On Magic Strength of Graph

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ABSTRACT. Let $G = (V, E)$ be a finite simple graph. G is said to be a magic graph iff there exists a magic assignment of G, which is a mapping L from E to $N = \{1, 2, \dots\}$ such that the sums of the labels of all edges incident to the vertices in V are identical. Let $M(G)$ be the set of all magic assignments of G. For any L in $M(G)$, define $s(L) = \max\{L(e)\colon e \text{ in } E\}$. Then, the magic strength of G is defined as $m(G) = \min\{s(L)\colon L \text{ in } M(G)\}$. In this paper, we determine the magic strengths of several classes of graphs and introduce some constructions of magic graphs. We also show that every connected graph is an induced subgraph of a magic graph.

1 Introduction

Let $G = (V, E)$ be a (p, q) graph, which is a finite simple graph with $|V| = p$ and $|E| = q$. Stanley [16] defined a *magic labeling of G of index t* to be an assignment $L\colon E \to \{0, 1, 2, \dots\}$ of a nonnegative integer $L(e)$ to each edge e of G such that for each vertex v of G the sum of the labels of all edges incident to v is t. He pointed out in [16] that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations. Jeurissen [9,10,11] called a magic labeling L *pseudo-magic* if the 'labels' $L(e)$ are pairwise different.

The original concept of a magic graph is due to J. Sedlacek [14,15], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Necessary and sufficient conditions for the existence of magic graphs in the sense of Sedlacek were given in [12]. Stewart [17,18] considered the case with $L(E) = \{1, 2, \dots, q\}$ or a set of consecutive integers, and Doob [2,3,4] considered the assignment into an Abelian group and an integral domain. Recently, Lee, Seah and Tan [13] considered *edge-magic* labeling with $L(E) = \{1, 2, \dots, q\}$ such that for each vertex v of G the sum of the labels of all edges incident to v modulo p are identical.

Chartrand et al. [1] considered a problem quite different from magic labeling. They required the assignment L with property that for distinct vertices u and v, the sums of the labels of all edges incident to u and incident to v are distinct. Denote the set of all such mappings by $Irr(G)$, they called $s(L) = \max\{L(e)\colon e \text{ in } E\}$ the *strength* of a labeling L, and $s(G) = \min\{s(L)\colon L \text{ in } Irr(G)\}$ the *irregular strength* of a graph G. The value of $s(G)$ has been determined for several graphs in [5,6,7].

We consider in this paper the set $M(G)$ of magic assignments L from E to $N = \{1, 2, 3, \dots\}$, where all the vertices have identical sum. The *magic strength* $m(G)$ of G is defined to be the $\min\{s(L)\colon L \text{ in } M(G)\}$. We determine magic strengths of certain graphs in this paper and also introduce some constructions of magic graphs. We show that any graph G with any assignment f on G can always be embedded into a magic graph G^* which contains G as an induced subgraph.

2 Some Simple Observations

The following observations are immediate.

Observation 1: A graph G is regular if and only if $m(G) = 1$. If a graph G does not admit any magic labeling, then we set $m(G) = 0$.

Observation 2: All connected graphs G which have vertices of degree 1 except K_2 have magic strengths 0.

Observation 3: Given a pair (G, f), where G is a (p, q)-graph and f is a magic assignment of G. We can associate a $p \times p$ matrix $A(f)$, which is related to the adjacency matrix of G, such that $A(f)$ has value $f((v_i, v_j))$ at the (i, j)-position if (v_i, v_j) in $E(G)$ and 0 otherwise. The matrix $A(f)$ has identical row sums and column sums which is equal to the index of f.

Observation 4. Given a graph G, the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [16]. In general, this is a very difficult problem in number theory. At present, no efficient algorithm is known for finding magic labelings for general graphs.

3 Magic strengths of some graphs

Hartnell and Kocay [8] introduced a class of graphs which are formed by stars. For each $k > 1$, they take two isomorphic copies of star $St(k)$ with $k+1$ vertices and connect k pairs of corresponding leaf vertices with edges. The resulting graph is called a double star $DS(k)$. We have

Theorem 1. *For $k > 1$, the magic strength of the double k-star $DS(k)$ is $k-1$.*

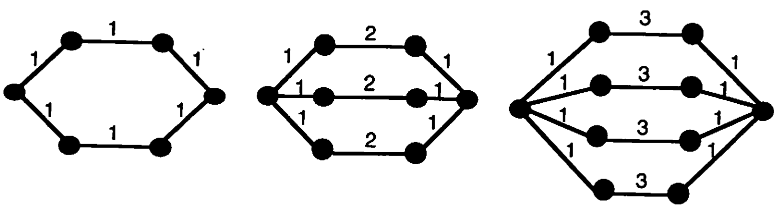

Figure 1. Magic labelings for $DS(2)$, $DS(3)$ and $DS(4)$.

