A Local Neighbourhood Condition for Cycles

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ABSTRACT. Let G be a connected graph with $\nu \geq 3$. Let $v \in V(G)$. We define $N_k(v) = \{u|u \in V(G) \text{ and } d(u,v) = k\}$. It is proved that if for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S| + 1$, then G is hamiltonian. Several previously known sufficient conditions for hamiltonian graphs follow as corollaries. It is also proved that if for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S| + 2$, then G is pancyclic.

1 Introduction and terminology

All graphs considered are finite, undirected, connected and simple.

Let G be a connected graph. Let v be a vertex in $V(G)$. We define $N_k(v) = \{u|u \in V(G) \text{ and } d(u,v) = k\}$. When $k = 1$, $N_1(v) = N(v)$. For a pair of vertices u and v of G, we use $I(u,v)$ to denote $|N(u) \cap N(v)|$. Let u and v be two vertices of G such that $d(u,v) = 2$. We define the *divergence* $\alpha^*(u,v)$ as follows: $\alpha^*(u,v) = \max_w\{|S|$ | for each $w \in N(u) \cap N(v)$, S is a maximum independent set in $N(w)$ containing u and $v\}$.

A graph G is said to be *pancyclic* if G has a cycle of length n for each n such that $3 \leq n \leq \nu(G)$.

Let C be a cycle of G. Let u be a vertex in $V(C)$. We give C an orientation. Then $u^+(C)$ denotes the successor of u on C in the orientation and $u^-(C)$ denotes the predecessor of u on C in the orientation. Let $S \subseteq V(C)$. Then $S^+(C) = \{x^+(C)|x \in S\}$ and $S^-(C) = \{x^-(C)|x \in S\}$. When there is no confusion about C, we simply write u^+, u^-, S^+ and S^- for $u^+(C)$, $u^-(C)$, $S^+(C)$ and $S^-(C)$. Let v be a vertex in $V(G)\backslash V(C)$. $N_C(v)$ denotes $N(v) \cap V(C)$. Suppose $N_C(v) \neq \emptyset$. An *A-structure* on $N_C(v)$ is a pair of vertices x and y such that $x, y \in N_C(v)$ and $x^+ = y$. Let $S \subseteq V(C)$.

A *suc-J-structure* on S is an edge x^+y^+ such that $x, y \in S$, $x^+ \neq y$ and $y^+ \neq x$. A *pre-J-strucure* on S is an edge x^-y^- such that $x, y \in S$, $x^- \neq y$ and $y^- \neq x$. Both suc-J-structures and pre-J-structures are called *J-structures*. Because of the obvious similarity between suc-J-structures and pre-J-structures, and for ease of notation and presentation, we frequently give proofs only using suc-J-structures (or pre-J-structures). Let $u, v \in V(C)$. We denote by $C^+[u, v]$ the path on C from u to v in the orientation and by $C^-[u, v]$ the path on C from u to v in the reverse orientation. Two J-structures uv and xy on a set S are said to be *independent* if $\{u, v\} \cap \{x, y\} = \emptyset$. Two independent J-structures uv and xy on a set S are said to be *crossed* if u is on $C^+[x, y]$ and v is on $C^+[y, x]$. If two independent J-structures on a set are not crossed, we say they are noncrossed.

Let G and H be two graphs such that $E(G) \cap E(H) = \emptyset$. We use $G + H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let $S \subseteq V(G)$. We use $G[S]$ to denote the induced subgraph of G on S.

For terminology and notation not defined in this paper, the reader is referred to [2].

Since Hasratian and Khachatrian [3] obtained the first local condition for hamiltonian graphs, some graph theorists have proposed different kinds of local conditions for hamiltonian graphs. Shi [4] introduced the concept of divergence and gave a condition for hamiltonian graphs using the concept of divergence. The Shi condition implies many known sufficient conditions for hamiltonian graphs. In [1], Aldred, Holton, Lou and Shi proved that under the Shi condition a graph is pancyclic or $K_{n,n}$. In this paper, we give a new local neighbourhood condition for hamiltonian graphs which implies the Shi condition. We conjecture that under this condition a graph is pancyclic or $K_{n,n}$. We then prove a result that under a little stronger condition a graph is pancyclic. In Section 3, we list some known results which our theorem implies.

