The Coexistence of Some Binary and N-Ary Designs

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ABSTRACT. It is shown that the existence of a semi-regular automorphism group of order m of a binary design with v points implies the existence of an n-ary design with v/m points. Several examples are described. Examples of other n-ary designs are considered which place such n-ary designs in context among n-ary designs generally.

1 Introduction

The concept of n-ary designs was introduced by Tocher [10] and described, under a more restrictive definition, by Billington [1]. We will follow the definition and notation of [1]. Thus an n-ary design is defined to be a set of V points and a collection of B multisets, called blocks, formed from the points so that each point occurs one of $0, 1, \ldots, n-1$ times in any block, each block contains K points, each point occurs R times among all the blocks and any pair of distinct points occurs together Λ times. An incidence matrix for an n-ary design has rows and columns indexed by the points and blocks, respectively, where the ij entry equals the number of times the point i occurs in the block j. In what follows we often identify an incidence matrix with the design to which it belongs.

When $n = 2$ the design is said to be binary and is the same as the usual balanced incomplete block design. Following [1], we use lower case letters for the parameters of a binary design, writing them in the form (v, b, r, k, λ) or, if the design is symmetric, in the form (v, k, λ). An n-ary design is regular if the number of blocks containing a point i times is independent of the point for any i, when this number is denoted ρ_i. Ternary designs ($n = 3$) are always regular.

Note that we do not assume that in an n-ary design an $(n-1)$-fold repetition of a point actually occurs. This allows us to use n-ary design as a generic term, which greatly facilitates expression. However, terms such as ternary ($n = 3$) or quaternary ($n = 4$) do imply that the value of n involved is the best possible.

An automorphism of an n-ary design is a permutation of the points and of the blocks which preserves incidence, and an automorphism group is any subgroup of the full automorphism group. A permutation group is semi-regular if only the identity fixes anything, so that the length of any orbit equals the order of the group.

We use the term residual of a symmetric binary design in the sense in which it is used in [8] and the residual of a design $\mathcal{D}$ at a block x will be written $\mathcal{D}_x$.

We denote by I_m and J_m, respectively, the identity and all-one $m \times m$ matrices and by $J_{m,n}$ the all-one $m \times n$ matrix. Other notations and terminology, if not otherwise referenced, will be found in [1] or [3].

2 The main result

Our purpose is to combine and generalize several known methods of constructing n-ary designs from binary designs. This is done in the following theorem.

Theorem 2.1. *If there exists a binary (v, b, r, k, λ) design admitting an automorphism group G of order m which is semi-regular on both points and blocks then there exists an n-ary design with $V = v/m$, $B = b/m$, $R = r$, $K = k$ and $\Lambda = \lambda m$.*

Proof: Clearly G has V point orbits and B block orbits. Let these be denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_V$ and $\mathcal{B}_1, \ldots, \mathcal{B}_B$ (resp.). Denote by a_{ij} the number of points of $\mathcal{P}_i$ on a given block of $\mathcal{B}_j$. Then a_{ij} is independent of the choice of block and is also the number of blocks of $\mathcal{B}_j$ through any point of $\mathcal{P}_i$. Clearly we have

$$\sum_{i=1}^{V} a_{ij} = k \text{ and } \sum_{j=1}^{B} a_{ij} = r \tag{1}$$

If $\mathcal{P}_i$ and $\mathcal{P}_{i'}$ denote distinct point orbits and we count the triples (P, Q, x) where P is fixed in $\mathcal{P}_i$, Q is arbitrary in $\mathcal{P}_{i'}$ and x is a block incident with both P and Q we obtain

$$\sum_{j=1}^{V} a_{ij} a_{i'j} = \lambda m \tag{2}$$

But (1) and (2) are precisely the conditions that $A = (a_{ij})$ be an incidence matrix for the required n-ary design.

