

A Class Of Well-Covered Graphs With Girth Four

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ABSTRACT. A graph is well-covered if every maximal independent set is also a maximum independent set. A 1-well-covered graph G has the additional property that $G - v$ is also well-covered for every point v in G . Thus, the 1-well-covered graphs form a subclass of the well-covered graphs. We examine triangle-free 1-well-covered graphs. Other than C_5 and K_2 , a 1-well-covered graph must contain a triangle or a 4-cycle. Thus, the graphs we consider have girth 4. Two constructions are given which yield infinite families of 1-well-covered graphs with girth 4. These families contain graphs with arbitrarily large independence number.

Introduction

A set of points in a graph is *independent* if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the *independence number of G* and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a *maximum independent set*. A set S of independent points in a graph is *maximal* (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [13] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a *well-covered* graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [4] proved that there are only four 3-connected cubic planar well-covered graphs. Campbell, Ellingham and Royle [3] have recently completed the

picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called *very well-covered* graphs. The subclass of very well-covered graphs was characterized by Staples [16] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [14] characterized bipartite well-covered graphs and Favaron [7] characterized the very well-covered graphs. Recently, Dean and Zito [5] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.

Finbow and Hartnell [8] and Finbow, Hartnell, and Nowakowski [9] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [10].

A well-covered graph (with at least two points) is *1-well-covered* if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph (with at least two points) is in the class W_2 if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [17] showed that a well-covered graph is 1-well-covered if and only if it is in W_2 . Since we will appeal mostly to the notion of extending two disjoint independent sets to disjoint maximum independent sets, henceforth we use the W_2 nomenclature instead of referring to 1-well-covered graphs.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are C_3 , C_4 , C_5 , and C_7 . We note that all complete graphs with at least two points are also in W_2 , but no complete bipartite graphs (except $K_{1,1}$) are in W_2 . The cycles C_3 and C_5 are the only cycles in W_2 . Some additional results for W_2 graphs can be found in [11] and [12].

Preliminary Results

We assume that all graphs are connected, unless otherwise stated. The reader is referred to [1] for terminology and notation not defined here. Note that a disconnected graph is in W_2 if and only if each of its components is in W_2 . Suppose G is well-covered, $G \neq K_1$. Let v be a point in G and consider the graph $G - v$. Since $G \neq K_1$, there exists a point $u \sim v$. Since G is well-covered, the point u is contained in a maximum independent set I in G . Clearly, v is not in I . Thus, I is also independent in $G - v$. Consequently, $\alpha(G - v) = \alpha(G)$ for any point v . Hence, from a result of Erdős and Gallai [6] it follows that $\alpha(G) \leq |V(G)|/2$.

Since W_2 graphs are well-covered, then $\alpha(G - v) = \alpha(G)$ for any point v in a W_2 graph. Also, W_2 graphs inherit the bound on independence number, $\alpha(G) \leq |V(G)|/2$.

Staples [17] proved that a W_2 graph cannot have a point of degree one.

Theorem 1. *If $G \in W_2$ and $G \neq K_2$, then $\delta \geq 2$.*

If v is a point in the graph G , then denote the neighborhood of v by $N(v)$. Let G_v be the graph induced by $G - \{v \cup N(v)\}$. Campbell [2] found the following very useful necessary condition for a graph to be well-covered.

Theorem 2. *If a graph G is well-covered and is not complete, then G_v is well-covered for all v in G . Moreover, $\alpha(G_v) = \alpha(G) - 1$.*

Fortunately, we prove in Theorem 3 that we have a similar necessary condition for a well-covered graph to be in W_2 . We will reference Theorem 3 several times in this paper.