2 Hamiltonicity and pancyclicity

First, we give a new sufficient condition for hamiltonian graphs.

Lemma 1. *Let G be a connected graph, C be a cycle of G and u be a vertex in $V(G) \setminus V(C)$. If there is an A-structure on $N_C(u)$ or there is a J-structure on $N_C(u)$, then there is a cycle C' in G of length $|V(C)| + 1$ such that $V(C') = V(C) \cup \{u\}$.*

Theorem 2. *Let G be a connected graph with $\nu \geq 3$. If for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S| + 1$, then G is hamiltonian.*

Proof: Suppose G is not hamiltonian. Let C be a longest cycle of G. Give C an orientation. Let $u \in V(G) \setminus V(C)$ such that $N_C(u) \neq \emptyset$.

By Lemma 1, there is no A-structure on $N_C(u)$ and no suc-J-structure on $N_C(u)$ otherwise G has a cycle C' longer than C, a contradiction. Now $T = N_C^+(u)$ is an independent set and $T \subseteq N_2(u)$. By the hypothesis of this theorem, $|N(T) \cap N(u)| \geq |T| + 1 = |N_C(u)| + 1$. Then there are two vertices $w \in N(u) \setminus V(C)$ and $v^+ \in N_C^+(u)$ such that $wv^+ \in E(G)$. So G has a cycle C' such that $V(C') = V(C) \cup \{u, w\}$, a contradiction. □

The next lemma shows that under the hypotheses of Theorem 2, a graph has a triangle and a 4-cycle, or the graph is $K_{n,n}$.

Lemma 3. *Let G be a connected graph with $\nu \geq 4$. Suppose for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S| + 1$. Then G has a triangle unless G is $K_{\nu/2,\nu/2}$ and also has a 4-cycle.*

Proof: Suppose G satisfies the hypotheses of this lemma. If G is a complete graph, then the lemma holds. So suppose G is not a complete graph. Hence there are two vertices u and v in G such that $d(u, v) = 2$. But $\{u\}$ is an independent set in $N_2(v)$. So $|N(u) \cap |N(v)| \geq 2$ and then G has a 4-cycle.

Suppose G has no triangle. First, we claim that G is a regular graph. Let uv be an edge of G. Then $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ are independent sets. But $N(u) \setminus \{v\} \subseteq N_2(v)$. By the hypothesis of this lemma, there are at least $|N(u) \setminus \{v\}| + 1 = |N(u)|$ vertices in $N(v)$. So $|N(u)| \leq |N(v)|$. By a symmetric argument, we also have $|N(u)| \geq |N(v)|$. Hence $|N(u)| = |N(v)|$. Since G is a connected graph, G is a k-regular graph for $k = d(u)$.

Under the assumption that G has no triangle, we now prove that G is $K_{\nu/2,\nu/2}$ by the assumption that G has no triangle. Let $v \in V(G)$, u and w be two distinct vertices in $N(v)$. Suppose $(N(u)\setminus\{v\})\setminus(N(w)\setminus\{v\}) \neq \emptyset$. As $N(w) \setminus \{v\}$ is an independent set of order $k-1$ in $N_2(v)$ and $|N(v)| = k$, by the hypothesis of the lemma, $\{x\} \cup (N(w) \setminus \{v\})$ is not independent for each vertex $x \in (N(u)\setminus\{v\})\setminus(N(w)\setminus\{v\})$. But G has no triangle, x is adjacent to a vertex in $(N(w)\setminus\{v\})\setminus(N(u)\setminus\{v\})$. And $N((N(u)\setminus\{v\})\setminus(N(w)\setminus\{v\}))\cap N(w) \subseteq (N(w) \setminus \{v\}) \setminus (N(u) \setminus \{v\})$. But $(N(u) \setminus \{v\}) \setminus (N(w) \setminus \{v\})$ is an independent set in $N_2(w)$. By the hypothesis, $|(N(w)\setminus\{v\})\setminus(N(u)\setminus\{v\})| \geq |N((N(u) \setminus \{v\}) \setminus (N(w) \setminus \{v\})) \cap N(w)| \geq |(N(u) \setminus \{v\}) \setminus (N(w) \setminus \{v\})| + 1$. But, by the k-regularity, $|(N(w)\setminus\{v\})\setminus(N(u)\setminus\{v\})| = |(N(u)\setminus\{v\})\setminus(N(w)\setminus \{v\})|$, a contradiction. Hence for any two distinct vertices $u, w \in N(v)$, $(N(u)\setminus\{v\}) \cap (N(w)\setminus\{v\}) = N(v)\setminus\{v\} = N(w)\setminus\{v\}$. By the k-regularity, G is $K_{\nu/2,\nu/2}$. □