Let us consider a simple example. Writing $I_2 = I$, $J_2 = J$, $A = J - I$ and O for the all-zero 2×2 matrix we can form an incidence matrix for a binary (16, 6, 2) design as follows:

$$M = \begin{pmatrix} I & I & I & I & J & O & O & O \\ I & I & A & A & O & J & O & O \\ I & A & I & A & O & O & J & O \\ I & A & A & I & O & O & O & J \\ J & O & O & O & I & I & I & I \\ O & J & O & O & I & I & A & A \\ O & O & J & O & I & A & I & A \\ O & O & O & J & I & A & A & I \end{pmatrix}.$$

Since M is constructed from cyclic 2×2 matrices it clearly admits a semi-regular automorphism group of order 2. The corresponding n-ary design, with $V = B = 8$, $R = K = 6$ and $\Lambda = 4$, has the following incidence matrix:

$$N = \begin{pmatrix} J_4 & 2I_4 \\ 2I_4 & J_4 \end{pmatrix}.$$

Theorem 2.1 is related to several known results. For a symmetric design it is a special case of Theorem 2.1 of [6] and if $m = 2$ then it is essentially the same as the construction using involutory automorphisms described on pages 286–288 of [5]. For the case of a difference-set design with G a subgroup of the difference-set group, the theorem recalls the idea of the contraction of a difference set as described in [7]. What we lose is that the difference-set group induces a transitive automorphism group of the resulting n-ary design which we do not obtain in the more general case.

Where an n-ary design is constructed by means of Theorem 2.1 the determination of whether it is ternary, quaternary etc. is difficult in general. In a few special cases, however, it is possible. Thus we always obtain a ternary design if $m = 2$ or if $m = 3$ and $\Lambda = 1$ or 2. Taking the Singer group of a Desarguesian protective plane of order q with $q \equiv 1 \pmod 3$ and a subgroup of order 3 yields a family of ternary designs as described in Theorem 2.10 of [5]. Also if G has order 7 and is in the Singer group of a protective plane then we obtain either a ternary design with $\rho_2 = 3$ or a regular quaternary design with $\rho_3 = 1$ and $\rho_2 = 0$. Moreover, we obtain a quaternary design only if the plane contains a subplane of order 2. Now the Desarguesian plane of order q contains a subplane of order 2 only if q is a power of 2. Thus we have the following result.

Theorem 2.2. *Whenever q is an odd prime power such that $q \equiv 2, 4 \pmod 7$ there is a symmetric ternary design with $V = (q^2 + q + 1)/7$, $K = q + 1$, $\rho_1 = q - 5$, $\rho_2 = 3$ and $\Lambda = 7$.*

When q is a power of 2 a little elementary field theory will show that a subgroup of order 7 of a Singer group will always give a quaternary design, for if g is an element of order 7 in the multiplicative group of $GF(q^3)$ and if 7 does not divide $q-1$ then the minimum polynomial of g over $GF(q)$ is either x^3+x+1 or x^3+x^2+1 and thus 1, g and g^3 or 1, g^2 and g^3 are linearly dependent over $GF(q)$. Thus we have the following result.

Theorem 2.3. *If $q=2^t$ and 3 does not divide t then there is a symmetric quaternary design with $V=(q^2+q+1)/7$, $K=q+1$, $\rho_1=q-2$, $\rho_2=0$, $\rho_3=1$ and $\Lambda=7$.*

Examples of symmetric designs which admit semi-regular automorphism groups but not difference-set groups are less common; however, although, as we will show below, there is no (56,11,2) difference set, four of the known (56,11,2) designs, namely those described in [4], admit involutory automorphisms with no fixed points or blocks. The resulting ternary designs have $V=28$, $K=11$ and $\Lambda=4$. These represent solutions for the design #94 of the list in [2]. In fact the ternary designs obtained from the designs B_1 and B_2 of [4] are isomorphic while the other two are not isomorphic to this design or to one another. Thus we have three solutions for this ternary design.

So far we have considered only examples which are symmetric. One source of non-symmetric examples is the groups described in the following theorem.

Theorem 2.4. *Let D be a symmetric (v,k,λ) design admitting an automorphism group G of order m which fixes an incident point-block pair (P,x) and is semi-regular on the remaining points and blocks. Then there exists an n-ary design with $V=(v-k)/m$, $B=(v-1)/m$, $R=k$, $K=k-\lambda$ and $\Lambda=\lambda m$.*

Proof: Let $\mathcal{D}_x$ denote the residual of $\mathcal{D}$ at x. Then G induces a semi-regular automorphism group of $\mathcal{D}_x$ and the result follows from Theorem 2.1.

