frac{1}{2}$ *if $l = 2n+1$ for some positive integer n.*
The extremal matroid is $L(n,n,n)$.

(c) $n(4,l) = \frac{3}{2}l - 1$ *if $l = 2n$ for some positive integer n.*
The extremal matroid is $L(n-1, n-1, n)$.

□

The following result is combined with the previous theorem to establish parts (a), (b), and (c) of Theorem 3.

Lemma 1. *Let r and l be integers exceeding one. If there exists a connected matroid M with $n(r+2,l)-1$ elements, rank at most r, and each cocircuit of M has fewer than l elements, then $e(r, C^*(l)) = n(r+2,l)-1$. Moreover, the set of extremal matroids of $e(r, C^*(l))$ consists of the members of the set of extremal matroids of $n(r+2,l)$ which have rank at most r.*

Proof: It follows from $M \in C^*(l)$ with $rM \leq r$ that $e(r, C^*(l)) \geq |E(M)| = n(r+2,l)-1$. The reverse inequality follows from Theorem 2 so that $e(r, C^*(l)) = n(r+2,l)-1$. Let N be an extremal matroid of $e(r, C^*(I))$. Then $rN \leq r$ implies that each circuit of N has fewer than $r+2$ elements. Thus $n(r+2,l) > |E(N)| \geq |E(M)| = n(r+2,l)-1$. Hence $|E(N)| = n(r+2,l)-1$ and N is an extremal matroid of $n(r+2,l)$ with rank at most r.

Conversely, suppose that N is an extremal matroid of $n(r+2,l)$ having rank at most r. Then $|E(N)| = n(r+2,l)-1 = e(r, C^*(l))$. Thus N is an extremal matroid of $e(r, C^*(l))$. □

The complement of a cocircuit in a matroid is a hyperplane (see [17, (2.1.14)]). Hence we obtain the following useful lemma.

Lemma 2. *Let M be a matroid of positive rank and $l \geq 2$. Then each cocircuit of M has fewer than l elements if and only if each hyperplane of M contains at least $|E(M)| - l + 1$ elements.* □

The proof of Theorem 3(a), (b), and (c): The matroid $U_{1,l-1}$ is a connected matroid with rank one and each cocircuit having $l-1$ elements. It has $l - 1 = n(3, l) - 1$ elements by Theorem 7(a). Therefore, Theorem 3(a) follows immediately from Theorem 7(a) and Lemma 1.

Suppose $l = 2n+1$ for some integer $n \geq 1$. Each cocircuit of the connected rank-2 matroid $L(n, n, n)$ has $2n < l$ elements. This matroid has $3n = \frac{3}{2}l - \frac{3}{2} = n(4, l) - 1$ elements by Theorem 7(b). Thus Theorem 3(b) follows immediately from Theorem 7(b) and Lemma 1.

Suppose $l = 2n$ for some integer $n \geq 2$. Each cocircuit of the connected rank-2 matroid $L(n-1, n-1, n)$ has at most $2n - 1 < l$ elements. This matroid has $3n - 2 = \frac{3}{2}l - 2 = n(4, l) - 1$ elements by Theorem 7(c). Thus Theorem 3(c) follows immediately from Theorem 7(c) and Lemma 1. □

We remark that Lemma 1 may be used to determine $e(3, C^*(5))$ and its extremal matroids by the following argument. Reid [18] showed that $n(5,5) = 9$. Hurst and Reid [3] showed that $AG(3,2)$, $P(2,2,2,2)$, $P(2,2,2,2)^*$, and $M(H_8)$ are the extremal matroids of $n(5,5)$. The only one of these with rank at most three is $P(2,2,2,2)$. Thus $e(3, C^*(5)) = n(5,5) - 1 = 8$ and $P(2,2,2,2)$ is the extremal matroid of $e(3, C^*(5))$ by Lemma 1. However, Lemma 1 cannot be used to determine $e(3, C^*(6))$. This follows as $n(5,6) = 11$ [3], but $e(3, C^*(6)) = 9$ is shown in Theorem 3(e). Thus $e(3, C^*(6)) \neq n(5,6) - 1$ and Lemma 1 does not apply in general.

