The Number of Orientations of a Tree Admitting an Efficient Dominating Set

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ABSTRACT. We give recursive methods for enumerating the number of orientations of a tree which can be efficiently dominated. We also examine the maximum number η_q of orientations admitting an efficient dominating set in a tree with q edges. While we are unable to give either explicit formulas or recursive methods for finding η_q, we are able to show that the growth rate of the sequence $\langle\eta_q\rangle$ stabilizes by showing that $\lim_{q\to\infty}\eta_q^{1/q}$ exists.

1 Introduction

For a directed graph D with vertex set V and arc set A, the *out-neighborhood* $\vec{N}_D(v)$ of a vertex v of D is $\{w \in V : \vec{vw} \in A\}$ while the *closed out-neighborhood* $\vec{N}_D[v]$ is $\vec{N}_D(v) \cup \{v\}$. We call a subset S of the vertices of D an *efficient dominating set* if $\{\vec{N}_D[v] : v \in S\}$ partitions the vertices of D. An *efficiency* of a graph G is a pair $(\vec{G}, S)$ where S is an efficient dominating set of the orientation $\vec{G}$ of G. The number of efficiencies, as pairs, of G is denoted by $\eta(G)$. We have shown [4] that every graph has at least one efficiency, found the extremal graphs which attain the maximum number and minimum number of efficiencies for graphs of given order and graphs of given size, and observed that any oriented tree has at most one efficient dominating set. See [1, 2, 3, 7] for results regarding efficient domination of graphs and [5, 6, 8] for additional results regarding efficient domination of directed graphs.

In this paper we give recursive methods for enumerating the number $\eta(T)$ of efficiencies of a labeled tree T. Since an oriented tree T has at most one efficient dominating set, $\eta(T)$ counts the number of orientations of T which can be efficiently dominated. We are also interested in the extreme value of $\eta(T)$ and define η_q to be the maximum of $\eta(T)$ among all trees with q edges. It follows that η_q enumerates the maximum number of orientations admitting an efficient dominating set in a tree with q edges. While we are unable to give either explicit formulas or recursive methods for finding η_q, we are able to show that the growth rate of the sequence $\langle \eta_q \rangle$ stabilizes by showing that $\lim_{q\to\infty} \eta_q^{1/q}$ exists.

2 The Number of Orientations of a Tree Admitting an Efficient Dominating Set

Let G be a labeled graph. For any vertex v in G, define the ordered triple (a, b, c) as follows:

$a =$ the number of efficiencies of G which include v in the dominating set (dominated using v).

$b =$ the number of efficiencies of the graph $G + u$ that include u in the dominating set, where u is a new vertex of degree one adjacent to v and edge uv is directed from u to v (dominated from outside).

$c =$ the number of efficiencies of G in which v is dominated by a neighbor in G (dominated from inside).

Observation. Note that for any graph G and any vertex v with triple (a, b, c), $\eta(G) = a + c$.

For example, the path P_1 of length one has triple $(1,1,1)$ at either end and $\eta(P_1) = 2$; K_3 has triple $(2,4,4)$ at any vertex and $\eta(K_3) = 6$. We also note that for any vertex of any graph, $a \geq 1$, $b \geq 1$, $c \geq 0$, and for nontrivial graphs $c \geq 1$. We next show how to compute triples, hence η, recursively for certain graphs including trees.

Theorem 1 (Merging theorem). *Let G_1 have triple (a_1, b_1, c_1) at vertex v_1 and let G_2 have triple (a_2, b_2, c_2) at vertex v_2. Form a separable graph G by merging v_1 and v_2 into a single vertex v. Then the triple for G at v is $(a_1a_2, b_1b_2, b_1c_2 + c_1b_2)$ so that $\eta(G) = a_1a_2 + b_1c_2 + c_1b_2$.*

Proof: Any pair $(\vec{G}, S)$ is an efficiency which uses v if and only if the restriction of $(\vec{G}, S)$ to G_1 and G_2 produces efficiencies which use v_1 and v_2, respectively. Thus $a = a_1a_2$. Similarly $b = b_1b_2$ since v can be dominated from outside (in G) if and only if v_1 can be dominated from outside in G_1 and v_2 can be dominated from outside in G_2. To show $c = b_1c_2 + c_1b_2$, note that in any efficiency of G which does not use v, v must be dominated by exactly one neighbor in G, either from a vertex in G_1 (hence outside G_2), or from a vertex inside G_2 (hence outside G_1). □