Corollary 4. *Let G be a connected graph with $\nu \geq 4$. Suppose for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S| + 2$. Then G has a triangle and a 4-cycle.*

Proof: When G satisfies the hypotheses of this corollary, G is not $K_{\nu/2,\nu/2}$. By Lemma 3, the corollary follows. □

In light of Lemma 3, we propose the following conjecture.

Conjecture 1: Let G be a connected graph with $\nu \geq 3$. If for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S|+1$, then G is pancyclic or $K_{\nu/2,\nu/2}$.

In the following we prove a little weaker result. First, however, we give some obvious but useful lemmas.

Lemma 5. *Let G be a connected graph, C be a cycle of G and u and v be two different vertices in $V(G) \setminus V(C)$. If there are two independent suc-J-structures (pre-J-structures) on $N_C(u) \cap N_C(v)$, then there is a cycle C' of length $|V(C)|+2$, where $V(C') = V(C) \cup \{u, v\}$.*

Lemma 6. *Let G be a connected graph, C be a cycle of G and u and v be two different vertices in $V(G) \setminus V(C)$. Suppose there is no A-structure on $N_C(u)$ and no A-structure on $N_C(v)$. If there are two noncrossed independent J-structures on $N_C(u) \cap N_C(v)$, then G has a cycle C' of length $|V(C)|+2$, where $V(C') = V(C) \cup \{u, v\}$.*

Theorem 7. *Let G be a connected graph with $\nu \geq 3$. If for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(S) \cap N(v)| \geq |S|+2$, then G is pancyclic.*

Proof: Suppose G satisfies the hypotheses of this theorem but is not pancyclic. By Corollary 6, we assume that m is the minimum number such that $3 \leq m \leq \nu - 2$ and G has a cycle of length m but does not have any cycle of length $m+2$. Note that if $\nu = 3$, then G is a triangle.

Let C be an oriented cycle of length m.

Claim 1: For all u in $V(G) \setminus V(C)$ that are adjacent to some vertex on C, there is no edge from $N_C^+(u) \cup N_C^-(u)$ to $N(u) \setminus V(C)$.

The proof of this claim is straightforward.

By Theorem 2, G is 2-connected and $m \leq \nu - 3$, hence there are two different vertices u and v in $V(G) \setminus V(C)$ each adjacent to vertices on C.

Case 1. There is an A-structure xy on $N_C(u)$ and there is an A-structure wz on $N_C(v)$.

If $|\{x, y\} \cap \{w, z\}| \leq 1$, then we have an $(m+2)$-cycle C' such that $V(C') = V(C) \cup \{u, v\}$, a contradiction. So we may assume that each of $N_C(u)$ and $N_C(v)$ has just one common A-structure, xy say, and no other A-structures.

Subcase (1.1) $uv \in E(G)$.

Then G has an $(m+2)$-cycle C' such that $V(C') = V(C) \cup \{u, v\}$, a contradiction.

Subcase (1.2) $uv \notin E(G)$.

Suppose $y = x^+$. Then there is a suc-J-structure w^+z^+ on $N_C(u) \setminus \{x\}$. Otherwise $T = (N_C(u) \setminus \{x\})$ is an independent set in $N_2(u)$ of order

$|N_C(u)| - 1$. But $N(T) \cap N(u) \subseteq N_C(u)$ and $|N(T) \cap N(u)| \leq |T| + 1$, contradicting the hypothesis. By Lemma 1, there is a cycle C' of length $m+1$ such that $V(C') = V(C) \cup \{u\}$. However, xy is also an A-structure on $N_{C'}(v)$, and then G has an $(m+2)$-cycle, a contradiction.

Case 2. There is an A-structure xy on $N_C(u)$ but no A-structure on $N_C(v)$.