Suppose $\mathcal{L}$ is a collection of subsets of the ground set of a matroid M. Then $\Sigma(\mathcal{L}, E(M))$ denotes the number of ordered pairs (l, e) such that $l \in \mathcal{L}$ and $e \in l$.

The proof of Theorem 3(d) and (e): Suppose $l = 2n + 1$ for some integer $n \geq 1$. Each cocircuit of the connected rank-3 matroid $P(n, n, n, n)$ has at most $2n < l$ elements. Thus $e(3, C^*(l)) \geq 4n = 2l - 2$ if l is odd.

Suppose $l = 2n$ for some integer $n \geq 2$. Each cocircuit of the connected rank-3 matroid $P(n-1, n-1, n-1, n)$ has at most $2n - 1 < l$ elements. Thus $e(3, C^*(l)) \geq 4n - 3 = 2l - 3$ for l even.

Suppose M is an extremal matroid of $e(3, C^*(l))$, where l may be even or odd. We obtain the following two statements by Theorem 7(b) and (c), respectively. If l is odd, then $|E(M)| \geq 2l - 2 \geq \frac{3}{2}l - \frac{1}{2} = n(4, l)$. If l is even, then $|E(M)| \geq 2l - 3 \geq \frac{3}{2}l - 1 = n(4, l)$. Hence $|E(M)| \geq n(4, l)$, and M contains a circuit with four or more elements. The maximum circuit size of M is four as $rM \leq 3$. Thus M contains a 4-element circuit $C = \{a, b, c, d\}$.

Let $\mathcal{L}$ denote the set of six distinct lines of M determined by each pair of elements of C. Each line of $\mathcal{L}$ is a hyperplane of M which contains at least

$|E(M)|-l+1$ elements by Lemma 2. Hence $\Sigma(\mathcal{L}, E(M)) \geq 6(|E(M)|-l+1)$. The elements a, b, c, and d are in exactly three lines of $\mathcal{L}$. Thus it is straightforward to show that an element e of M is in at most three lines of $\mathcal{L}$. Moreover, e is in exactly three lines of $\mathcal{L}$ if and only if e is in a parallel class with a, b, c, or d. It follows that $\Sigma(\mathcal{L}, E(M)) \leq 3|E(M)|$ with equality if and only if every element of M is in a parallel class with a, b, c, or d. Thus

$$6(|E(M)| - l + 1) \leq \sum(\mathcal{L}, E(M)) \leq 3|E(M)|. \tag{1}$$

This implies that $|E(M)| \leq 2l - 2$. It follows that if l is odd, then $e(3, C^*(l)) = 2l - 2$. If l is even, then $e(3, C^*(l))$ is $2l - 3$ or $2l - 2$.

Suppose $|E(M)| = 2l - 2$. From substituting $|E(M)| = 2l - 2$ into (1), we obtain equality throughout so that $\Sigma(\mathcal{L}, E(M)) = 3|E(M)|$. Thus each element of M is a member of $cl(a)$, $cl(b)$, $cl(c)$, or $cl(d)$. For each distinct pair of elements x and y of C, $cl(x,y)$ is a hyperplane of M so that $|cl(x)| + |cl(y)| = |cl(x,y)| \geq l-1$ by Lemma 2. Also, $2l-2 = |E(M)| = |cl(a)|+|cl(b)|+|cl(c)|+|cl(d)|$. Thus $|cl(a)|+|cl(b)| = |cl(c)|+|cl(d)| = l-1$. Likewise, $|cl(a)| + |cl(c)| = |cl(b)| + |cl(d)| = l - 1$ and $|cl(a)| + |cl(d)| = |cl(b)|+|cl(c)| = l-1$. From using these equations we obtain that $|cl(a)| = |cl(b)| = |cl(c)| = |cl(d)|$. Thus $4|cl(a)| = |E(M)| = 2l-2$. It follows that $l = 2|cl(a)|+1$ is odd. Hence l even implies that $|E(M)| = 2l-3$. Moreover, $l = 2n+1$ odd implies that $|cl(a)| = |cl(b)| = |cl(c)| = |cl(d)| = \frac{1}{4}|E(M)| = \frac{1}{4}(2l-2) = \frac{1}{2}(l-1) = n$. Thus $M \cong P(n,n,n,n)$ and $P(n,n,n,n)$ is the only extremal matroid of $e(3, C^*(l))$ for l odd and $l = 2n+1$. It remains to determine the extremal matroid of $e(3, C^*(l))$ when l is even.