Theorem 2 (Lifting theorem). *Let G' be a graph with triple (a', b', c') at vertex v. Form a graph G by joining a vertex u of degree one to vertex v. Then the triple for G at u is $(b' + c', a' + 2c', a')$ and $\eta(G) = a' + b' + c'$.*

Proof: Let the triple for G at u be given by (a, b, c). To efficiently dominate G using vertex u, we must either direct edge uv from u to v, in which case there are b' ways to dominate G using u, or direct edge uv from v to u, in which case there are c' ways to dominate G using u. Thus $a = b' + c'$. Similarly, if edge uv is directed from u to v there are $a'+c'$ ways to dominate G efficiently from outside at u, while if edge uv is directed from v to u there are c' ways to do it. Thus $b = a' + 2c'$. The only way to dominate u from inside is to direct edge uv from v to u and include vertex v in the dominating set. Thus $c = a'$. □

In what follows, we find it convenient to let P_q denote the path of length q. As simple examples of the preceding theorems, note that by lifting $P_1(1,1,1)$ we see P_2 has triple $(2,3,1)$ at either end, and by merging two copies of $P_1(1,1,1)$ we see P_2 has triple $(1,1,2)$ at its center. We next give an asymptotic formula for $\eta(P_q)$.

Corollary 3. *For $q \geq 3$, $\eta(P_q) = 2\eta(P_{q-2}) + 2\eta(P_{q-3})$ where $\eta(P_0) = 1$, $\eta(P_1) = 2$ and $\eta(P_2) = 3$. Hence, $\eta(P_q) \sim ar^q$ where $a \doteq .58644$ and $r \doteq 1.76929$ is the real root of $x^3 - 2x - 2$.*

Proof: Let (a_q, b_q, c_q) be the triple for P_q at an endvertex. By the lifting theorem, $a_q = b_{q-1} + c_{q-1}$, $b_q = a_{q-1} + 2c_{q-1}$ and $c_q = a_{q-1}$. For $q \geq 3$, we have

$$\begin{aligned}\eta(P_q) &= b_{q-1} + c_{q-1} + a_{q-1} = a_{q-2} + 2c_{q-2} + a_{q-2} + b_{q-2} + c_{q-2} \\ &= 2\eta(P_{q-2}) + b_{q-2} + c_{q-2} = 2\eta(P_{q-2}) + a_{q-3} + 2_{q-3} + a_{q-3} \\ &= 2\eta(P_{q-2}) + 2\eta(P_{q-3})\end{aligned}$$

Hence, using standard methods for solving linear recurrence relations, $\eta(P_q) = ar^q + bs^q + ct^q$ where r, s, t are the roots of $x^3 - 2x - 2$ with r real. Then $a \doteq .58644$ and $v > |s|$, $|t|$ so that $\eta(P_q) \sim ar^q$. □

In [5], both the maximal trees and η_q were determined for all $q \leq 69$. Examination of this data showed that the sequence $\langle \eta_q^{1/q} \rangle$ behaved erratically and, in particular, was not monotonic. Subsequently, we showed that the sequence $\langle \eta_q \rangle$ did not grow too quickly and was supermultiplicative (see Lemmas 6, 7). Some time after proving the next theorem we discovered that it is a slight extension of a result of Fekete [9] (see also [10; p. 85]). This result also plays an important role in Moon's proof [11; p. 27] of a result of Szele [14]; the existence of the Shannon capacity of a graph [13]; and the existence of certain parameters concerning permanents [12].

Theorem 4. [9]. *Let $\langle a_n \rangle$ be a sequence of positive real numbers and let m be a nonnegative integer. If*

(1) there exists a positive constant c such that $a_{n-1} \le a_n \le c a_{n-1}$,

(2) for all positive integers i and j, $a_{i+j+m} \ge a_i a_j$,

then $\lim_{n\to\infty} a_n^{1/n}$ exists and is finite.