Claim 2. If $w \in V(G) \setminus V(C)$, $N_C(w) \neq \emptyset$ and there is no A-structure on $N_C(w)$, then there are a suc-J-structure and a pre-J-structure on $N_C(w)$ though they may be the same one.

Otherwise $T = N_C^+(w)$ is an independent set in $N_2(w)$. By Claim 1, $N(T) \cap N(w) \subseteq N_C(w)$ and $|N(T) \cap N(w)| \leq |T|$, a contradiction.

This completes the proof of Claim 2.

Assume $y = x^+$. By Clalm 2, there is a suc-J-structure wz on $N_C(v)$. Then $y = w$ or z otherwise $V(C)$, u and v form an $(m+2)$-cycle. By Claim 2, there is a pre-J-structure $w'z'$ on $N_C(v)$. Then $x = w'$ or z' otherwise $V(C)$, u and v form a cycle of length $m+2$. But this means $N_C(v)$ has an A-structure xy, a contradiction.

Case 3. Neither $N_C(u)$ nor $N_C(v)$ have an A-structure.

Claim 3. For any suc-J-structure (pre-J-structure) J on $N_C(u)$, J is on $N_C(u) \cap N_C(v)$ and $uv \notin E(G)$.

This claim takes a little proof. By Claim 2, let x^+y^+ be a suc-J-structure on $N_C(u)$. By Lemma 1, $C' = C^+[y^+, x] + xuy + C^-[y, x^+] + x^+y^+$ is an $(m+1)$-cycle. We give C' an orientation such that C' and C have the same orientation on $C^+[y^+, x]$.

If $N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\} = \emptyset$, then $(N_{C'}(v) \setminus \{u\})^+(C') \subseteq N_C^+(v) \cup N_C^-(v)$ and by Claim 1 there is no edge from $(N_{C'}(v) \setminus \{u\})^+(C')$ to $N(v) \setminus V(C')$. If $(N_{C'}(v) \setminus \{u\})^+(C')$ is an independent set, since $T = (N_{C'}(v) \setminus \{u\})^+(C') \subseteq N_2(v)$ and $N(T) \cap N(v) \subseteq N_C(v)$, $|N(T) \cap N(v)| \leq |N_C(v)| = |T|$, contradicting the hypothesis. So there is a suc-J-structure on $N_{C'}(v)$. By Lemma 1, G has a cycle of length $|V(C')| + 1 = m+2$, a contradiction.

Now we have $N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\} \neq \emptyset$. Then $uv \notin E(G)$, otherwise either we have a contradiction to Claim 1 or we have an $(m+2)$-cycle C'' such that $V(C'') = V(C) \cup \{u, v\}$. Note that $N_{C'}(v) = N_C(v)$. If $|N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\}| \leq 1$, then either $N_{C'}^+(v) \subseteq N_C^+(v) \cup N_C^-(v)$ or $N_{C'}^-(v) \subseteq N_C^+(v) \cup N_C^-(v)$. By the same argument as above, there is either a suc-J-structure on $N_{C'}(v)$ or a pre-J- structure on $N_{C'}(v)$ and by Lemma 1, G has an $(m+2)$-cycle, a contradiction. Hence $|N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\}| \geq 2$. As there is no A-structure on $N_C(v)$, we have only four subcases to discuss.

Subcase (3.1) $N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\} = \{x, y^+(C)\}$.

Let $y^+(C) = z$. Then $z^-(C') = x^+(C)$. By Claim 1, there is no egde from $x^+(C) = z^-(C')$ to $N(v) \setminus V(C) = N(v) \setminus V(C')$. Also $(N_{C'}(v) \setminus$

$\{y^+(C)\})^-(C') \subseteq N_C^+(v) \cup N_C^-(v)$. By Claim 1, there is no edge from $N_{C'}^-(v)$ to $N(v) \setminus V(C')$. By the hypothesis of this theorem, there is a pre-J-structure on $N_{C'}(v)$. By Lemma 1, G has a cycle of length $|V(C')|+1 = m+2$, a contradiction.

Subcase (3.2) $N_{C'}(v) \cap \{x.x^+(C), y, y^+(C)\} = \{x^+(C), y\}$.