Suppose $l = 2n$. Then $e(3, C^*(l)) = 2l-3$. Hence each line of M contains at least $|E(M)|-l+1 = l-2$ elements. If $l = 4$, then $|E(M)| = e(3, C^*(l)) = 5$, and each line of M need only contain $l - 2 = 2$ elements. Thus any connected rank-3 matroid with five elements is an extremal matroid of $e(3, C^*(4))$. The four such matroids are $U_{3,5}$, $P(1,1,1,2)$, $P^+(1,1,1,1)$, and the matroid P_5 of Figure 1.

Suppose that $l \geq 6$. First assume that every pair of distinct lines of $\mathcal{L}$ meet. Then there exist distinct elements e, f, and g of $E(M) - C$ such that $M|C \cup \{e,f,g\}$ is isomorphic to the Fano or non-Fano matroid.

There are seven and nine distinct lines contained in F_7 and F_7^-, respectively. Suppose that $M|C \cup \{e,f,g\} = F_7$ is as given in Figure 2. Let $\mathcal{L}_1$ denote the set of seven distinct lines of M determined by pairs of elements of $C \cup \{e,f,g\}$. Then each line of $\mathcal{L}_1$ contains at least $l-2$ elements so that $\Sigma(\mathcal{L}_1, E(M)) \geq 7(l-2) = 7l-14$. Each element of $E(M)$ is in at most three lines of $\mathcal{L}_1$. The maximum is attained when an element is in a parallel class with a, b, c, or d. Thus $\Sigma(\mathcal{L}_1, E(M)) \leq 3|E(M)| = 3(2l-3) = 6l-9$. Hence $7l - 14 \leq 6l - 9$ or $l \leq 5$; a contradiction.

Suppose that $M|C \cup \{e, f, g\} = F_7^-$ is as given in Figure 2. Let $\mathcal{L}_2$ denote the set of nine distinct lines of M determined by pairs of elements of the set $C \cup \{e, f, g\}$. These consist of the six lines with three elements pictured and the lines $cl(e, f)$, $cl(e, g)$, and $cl(f, g)$. The elements e, f, and g are in four lines of $\mathcal{L}_2$, while the elements of C are in three lines of $\mathcal{L}_2$. Thus each element of M is in at most four lines of $\mathcal{L}_2$. It follows that $9(l-2) \leq \Sigma(\mathcal{L}_2, E(M)) = \Sigma(\mathcal{L}_2, C) + \Sigma(\mathcal{L}_2, E(M) - C) \leq 3 \cdot 4 + 4(2l - 7) = 8l - 16$. Hence $9l - 18 \leq 8l - 16$ and $l \leq 2$; a contradiction. Hence there exists a pair of disjoint lines of $\mathcal{L}$. Let $cl(a, b)$ and $cl(c, d)$ be disjoint lines of $\mathcal{L}$, without loss of generality. These two disjoint lines each contain at least $l - 2$ elements and $|E(M)| = 2l - 3$. Thus at most one element of M is in neither the line $cl(a, b)$ nor the line $cl(c, d)$.