We note that an involutory automorphism of a symmetric binary design cannot fix exactly one point, and so the group G in Theorem 2.4 must have odd order. On the other hand, if q is an odd prime power then the Desarguesian plane of order q admits such a group G with $m=q$, and so such a group of any order dividing q. Again the design is ternary if the group has order 3,5 or 7 (since we are dealing with planes of odd characteristic). Thus we get the following result.

Theorem 2.5. *Ternary designs exist for the following values of the parameters, with t an arbitrary positive integer:*

(i) $V=3^{2t-1}$, $B=3^{2t-1}+3^{t-1}$, $\rho_1=3^t-1$, $\rho_2=1$, $R=3^t+1$, $K=3^t$ and $\Lambda=3$;

(ii) $V = 5^{2t-1}$, $B = 5^{2t-1} + 5^{t-1}$, $\rho_1 = 5^t - 3$, $\rho_2 = 2$, $R = 5^t + 1$, $K = 5^t$ and $\Lambda = 5$;

(iii) $V = 7^{2t-1}$, $B = 7^{2t-1} + 7^{t-1}$, $\rho_1 = 7^t - 5$, $\rho_2 = 3$, $R = 7^t + 1$, $K = 7^t$ and $\Lambda = 7$.

Proof: For (i) we use Theorem 2.4 with $\mathcal{D}$ a Desarguesian plane of order 3^t and $m = 3$. For (ii) and (iii) we replace 3 with 5 and 7, respectively.

Of course there are many examples of binary designs admitting semi-regular automorphism groups besides those described above; for example, projective and affine geometries of arbitrary dimension admit such groups, and there are many other examples. Of these we mention just two. As noted in [5], taking a (36, 15, 6) difference set and a subgroup of order 2 yields a solution for the design #302 in the list in [2]. Also, it is not difficult to show that a solution for the design #153 of [2] cannot admit a transitive cyclic automorphism group, and thus taking a (45, 12, 3) difference set and a subgroup of order 3 will yield a quaternary design.

3 Recognizing the n-ary designs

Given an n-ary design we would like to be able to determine whether it can be constructed from a binary design via Theorem 2.1. To begin with, there are necessary parametric conditions. However, the design may not be constructible in this way even when it has appropriate parameters. We consider both situations.

Suppose we have a given n-ary design constructed from a binary design as in Theorem 2.1. We adopt the notation and hypotheses of Theorem 2.1. Then we have $V\Lambda - R(K-1) = (v/m).\lambda m - r(k-1) = v\lambda - r(k-1)$. Since the relation $\lambda(v-1) = r(k-1)$ holds in any binary design, we have $V\Lambda - R(K-1) = \lambda$. Let us write D for the expression $V\Lambda - R(K-1)$ in an arbitrary n-ary design. Then for the design to be constructible via Theorem 2.1, D must be positive and must divide Λ. This rules out most n-ary designs.

At this point we must consider the effect of taking complements. Still with the notation and hypotheses of Theorem 2.1, let A be the incidence matrix of the resulting n-ary design. Let $\mathcal{D}^c$ denote the complement of $\mathcal{D}$. Then $\mathcal{D}^c$ also admits G as a semi-regular automorphism group. If we apply the theorem to this design we get an n-ary design with incidence matrix $mJ_{V,B} - A$. This will be the complement of A in the sense of [1] when the value m occurs in A, but not otherwise. In the latter case, if h is the largest value occurring in A then $mJ_{V,B} - A = B + (m-h)J_{V,B}$ where 0 occurs in B and B is also an n-ary design. It can happen, therefore, that a given n-ary design is constructible in this way when the necessary conditions for it to be directly constructible are not met. It is a consequence of this that when testing whether an n-ary design meets the necessary parametric conditions, we usually have to consider the design and its complement separately. In what follows we in general consider only the possibility that an n-ary design is directly constructible via Theorem 2.1. However, in case (ii) below we give a simple condition under which a design is never constructible via Theorem 2.1, even indirectly.