Remark. It is easily seen that condition (2) is sufficient to ensure both that $\lim_{n\to\infty} a_n^{1/n}$ exists, though possibly infinite, and that $\lim_{n\to\infty} a_n^{1/n} \ge a_{d-m}^{1/d}$ for any fixed integer $d > m$. We note that while conditions (1), (2) imply that $\lim_{n\to\infty} a_n^{1/n}$ exists and is a finite real number L they do not imply that $a_n \sim L^n$. For example, fix $L \in (1,3)$ and let $a_n = L^n(1 - n^{-2/3})^n$ for all large n and $a_n = 1$, otherwise. Then conditions (1), (2) hold with $m = 0$ (so $m = 1$ since nondecreasing) and $c = 3$ but $a_n = o(L^n)$.

A tree having q edges will be denoted by T_q. If $\eta(T_q)$ equals η_q we call T_q *maximal* and denote it by T_q^*. We next show that the sequence $\langle \eta_q \rangle$ satisfies the two conditions of Theorem 4 and, hence, $\langle \eta_q^{1/q} \rangle$ has a finite limit as $q \to \infty$.

Lemma 5. *For any tree with at least one edge, there exists a vertex w such that the triple (a,b,c) at w has $c \ge a$.*

Proof: Let u be a leaf of a tree T and let v be the neighbor of u in T. Let T' be the tree obtained by deleting u from T. If the triple for T' at v is (a',b',c'), then by merging T' and the path $P_1(1,1,1)$ the triple for T at v is $(a',b',b'+c')$, while by lifting T' the triple for T at u is $(b'+c',a'+2c',a')$. If $a' \ge b'+c'$, let $w = u$; otherwise, let $w = v$. □

Lemma 6. *For $q \ge 1$, $\eta_{q-1} < \eta_q < 3\eta_{q-1}$.*

Proof: For the first inequality, assume that T_{q-1}^* is a maximal tree with triple (a,b,c) at a vertex. Then, lifting T_{q-1}^* from this vertex gives a tree T_q and, since b is positive, $\eta_{q-1} = a + c < a + b + c = \eta(T_q) \le \eta_q$.

For the second inequality, assume that T_q^* is a maximal tree with $q \ge 2$. Any tree with at least two edges must contain either two leaves with a common neighbor or a suspended path of length two ending at a leaf. In the first case, T_q^* has been formed by merging some tree T_{q-2} having triple (a,b,c) at vertex v with the center of a path $P_2(1,1,2)$, so that $\eta_q = a+2b+c$. In the second case, we have a similar merger with an end of a path $P_2(2,3,1)$, so that $\eta_q = 2a+b+3c$. In either case, $\eta_q < 3a+3b+3c$. Form a tree T_{q-1} by merging a path $P_1(1,1,1)$ with T_{q-2} at v. By the merging theorem, $\eta(T_{q-1}) = a+b+c$. Thus, we have in either case, $\eta_q < 3a+3b+3c = 3\eta(T_{q-1}) \leq 3\eta_{q-1}$. □

Lemma 7. *If i and j are positive integers, then $\eta_{i+j+1} > \eta_i\eta_j$.*

Proof: Let T_i^* be a maximal tree. By Lemma 5, we may choose vertex v_i in T_i^* with triple (a_i, b_i, c_i) so that $c_i \geq a_i$. Similarly, choose vertex v_j in a disjoint maximal tree T_j^* with triple (a_j, b_j, c_j) so that $c_j \geq a_j$. Then $\eta_i\eta_j = (a_i+c_i)(a_j+c_j)$. Construct tree T_{i+j+1} from T_i^* and T_j^* by adding the edge v_iv_j. From the lifting and merging theorems, we obtain $\eta(T_{i+j+1}) = a_i(b_j+c_j) + b_ia_j + c_i(a_j+2c_j) > a_ic_j + c_ia_j + 2c_ic_j \geq (a_i + c_i)(a_j+c_j)$. Thus $\eta_{i+j+1} \geq \eta(T_{i+j+1}) > \eta_i\eta_j$. □

Theorem 8. $\lim_{q\to\infty} \eta_q^{1/q}$ *exists and is finite.*

Proof: Lemmas 6 and 7 show that the sequence $\langle n_q \rangle$ satisfies the conditions of Theorem 4 with $m = 1$, and the result follows. □

Remark. In [4], we showed that the minimum of $\eta(G)$ among all graphs G, and hence among trees, with q edges is attained solely by the star. The remark after Theorem 4 implies that the limit L guaranteed by Theorem 8 satisfies $L \geq \eta_{69}^{1/70} > 1.8488$. Consequently, paths and stars are far from the maximal trees, unlike many properties of trees. This answers a question posed in a conversation with Paul Erdös.