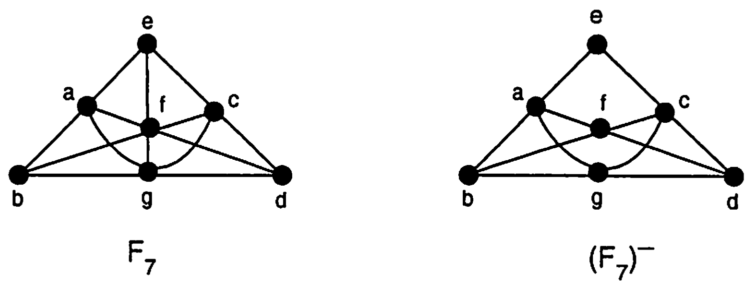


Figure 2

Suppose there exists an element x on the line $cl(a, b)$ that is in neither $cl(a)$ nor $cl(b)$. Then a Euclidean representation for $M|C \cup \{x\}$ is as given in P_5 of Figure 1. Let $\mathcal{L}_3$ denote the set of eight distinct lines of M determined by the pairs of elements of $C \cup \{x\}$. The elements of M which are in the maximum number of lines of $\mathcal{L}_3$, namely four, are those in $cl(c)$ or $cl(d)$. Let $S = cl(c) \cup cl(d)$. Then $8(l-2) \leq \Sigma(\mathcal{L}_3, E(M)) = \Sigma(\mathcal{L}_3, S) + \Sigma(\mathcal{L}_3, E(M) - S) \leq 4|S| + 3|E(M) - S| = 4|S| + 3(2l - 3 - |S|) = |S| + 6l - 9$. Thus $|S| \geq 2l - 7$, and there are at most four elements of M not in $S = cl(c) \cup cl(d)$. Each element of $cl(a, b)$ is not in $cl(c) \cup cl(d)$ as the lines $cl(a, b)$ and $cl(c, d)$ do not meet. Hence $4 \geq |cl(a, b)| \geq l - 2 \geq 4$ as $l \geq 6$. Thus equality holds throughout, $l = 6$, and $|E(M)| = 2l - 3 = 9$. It follows from $|S| \geq 2l - 7 = 5$ that every one of the five elements not in $cl(a, b)$ is in either $cl(c)$ or $cl(d)$. One of $cl(c)$ and $cl(d)$, say $cl(c)$, contains at most two elements. As the line $cl(a, b)$ contains only four elements, at least two of $cl(a)$, $cl(b)$, and $cl(x)$ have only one element. Suppose $|cl(a)| = 1$, without loss of generality. Then $l - 2 = 4 > 3 \geq |cl(a)| + |cl(c)| = |cl(a, c)|$; a contradiction. Thus every element on the line $cl(a, b)$ is in either $cl(a)$ or $cl(b)$. By symmetry, every element of the line $cl(c, d)$ is in either $cl(c)$ or $cl(d)$. It follows that every element of M, with at most one exception, is in a parallel class with a, b, c, or d.

First suppose that every element of M is in a parallel class with a, b, c, or d. Then $|cl(a)| + |cl(b)| + |cl(c)| + |cl(d)| = |E(M)| = 2l - 3$. We shall show that $M \cong P(n-1, n-1, n-1, n)$. Let u and v be distinct elements of C. Then $|cl(u)| + |cl(v)| = |cl(u,v)| \geq l-2$. Thus $|cl(u)| + |cl(v)|$ is $l-2$ or $l-1$ for each $u, v \in$C. Suppose $|cl(a)| \leq n-2$. For each $z \in \{b,c,d\}$, $|cl(a)| + |cl(z)| \geq l-2$. Thus $|cl(z)| \geq l-2-|cl(a)| \geq l-2-(n-2) = n$ as $l = 2n$. Hence $|cl(b)| + |cl(c)| \geq 2n = l$; a contradiction. Thus $cl(a)$, and likewise $cl(b)$, $cl(c)$, and $cl(d)$, contains at least $n-1$ elements. Hence $|cl(a)| + |cl(b)| \leq l-1$ implies that $|cl(a)| \leq l-1-|cl(b)| \leq l-1-(n-1) = n$. Hence $cl(a)$, and likewise $cl(b)$, $cl(c)$, and $cl(d)$, contains at most n elements. It follows from the facts that each of the four parallel classes of M contain either $n-1$ or n elements and $|E(M)| = 2l - 3 = 4n - 3$ that $M \cong P(n-1, n-1, n-1, n)$.