Theorem 3. *If a graph G is in W_2 and G is not complete, then G_v is in W_2 for all v in G .*

Proof: Let v be a point in G . Since G is not complete, then $G_v \neq \emptyset$. By Theorem 2, graph G_v is well-covered and $\alpha(G_v) = \alpha(G) - 1$. Suppose I_1 and I_2 are disjoint independent sets in G_v . Then $I_1 \cup \{v\}$ is an independent set in G , as is $I_2 \cup \{v\}$. Since G is in W_2 , there exists a maximum independent set $J_1 \supseteq I_1 \cup \{v\}$ such that $J_1 \cap I_2 = \emptyset$. Since $I_2 \cup \{v\}$ and $J_1 - v$ are disjoint independent sets in G , then there exists a maximum independent set $J_2 \supseteq I_2 \cup \{v\}$ such that $J_2 \cap (J_1 - v) = \emptyset$. Hence, $J_2 - v$ and $J_1 - v$ are disjoint independent sets in G_v . Since $|J_i| = \alpha(G)$, then $|J_i - v| = \alpha(G) - 1$, for $i = 1, 2$. Thus, $J_1 - v$ contains I_1 , $J_2 - v$ contains I_2 , and $J_1 - v$ and $J_2 - v$ are disjoint maximum independent sets in G_v . So any two disjoint independent sets in G_v can be extended to disjoint maximum independent sets in G_v . By definition of the class W_2 , we conclude that $G_v \in W_2$. \square

We prove in the following theorem that if a W_2 graph has a cutpoint, then the graph obtained by deleting the cutpoint is also a W_2 graph.

Theorem 4. *If $G \in W_2$ and v is a cutpoint of G , then $G - v \in W_2$.*

Proof: Let H_1, H_2, \dots, H_n be the components of $G - v$. Let $x \in V(H_1)$ and $y \in V(H_2)$ such that $x \sim v$ and $y \sim v$. By Theorem 3, the graphs $G_y = G - N[y]$ and $G_x = G - N[x]$ are in W_2 . Clearly, H_i is a component of G_y , for $i \neq 2$, and H_j is a component of G_x , for $j \neq 1$. Hence, H_i is a W_2 graph for all i . It follows that $G - v$ is also a W_2 graph. \square

In order to consider triangle-free W_2 graphs, we introduce some terminology given in [10]. A 5-cycle in a graph is called a *basic 5-cycle* provided that it contains no two adjacent points of degree ≥ 3 (that is, at most two points in the 5-cycle can have degree ≥ 3 and two such points must be

nonadjacent). A graph G is in the *family PC* if $V(G)$ can be partitioned into two sets $V(C)$ and $V(P)$ such that $V(C)$ contains points from basic 5-cycles in G and $V(P)$ contains points from pendant lines in G ; in addition, the lines induced by $V(P)$ must be independent. Two graphs in PC are given in Figure 1.

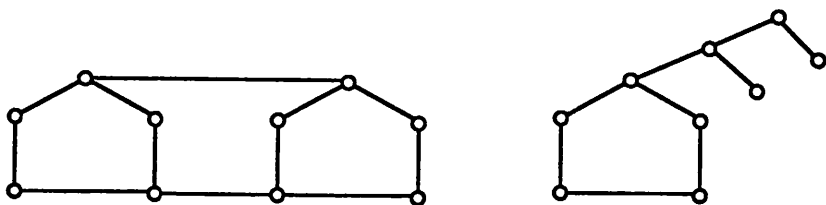


Figure 1

The *girth* of a graph is the size of a smallest cycle in the graph. We say a graph with no cycles has infinite girth. Finbow, Hartnell and Nowakowski [10] proved that the family PC described above contains all well-covered graphs with girth at least five, except K_1 , C_7 , and the four graphs shown in Figure 2. We state their result in the next theorem.