A fan $F(k)$ with k spokes is the graph $P_k + K_1$.

Theorem 2. *For $k \geq 1$, the magic strength of the fan $F(k)$ is 1 if $k = 1$ or 2; it is k if k is even and greater than 2, and it is 0 if k is odd and greater than 1.*

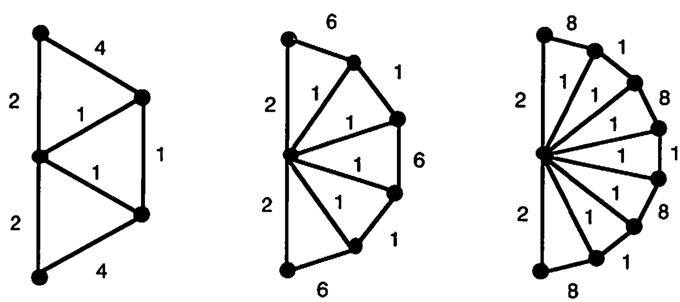

Figure 2. Magic labelings for $F(4)$, $F(6)$ and $F(8)$.

A wheel $W(k)$ with k spokes is the graph $C_k + K_1$.

Theorem 3. *For $k \geq 3$, the magic strength of the wheel $W(k)$ is $k/2$ if k is even and $(k-1)/2$ if k is odd.*

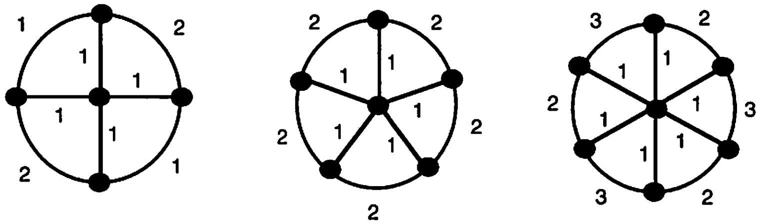

Figure 3. Magic labelings for $W(4)$, $W(5)$ and $W(6)$.

Theorem 4. *The magic strength of the grid graph $P_m \times P_n$ is given by*

$$m(P_m \times P_n) = \begin{cases} 1, & \text{if } m = n = 2, \\ 2, & \text{if } m = 2 \text{ and } n > 3 \text{ or vise versa,} \\ 3, & \text{if } m \text{ or } n \text{ is even and greater than } 2. \end{cases}$$

We illustrate this result with the following examples in Figure 4 and Figure 5.

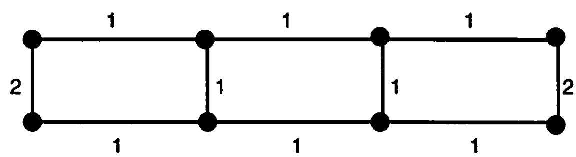

Figure 4. A magic labeling for $P_2 \times P_4$.

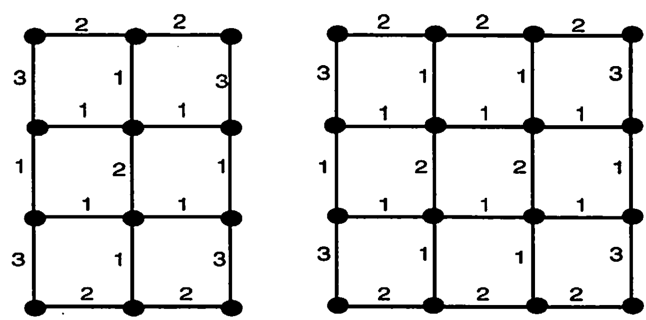

Figure 5. Magic labelings for $P_4 \times P_3$ and $P_4 \times P_4$.

For any $n \geq 1$, we identify n copies of K_3 with a common vertex, the resulting graph is called a Dutch windmill $K(3, n)$ or a friendship graph.

Theorem 5. *The magic strength of Dutch windmill $K(3, n)$ is 1 if n is 1 and $2n - 1$ for $n > 1$.*

4 Construction of magic graphs

In this section we give some methods of construction of magic graphs. We show that every finite graph is an induced subgraph of a magic graph and from which we derived that there is no Kuratowski type of characterization of magic graphs. For a given pair (G, f), where f is an assignment of G and v is a vertex in G, we denote the value of the sum of all integers assign to edges incident to v by $f^+(v)$.