Finally, suppose that there exists an element of M, say j, not in a parallel class determined by a, b, c, or d. Then we shall show that $M \cong P^+(n-1, n-1, n-1, n-1)$. Let $l(j)$ denote the number of lines of $\mathcal{L}$ containing j. Suppose $l(j) = 0$. Then $M|C \cup \{j\} \cong U_{3,5}$. Hence $C \cup \{j\}$ determines a set of ten distinct lines of M. Call this set $\mathcal{L}_4$. Each element of M is in four lines of $\mathcal{L}_4$ and hence $10(l-2) \leq \Sigma(\mathcal{L}_4, E(M)) = 4(2l-3)$. Thus $l \leq 4$; a contradiction. Suppose $l(j) = 1$. Then $M|C \cup \{j\}$ is isomorphic to the matroid P_5 in Figure 1. However, we have previously shown that there cannot exist two disjoint lines when one of the lines contains at least three distinct parallel classes. Thus $l(j) \geq 2$. But j is not in a parallel class with a, b, c, or d. Thus $l(j) < 3$. Hence $l(j) = 2$. Suppose that $j \in cl(a,b) \cap (c,d)$, without loss of generality. The equations $|cl(a)| + |cl(b)| + |cl(c)| + |cl(d)| = |E(M) - \{j\}| = 2l - 4$, $|cl(a)| + |cl(c)| \geq l-2$, $|cl(a)| + |cl(d)| \geq l-2$, $|cl(b)| + |cl(c)| \geq l-2$, and $|cl(b)| + |cl(d)| \geq l-2$ can be used to show that $|cl(a)| = |cl(b)|$ and $|cl(c)| = |cl(d)|$. Suppose $|cl(a)| \leq \frac{1}{2}l - 2$. Then $|cl(a,b)| = |cl(a)| + |cl(b)| + 1 \leq l-3$; a contradiction. Thus $|cl(a)|$, and likewise $|cl(b)|$, $|cl(c)|$, and $|cl(d)|$, all exceed $\frac{1}{2}l-2$. It follows from the fact that there are $2l-4$ elements in the parallel classes determined by a, b, c, and d that each of these parallel classes contains exactly $\frac{1}{2}l - 1 = n-1$ elements. A Euclidean representation for $M|C \cup \{j\}$ is given in $P^+(1,1,1,1)$ of Figure 1. From adding points in parallel to a, b, c, and d we obtain that $M \cong P^+(n-1, n-1, n-1, n-1)$. □

The proof of Theorem 4: By Theorem 3, $e(r-2, C^*(l)) + 2 \geq e(1, C^*(3)) + 2 \geq 4$. Thus the result holds if $|E(M)| \leq 4$. Suppose that $|E(M)| > 4$. Then M is simple (see [17, (8 1.6)]).

First suppose that every pair of distinct elements of M is in a triangle of M. Let H be a hyperplane of M. Then $E(M) - H$ is a cocircuit of M. Thus $|E(M) - H| \leq l - 1$.

Let $x \in E(M) - H$. If $y \in H$, then there exists a triangle T containing x and y. Let $z \in T - \{x, y\}$. Then $z \notin H$ as $z \in H$ would imply that $x \in cl(H) = H$; a contradiction. Hence each element y of H may be associated with an element z of $(E(M) - H) - \{x\}$ chosen from the line $cl(x, y)$. Let $f \colon H \to (E(M) - H) - \{x\}$ be a function induced in this manner.