Let us now consider examples of n-ary designs not constructible via Theorem 2.1. There are several possibilities, and for completeness we give an example of each. We draw on [1] and [2] for most of our examples. Original references, where appropriate, are given there.

(i) $D \leq 0$. The ternary designs constructed as example (4.8) in [1] have $V = K = \Lambda + 1$ and so $D = 0$. These have as points the elements of the Galois field $GF(q)$ for q odd. An initial block is formed by taking 0 once and each non-zero square in $GF(q)$ twice. The other blocks are the images of this under addition in $GF(q)$. If $q \equiv 3 \pmod 4$ and in the initial block we take each non-zero square t times, where t is an integer greater than two, we get an n-ary design with $D < 0$.

For symmetric ternary designs the condition $D < 0$ seems to be rare. Among those constructed in [2] only one meets this condition, namely #24, which has $D = -2$. (There are several non-symmetric examples.) A second symmetric ternary design with D negative is the following solution for #164 of [2] with $V = 32$, $K = 12$ and $\Lambda = 4$. Let X be the matrix of a Hadamard system $H(4,1)$ (see [9] page 65) and let $Y = I_4 \otimes J_4$. Then an incidence matrix for the required design is M where

$$M = \begin{pmatrix} 2X & Y \\ Y & 2X \end{pmatrix}.$$

This design has $D = -4$. Its complement belongs to case (ii) below.

(ii) $D > 0$ but D does not divide Λ. This is the most common situation; most of the designs in [2] belong to this category. This case serves to show that certain n-ary designs can never be constructed, even indirectly, via Theorem 2.1. Let M be a symmetric n-ary design for which V and K have a common factor d which does not divide Λ. Then d divides D, so D is not a divisor of Λ and the design belongs to this case or the previous one. However the design $J_V + M$ has parameters $V^* = V$, $K^* = V + K$ and $\Lambda^* = V + 2K + \Lambda$ and so meets the same condition. Inductively, this condition holds in the design $cJ_V + M$ for any positive integer c, and so none of these are constructible via Theorem 2.1. The ternary designs on the points of $GF(q)$ which were described in case (i) above have this property.

(iii) $D > 0$ and D divides Λ but the required binary design does not exist. There are many examples of this; one is the design #18 of [2] for which the binary design would be a symmetric (22,7,2) design, which is non-existent since v is even but $k - \lambda$ is non-square.

(iv) There is a t-fold repetition of a point in a block where $t > \Lambda/D$. In this case the n-ary design is not constructible via Theorem 2.1 because Λ/D would be the order of the group and so the largest value which could occur in the incidence matrix of an n-ary design so constructed. As an example let M be an incidence matrix for the ternary design on 32 points described in case (i) above and let $N = 2J_{32} + M$. Then N is an n-ary design with $V = 32$, $K = 76$ and $\Lambda = 180$. Here $D = 60$ and $\Lambda/D = 3$ so the parameters are right for the design to be constructible from a (96,76,60) design and a group of order 3. This design is the complement of a (96,20,4) design for which there is a difference set (see [7] pp 123–124), so examples exist; however the value 4 occurs in N so it is not constructible in this way.

(v) $D > 0$, D divides Λ and the required binary design exists but does not admit an automorphism group of the required type. Examples of this type, where the n-ary design is known, are less easy to establish. The clearest examples, perhaps, are those for residuals of biplanes (binary designs with $\lambda = 2$) when m is even. These would be special cases of Theorem 2.4, since any residual biplane is embeddable in a biplane and inherits all of its automorphisms from the biplane. But an involutory automorphism of a biplane cannot fix exactly one point so there are no examples with m even. Nevertheless, the n-ary design may exist as is shown by a number of the designs in [2], of which #56 is one.

The proof of Theorem 5.1 below will yield a further example belonging to this case.