3 Conclusion

The limit L guaranteed by Theorem 8 is the growth rate of the sequence $\langle \eta_q \rangle$. We would like to know the value of L. We have done considerable work [5] on forbidden structures in maximal trees, but as yet have no structural characterization for maximal trees. Using known forbidden structures, we have generated the maximal trees for $q \leq 69$ and these are cataloged in the accompanying figure and table. The apparent patterns in the table cannot continue: all known maximal trees have diameter at most six, but in [5] we prove that the diameter of maximal trees goes to infinity with q. We do not know whether anomalies such as T_{26}^* reoccur for large values of q.

A lengthy analysis in [5] shows that any maximal tree T_{q+6}^* having $q+6 \geq 6$ edges is obtained by merging a vertex of some tree T_q having q edges with the center of a fan having 6 edges. Merging a path of length one with that vertex of T_q produces a tree T_{q+1} having $q+1$ edges where $\eta_{q+6} = \eta(T_{q+6}^*) \leq 27\eta(T_{q+1}) \leq 27\eta(T_{q+1}^*) = 27\eta_{q+1}$ and, hence, $L \leq 27^{1/5}$. In addition, merging k copies of the palm having fifteen edges at the endvertex adjacent to its center gives an infinite family of trees which imply $L \geq 10334^{1/15}$. Although we are able to slightly improve both bounds given above, neither improvement is optimal and, hence, we will not give the details here (see [5] for details).

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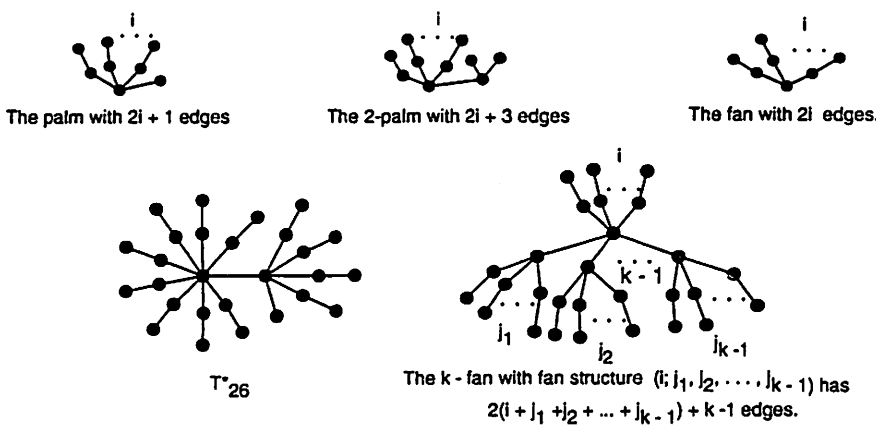

Figure 1. Maximal trees

Tree Type	Number of edges (and fan structure)
Palms	3, 5, 7, 9, 11
2-palms	13, 15
Fans	2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24
T^*_{26}	26
2-fans	17(4;4), 19(5;4), 21(5;5), 23(6;5), 25(6;6), 27(7;6), 29(7;7), 31(8;7), 33(8;8), 35(9;8), 37(9;9), 39(10;9), 41(10;10)
3-fans	28(4;5,4), 30(5;5,4), 32(5;5,5), 34(6;5,5), 36(6;6,5), 38(6;6,6), 40(7;6,6), 42(7;7,6), 44(7;7,7), 46(8;7,7), 48(8;8,7), 50(8;8,8), 52(9;8,8), 54(9;9,8), 56(9;9,9)
4-fans	43(5;5,5,5), 45(6;5,5,5), 47(6;6,5,5), 49(6;6,6,5), 51(6;6,6,6), 53(7;6,6,6), 55(7;7,6,6), 57(7;7,7,6), 59(7;7,7,7), 61(8;7,7,7), 63(8;8,7,7), 65(8;8,8,7), 67(8;8,8,8), 69(9;8,8,8)
5-fans	58(6;6,5,5,5), 60(6;6,6,5,5), 62(6;6,6,6,5), 64(6;6,6,6,6), 66(7;6,6,6,6), 68(7;7,6,6,6)

Table 1. Maximal Trees with q edges

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