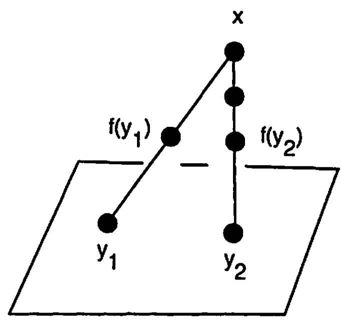


Figure 3

Let y_1 and y_2 be distinct elements of H. Suppose $f(y_1) = f(y_2)$. Then $y_2 \in cl(x, f(y_2)) = cl(x, f(y_1)) = cl(x, y_1)$. Thus $cl(x, y_1) = cl(y_1, y_2) \subset H$. Hence $x \in H$; a contradiction. It follows that $f(y_1) \neq f(y_2)$ and f is an injection from H to $(E(M) - H) - \{x\}$. Hence $|H| \leq |(E(M) - H) - \{x\}| = |E(M) - H| - 1$. Thus $|E(M)| = |H| + |E(M) - H| \leq 2|E(M) - H| - 1 \leq 2(l - 1) - 1 = 2l - 3$. It follows from Theorem 3(d) and (e) that $|E(M)| \leq 2l - 3 \leq e(3, C^*(l)) \leq e(r - 2, C^*(l)) < e(r - 2, C^*(l)) + 2$.

Suppose there exists a pair of elements of M which is in no triangle of M. Then, by Theorem 6, there exist $e, f \in E(M)$ such that $M/e, f$ is connected. The set $\{e, f\}$ is independent as M is simple. It follows that the matroid $M/e, f$ has rank at most $r - 2$ and is in $C^*(l)$. Thus $e(r-2, C^*(l)) \geq |E(M/e, f)| = |E(M)| - 2$. Hence $|E(M)| \leq e(r - 2, C^*(l)) + 2$. □

References

[1] S. Akkari and J.G. Oxley, Some extremal connectivity results for matroids, *J. Combin. Theory Ser. B* **52** (1991), 301–320.

[2] J.W. Hipp, The maximum size of combinatorial geometries excluding wheels and whirls as minors, Doctoral dissertation, University of North Texas, Denton, Texas, 1989.

[3] F.B. Hurst and T.J. Reid, Some small circuit-cocircuit Ramsey numbers for matroids, *Combinatorics, Probability, and Computing*, to appear.

[4] J.P.S. Kung, Growth rates and critical exponents of minor-closed classes of binary geometries, *Trans. Amer. Math. Soc.* **293**(1986), 837–857.

[5] J.P.S. Kung, Excluding the cycle geometries of the Kuratowski graphs from binary geometries, *Proc. London Math. Soc.* **55**(1987), 209–242.

[6] J.P.S. Kung, The long-line graph of a combinatorial geometry. I. Excluding $M(K_4)$ and the $(q+2)$-point line as minors, *Quart. J. Math. Oxford (2)* **39**(1988), 223–234.

[7] J.P.S. Kung, The long-line graph of a combinatorial geometry. II. Geometries representable over two fields of different characteristics, *J. Combin. Theory Ser. B* **50**(1990), 41–53.

[8] J.P.S. Kung, Combinatorial geometries representable over $GF(3)$ and $GF(q)$. I. The number of points, *Discrete Comput. Geom.* **5** (1990), 84–95.

[9] J.P.S. Kung, Extremal matroid theory. In "Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference on Graph Minors", University of Washington, Seattle, WA, 1991 (eds. N. Robertson and P. Seymour) *Contemporary Math.* **147**(1993), 21–61.

[10] J.P.S. Kung and J.G. Oxley, Combinatorial geometries representable over $GF(3)$ and $GF(q)$. II. Dowling geometries, *Graphs and Combin.* **4**(1988), 323–332.

[11] J. Lee, Turán's triangle theorem and binary matroids, *European J. Combin.* **10**(1989), 85–90.

[12] J.G. Oxley, A characterization of the ternary matroids with no $M(K_4)$-minor, *J. Combin. Theory Ser. B* **42**(1987), 212–249.

[13] J.G. Oxley, On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300**(1987), 663–679.

[14] J.G. Oxley, The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.* **301**(1987), 63–75.

[15] J.G. Oxley, The regular matroids with no 5-wheel minor, *J. Combin. Theory Ser. B* **46**(1989), 292–305.

[16] J.G. Oxley, On an excluded-minor class of matroids, *Discrete Math.* **82**(1990), 35–52.

[17] J.G. Oxley, "Matroid Theory", Oxford University Press, New York, 1992.

[18] T.J. Reid, Ramsey numbers for matroids, submitted.

[19] W.T. Tutte, Connectivity in matroids, *Canad. J. Math.* 18(1966), 1301–1324.