(vi) $D > 0$, D divides Λ and examples of the required binary design with a group of the required type exist but do not yield the given n-ary design. For this case to occur the n-ary design must have non-isomorphic solutions. We take as our example the ternary design #49 in [2] with $V = K = 5$, $B = R = 9$ and $\Lambda = 8$. Here $D = 4$ so we are looking for a (10,18,9,5,4) design with an involutory automorphism moving all points and blocks. It is easily established that the example given in [2] cannot be constructed from such a binary design. However the following is a second solution for this design.

$$X = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

Writing $I_2 = I$, $J_2 = J$, $A = J - I$ and O for all-zero 2×2 matrix we can replace the entries in X to give the required binary design as follows:

$$Y = \begin{pmatrix} J & J & I & I & I & I & I & O & O \\ I & I & J & I & A & O & O & J & I \\ I & I & O & A & I & J & O & I & J \\ A & O & I & J & O & A & J & I & I \\ O & A & A & O & J & I & J & I & I \end{pmatrix}.$$

This design is quasi-residual. According to [8] every such quasi-residual is embeddable in a (19,9,4) design. However the involutory automorphism exhibited in the structure of Y cannot act on the full symmetric design since it would fix just one block, which is impossible.

4 A further generalization

There is no reason why the construction given in Theorem 2.1 has to begin with a binary design. Indeed Theorem 2.1 is a special case of the following theorem which can be proved in the same way.

Theorem 4.1. *Let $\mathcal{D}$ be an n-ary design with parameters V, B, R, K and Λ and let G be an automorphism group of $\mathcal{D}$ of order m which is semi-regular on points and blocks. Then there exists an n-ary design with parameters $V^* = V/m$, $B^* = B/m$, $R^* = R$, $K^* = K$ and $\Lambda^* = \Lambda m$.*

In terms of known examples Theorem 2.1 is more useful than Theorem 4.1. Nevertheless examples of n-ary designs meeting the hypotheses of Theorem 4.1 but not themselves constructible using Theorem 2.1 do exist. The ternary designs on the points of $GF(q)$, q odd, given in case (i) of Section 3 are not constructible even indirectly via Theorem 2.1, but they admit a regular automorphism group induced by the addition in $GF(q)$, and thus semi-regular automorphism groups of order any divisor of q. Thus they provide examples of Theorem 4.1 whenever q is not prime.

5 A theorem on (56,11,2) designs

In this section we prove a theorem for binary (56,11,2) designs which was quoted in Section 2 and Section 3.

Theorem 5.1. *A symmetric (56,11,2) binary design cannot admit a semi-regular automorphism group of order 8.*

Proof: Let G be an automorphism group of order 8 of a 2-(56,11,2) design with G semi-regular on points. Then G is semi-regular on blocks also. Let A be an incidence matrix of the corresponding n-ary design given by Theorem 2.1. Then A is a 7×7 matrix with constant row and column sum equal to 11, and the inner product of any two distinct rows or any two distinct columns equal to 16. Moreover, by Lemma 2.3 of [1], the sum of the squares of the entries in any row or column is $RK - \Lambda(V-1)$ which here equals 25. Thus any row or column of A is a collection of seven non-negative integers $x_1, \ldots, x_7$ such that $x_1 + \cdots + x_7 = 11$ and $x_1^2 + \cdots + x_7^2 = 25$. There are just three solutions to these equations. They are (ignoring the order of the terms)

$$4,2,1,1,1,1,1 \qquad \text{(a)}$$

$$3,3,2,1,1,1,0 \qquad \text{(b)}$$

$$3,2,2,2,2,0,0 \qquad \text{(c)}$$

Thus any row or column of A must be of type (a), (b) or (c). Let us write $A = (a_{i,j})$. We claim that no row or column of A is of type (a). Otherwise we can assume that the first row is of type (a) (if necessary by permuting the rows or transposing A, which is permissible because the hypotheses are self-dual).We can set $a_{11} = 4$, $a_{12} = 2$ and $a_{1i} = 1$ for $i = 3, \ldots, 7$. Then the first column of A is also of type (a) and we can take $a_{21} = 2$. We now find that there is no way to complete row 2 so that its inner product with row 1 is 16.