Let $x^+(C) = z$. Then $z^+(C') = y^+(C)$. By Claim 1, there is no edge from $y^+(C) = z^+(C')$ to $N(v) \setminus V(C) = N(v) \setminus V(C')$. Also $(N_{C'}(v) \setminus \{x^+(C)\})^+(C') \subseteq N_C^+(v) \cup N_C^-(v)$. By Claim 1, there is no edge from $N_{C'}^+(v)$ to $N(v) \setminus V(C')$. By the hypothesis of this theorem, there is a suc-J-structure on $N_{C'}(v)$. By Lemma 1, G has a cycle of length $|V(C')|+1 = m+2$, a contradiction.

Subcase (3.3) $N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\} = \{x^+(C), y^+(C)\}$.

Then $C'' = C^+[y^+(C), x] + xuy + C^-[y, x^+(C)] + x^+(C)vy^+(C)$ is an $(m+2)$-cycle, a contradiction.

Hence there remains the following case which is stated by Claim 3.

Subcase (3.4) $N_{C'}(v) \cap \{x, x^+(C), y, y^+(C)\} = \{x, y\}$.

Claim 4. Either there are two independent suc-J-structures on $N_C(u)$ or there is a triangle in $G[N_C^+(u)]$.

Otherwise there is an independent set $S \subseteq N_C^+(u)$ of order at least $|N_C(u)| - 1$. But $S \subseteq N_2(u)$ and $|N(S) \cap N(u)| \leq |N_C(u)| \leq |S| + 1$, contradicting the hypothesis and completing the proof of the claim.

Subcase (3.4.1) There are two independent suc-J-structures on $N_C(v)$.

By Claim 3, the two suc-J-structures are on $N_C(u) \cap N_C(v)$. By Lemma 5, G has an $(m+2)$-cycle, a contradiction.

Subcase (3.4.2) There is a triangle $x^+y^+z^+x^+$ in $G[N_C^+(u)]$.

Assume x is followed by y on C which is followed by z on C. By Claim 2, there is a pre-J-structure a^-b^- on $N_C(u)$. By Claim 3 x^+y^+, y^+z^+, x^+z^+ and a^-b^- are on $N_C(u) \cap N_C(v)$. By Lemma 6, a^-b^- shares a vertex with or crosses each of x^+y^+, y^+z^+, z^+x^+. Without loss of generality, assume that $a^- = x^+$ and a^-b^- crosses y^+z^+. Then $C' = C^-[x, z^+] + z^+x^+y^+ + C^+[y^+, z] + zvy + C^-[y, a] + aux$ is a cycle of length $m+2$, a contradiction.

This final contradiction completes the proof of Theorem 7. □

3 Some consequential results of Theorem 2

Although the proof of Theorem 2 is very simple, it turns out to ba a strong result. In this section, we show some known sufficient conditions for hamiltonian graphs which follow as corollaries of Theorem 2. The first corollary was due to Shi [4].

Corollary 8. *Let G be a connected graph with $\nu \geq 3$. If for each pair of* *vertices u and v a distance 2 apart, $I(u,v) \geq \alpha^*(u,v)$, then G is hamiltonian.*

Proof: Let G be a graph satisfying the hypotheses. We shall prove G satisfies the hypotheses of Theorem 2.

Let v be a vertex of G and S be an independent subset of $N_2(v)$. Let $S = \{w_1, w_2, \ldots, w_s\}$ and $T = N(S) \cap N(v) = \{v_1, v_2, \ldots, v_t\}$. And let $k_i = |\{w_j | v_i w_j \in E(G), w_j \in S\}| (i = 1, 2, \ldots, t)$. Without loss of generality, assume $k_1 \leq k_2 \leq \cdots \leq k_t$. Let $k_{m_j} = \max\{k_i | v_i w_j \in E(G) \text{ and } v_i \in T\}$ $(j = 1, 2, \ldots, s)$. Without loss of generality, assume $k_{m_1} \leq k_{m_2} \leq \cdots \leq k_{m_s}$.