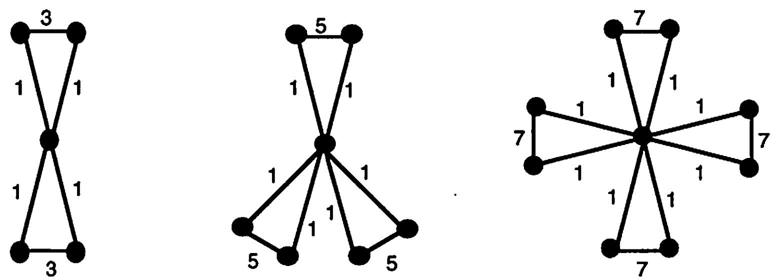

Figure 6. Magic labelings for $K(3,2)$, $K(3,3)$ and $K(3,4)$.

Theorem 6. *If G_1 and G_2 are magic graphs, then $G_1 \times G_2$ is also magic with $m(G_1 \times G_2) = \max\{m(G_1), m(G_2)\}$.*

Proof: Let f_i be a magic labeling of G_i such that $m(G_i) = s(f_i)$ for $i = 1, 2$. Assume $V(G_1) = \{v_1, v_2, \ldots, v_a\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_b\}$. We define an assignment $F\colon E(G_1 \times G_2) \to N$ as follows:

$$F(\{(v_1, u_1), (v_2, u_2)\}) = \begin{cases} f_1((v_1, v_2)), & \text{if } u_1 = u_2, \\ f_2((u_1, u_2)), & \text{if } v_1 = v_2. \end{cases}$$

We see that if G_i has index t_i under f_i, $i = 1, 2$, then the cartesian product $G_1 \times G_2$ has magic index $t_1 + t_2$ under the assignment F.

Theorem 7. *Given any pair (G, f) of a graph G, not necessarily connected, with assignment $f\colon E(G) \to N$, there exists a graph G^* and assignment f^* of G^* such that G^* contains G as an induced subgraph and f^* is a magic extension of f.*

Proof: Let (G, f) be a given pair with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Consider the graph $G^* = G \times K_2$ with $V(G \times K_2) = \{(v_1, a), (v_2, a), \ldots, (v_p, a), (v_1, b), (v_2, b), \ldots, (v_p, b)\}$. We have $((v_i, x), (v_j, y))$ in $E(G \times K_2)$ if and only if (v_i, v_j) in $E(G)$ and $x = y$, or $x = a$, $y = b$, and $v_i = v_j$. Assume that $t = \max\{f^+(v_1), \ldots, f^+(v_p)\}$. We define an assignment f^* on $E(G^*)$ as follows:

$$f^*(\{(v_i,x),(v_j,y)\}) = \begin{cases} f((v_i,v_j)), & \text{if } x=y, \\ t-f^+(v_i)+1, & \text{if } v_i=v_j \text{ and } x=a, y=b. \end{cases}$$

We see that (G^*, f^*) is a magic extension of (G, f) with index $t+1$.

Corollary 8. *Every connected graph G is an induced subgraph of a magic graph with magic strength $\leq \Delta(G)-\delta(G)+1$, where $\Delta(G)$ is the maximum degree, and $\delta(G)$ is the minimum degree, of G.*

Proof: For a given graph G, let $f\colon E(G) \to N$ be the mapping $f(e) = 1$ for all e in $E(G)$. From the construction of Theorem 7, it implies that the magic strength of G^* is less than or equal to $\Delta(G) - \delta(G) + 1$.

For a pair (G, f) where f is an assignment of G, we say that a pair $(G+K_1, g)$ is an *one-vertex extension of* (G, f) if $g|_{E(G)} = f$. The following is a necessary and sufficient condition for an one-vertex extension of (G, f) to be magic.

Theorem 9. *For any (p,q)-graph G and any assignment $f\colon E(G) \to N$ with $a_i = f^+(v_i)$ for $i = 1, 2, \ldots, p$. The one-vertex extension $G + K_1$ is magic if and only if there exists p nonzero natural numbers $b_1, \ldots, b_p$ such that $a_1 + b_1 = a_2 + b_2 = \cdots = a_p + b_p = b_1 + b_2 + \cdots + b_p$.*

Corollary 10. *If a (p,q)-graph G with $p > 2$ has an assignment $f^+(v_i) = 1$ for some i, then its one-vertex extension is not magic.*

Corollary 11. *The fan $F(3) = P_3 + K_1$ has magic strength* 0.

Proof: Let $f\colon E(P_3) \to N$ be any assignment with $f^+(v_1) = a_1$, $f^+(v_2) = a_1 + a_2$ and $f^+(v_3) = a_2$. By Theorem 9, the one-vertex extension $(F(3), g)$ will provide a system of equations: $a_1 + b_1 = a_2 + b_2 = a_1 + a_2 + b_3 = b_1 + b_2 + b_3$, from which we deduce that $b_3 = 0$, which is impossible.

5 Open Problem

We conclude with a conjecture.

Conjecture. Almost all connected graphs are not magic.

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