We note that just two of the values in type (b) are even, and if every row were of type (b) then there would be distinct rows such that no position contained an even term in both rows. The inner product of those two rows would then be odd. Since this cannot occur, there must be a row of type (c), and without loss of generality we set $a_{11} = 3$, $a_{1j} = 2$ for $j = 2, \ldots, 5$ and $a_{16} = a_{17} = 0$. If now we have a_{i1} odd, for any $i > 1$, then the inner product of the ith row with the first would be odd. Since this cannot occur, the first column must be of type (c) and we can set $a_{i1} = 2$ for $i = 2, \ldots, 5$ and $a_{61} = a_{71} = 0$.

Now the inner product of the sixth or seventh row with the first will be twice the sum of its entries in positions 2 to 5, so this sum must be 8. For a row of type (b) these values must be (in some order) 3,3,1,1 while for a row of type (c) they must be 2,2,2,2. We may try various combinations of these and then try to complete A. We find that we meet a contradiction unless (up to equivalence) we set row 6 to be 0,3,3,1,1,2,1 and row 7 to be 0,1,1,3,3,1,2. In that case we can complete A uniquely, so that, up to permutation of the rows and columns, there is a unique solution for A, namely

$$A = \begin{pmatrix} 3 & 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 & 3 & 3 & 1 \\ 2 & 3 & 0 & 1 & 1 & 1 & 3 \\ 2 & 0 & 3 & 1 & 1 & 1 & 3 \\ 0 & 3 & 3 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}.$$

Now let H be a subgroup of G of order 4. Then H is also semi-regular. Each orbit of G is decomposed into two orbits under H, and the n-ary design corresponding to H can be obtained from A by replacing each a_{ij} with a 2×2 matrix with row and column sum equal to a_{ij}. Moreover, in such a matrix B the sum of the squares of the entries in any row or column will be 17 (again using Lemma 2.3 of [1]). Thus it must be possible to split each entry in any row of A into the sum of two terms so that the sum of the squares of the fourteen integers thus obtained is 17. Now for a row of type (b) this can only be done by decomposing each 3 as $2+1$, the unique 2 as $2+0$ and each 1 as $1+0$. This means that each entry 2 in the first column of A must be replaced either by $2I_2$ or by $2J_2 - 2I_2$. We now find that no possible 2×2 matrix can replace the 3 in the first column of A without making the sum of the squares in the first (or second) column of B too large. This contradiction shows the group G which we first postulated to be impossible.

Corollary. *There is no difference set for a binary (56,11,2) design.*

We note that the above corollary is known for the case of abelian difference sets (see [7], page 228).

References

[1] Elizabeth J. Billington, Balanced n-ary designs: a combinatorial survey and some new results, *Ars Combinatoria* **17A** (1984), 37–72.

[2] Elizabeth J. Billington and Peter J. Robinson, A list of balanced ternary designs with $R \leq 15$, and some necessary existence conditions, *Ars Combinatoria* **16** (1983), 235–258.

[3] H.P. Dembowski, "Finite Geometries" Berlin-Heidelberg-New York, Springer Verlag, 1968.

[4] R.H.F. Denniston, On biplanes with 56 points, *Ars Combinatoria* **9** (1980), 167–179.

[5] J.F. Dillon and M.A. Wertheimer, Balanced ternary designs derived from other combinatorial designs, *Congressus Numerantium* **47** (1985), 285–298.

[6] D. R. Hughes, Collineations and generalized incidence matrices, *Trans. Amer. Math. Soc.* **86** (1957), 284-296.

[7] E. S. Lander, Symmetric designs: an algebraic approach, *L.M.S. Lecture Notes Series* **74**, Cambridge University Press, 1983.

[8] J.H. van Lint, H.C.A. van Tilborg and J.R. Wiekema, Block designs with $v = 10$, $k = 5$, $\lambda = 4$, *J. Comb. Theory (A)* **23** (1977), 105–115.

[9] D.P. Rajkundlia, Some techniques for constructing infinite families of BIBD's *Discrete Math.* 44 (1983), 61–96.

[10] K.D. Tocher, The design and analysis of block experiments, *J. Roy. Statist. Soc. (B)* **14** (1952), 45–100.