If v_j is adjacent to w_i, considering v, by the hypothesis of this corollary, w_i and v have at least $k_j + 1$ common neighbours in T. Considering all vertices in T adjacent to w_i, w_i has at least $k_{m_i} + 1$ neighbours in T by the definition of k_{m_i}. The vertices in T send a total of $k_1 + k_2 + \cdots + k_t$ edges to S, whereas the vertices in S send at least $(k_{m_1}+1)+(k_{m_2}+1)+\cdots+(k_{m_s}+1)$ edges to T. So

$$k_1 + k_2 + \cdots + k_t \geq (k_{m_1} + 1) + (k_{m_2} + 1) + \cdots + (k_{m_s} + 1) \qquad (1)$$

By (1),

$$\sum_{i=1}^{t} k_i \geq \sum_{j=1}^{s} k_{m_j} + s \qquad (2)$$

In the following, we prove that

$$k_i \leq k_{m_i} \quad (i = 1, 2, \ldots, s) \qquad (3)$$

By the definition of k_{m_i}, $k_{m_1} \geq K_1$. Assume that $k_{m_i} \geq k_i$ for all i such that $i < j$. Now suppose $i = j$. If there is a $w_p \in \{w_1, w_2, \ldots, w_j\}$ such that $w_p v_q \in E(G)$ for some $q \geq j$, then $k_{m_j} \geq k_{m_p} \geq k_q \geq k_j$. Otherwise $N(\{w_1, w_2, \ldots, w_j\}) \cap T \subseteq \{v_1, v_2, \ldots, v_{j-1}\}$. Then $k_1 + k_2 + \cdots + k_{j-1} \geq (k_{m_1}+1)+(k_{m_2}+1)+\cdots+(k_{m_j}+1)$. By the induction hypothesis, $k_{m_i} \geq k_i$ $(i = 1, 2, \ldots, j-1)$ and $k_{m_j} \geq 1$, so $k_{m_1} + k_{m_2} + \cdots + k_{m_{j-1}} + k_{m_j} > k_1 + k_2 + \cdots + k_{j-1}$, a contradiction.

By (3) and (2), we have $t \geq s+1$. And hence $|N(S) \cap N(v)| \geq |S| + 1$. □

The following corollaries were proved to be consequences of Corollary 8 in [1]. Corollary 9 is the well known Ore's condition. Corollary 10 was due to Hasratian and Khachatrian [3]. Corollaries 11 and 12 were duo to Shi [5] and Lou respectively. Let $M_3(u) = \{v | d(u,v) \leq 3 \text{ and } v \in V(G)\}$.

Corollary 9. *If for each pair of nonadjacent vertices u and v of a graph G, $d(u) + d(v) \geq \nu(G)$, then G is hamiltonian.*

Corollary 10. *Let G be a connected graph with $\nu \geq 3$. If for each vertex $v \in V(G)$, $d(v) \geq |M_3(v)|/2$, then G is hamiltonian.*

Corollary 11. *Let G be a connected graph with $\nu \geq 3$. If for each vertex $w \in V(G)$, $G' = G[\{w\} \cup N(w)]$ and for each pair of nonadjacent vertices u and v in $V(G')$, $d_{G'}(u) + d_{G'}(v) \geq d_G(w) + 1$, then G is hamiltonian.*

Corollary 12. *Let G be a connected graph. If for each vertex $w \in V(G)$, $\epsilon(G[\{w\} \cup N(w)]) > (d(w) - 1)(d(w) - 2)/2 + d(w)$, then G is hamiltonian.*

Remark 1. Theorem 2 is sharp in the sense that we cannot replace the hypothesis by $|N(S) \cap N(v)| \geq |S|$. $K_{n,n+1}$ ($n \geq 2$) is a counterexample. It is not difficult to find other counterexamples.

Remark 2. Corollary 8 cannot imply Theorem 2. Let $G_1 = K_5$ with $V(G_1) = \{v, v_1, v_2, v_3, v_4\}$. Let $H = K_{2,3}$ with bipartition $(\{w_1, w_2\}, \{u_1, u_2, u_3\})$. Let $G_2 = H + w_1w_2$. We construct a graph $G = G_1 \cup G_2 \cup \{v_1u_i | i = 1, 2, 3\} \cup \{v_2u_1, v_2u_3, v_3u_2, v_3u_3, v_4u_1, v_4u_2\}$. Then G satisfies the hypotheses of Theorem 2, but G does not satisfy the hypothesis of Corollary 8. $d(v, u_1) = 2$. Considering v_1, $\alpha^*(v, u) = 4$. But $I(v, u_1) = 3$.

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