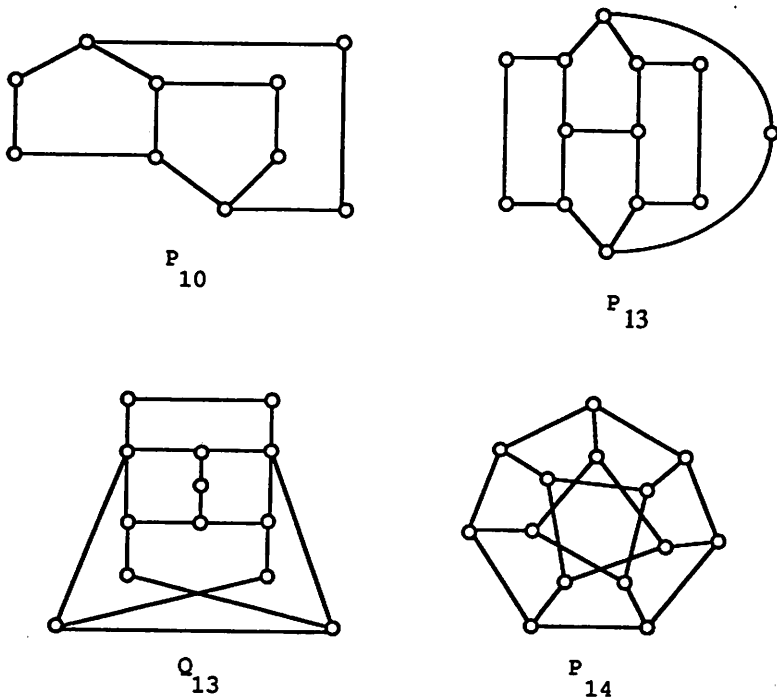


Figure 2

Theorem 5. *Suppose G is well-covered with girth ≥ 5 . Then $G \in PC$ or $G \in \{K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$.*

We need Lemma 6 to show that K_2 and C_5 are the only W_2 graphs in PC . Consequently, we prove in Theorem 7 that a W_2 graph other than K_2 and C_5 has girth at most four.

Lemma 6. *If G is in PC with girth ≥ 5 ($G \neq K_2$ or C_5), then $G \notin W_2$.*

Proof: Suppose $G \in W_2$. As mentioned earlier, we assume that G is connected. By Theorem 1, we have $\delta \geq 2$. So $G \in PC$ and $\delta \geq 2$ together imply that $\{C_i\}$, $i = 1, \dots, n$, partitions $V(G)$, where each C_i is a basic 5-cycle. Since $G \neq C_5$, then $i \geq 2$.

Now C_1 is joined to one or more of the C_i ($i \geq 2$) by one or more lines. Without loss of generality, assume C_1 is connected to C_2 by line $e = uv$. Let $C_1 = uabcd$ and $C_2 = vwxyz$. Since C_1 is a basic 5-cycle then either v is not adjacent to b or v is not adjacent to c . We can assume that v is not adjacent to c . Since $v \sim u$, then $\deg(a) = 2$. Thus, $\{v, c\}$ is independent and so $\{v, c\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G , a contradiction since $G \in W_2$. \square

Theorem 7. *If $G \in W_2$ ($G \neq K_2$ or C_5), then girth $G \leq 4$.*

Proof: Suppose girth $G \geq 5$ and G is well-covered. By the preceding lemma, if $G \in PC$ then $G \notin W_2$. From Theorem 5, if $G \notin PC$, then $G \in \{K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$. It is straightforward to check that each of these 6 graphs is not in W_2 by finding a pair of disjoint independent sets that do not extend to disjoint maximum independent sets. Thus, if G is well-covered with girth ≥ 5 , then $G \notin W_2$. \square

Hence, a W_2 graph (other than K_2 and C_5) must contain a triangle or a 4-cycle. Thus, a triangle-free W_2 graph (other than K_2 and C_5) has girth 4. We study W_2 graphs of girth four for the remainder of this paper.

A line in a graph G is a *critical* line if its removal increases the independence number. A *line-critical* graph is a graph with only critical lines. Staples proved in [16] that a triangle-free W_2 graph is line-critical. Hence, all graphs given in the following constructions are line-critical.

Constructions

The following constructions show how to build a larger (in size and independence number) W_2 graph of girth four from a given such graph with some additional properties. The fact that the constructions yield W_2 graphs can be verified directly from the definition of a W_2 graph by showing that every two disjoint independent sets can be extended to two disjoint maximum independent sets.

An alternative means of verifying that a graph is in W_2 uses the notion of an extendable point. Finbow, Hartnell and Nowakowski [10] defined a point v in a well-covered graph to be an *extendable* point if $G - v$ is well-covered and $\alpha(G - v) = \alpha(G)$. Since every point in a W_2 graph has this property, then every point in a W_2 graph is extendable. See [10] for details on the use of extendable points.

Construction 1. Suppose H is a W_2 graph of girth 4 and C is a 4-cycle in H such that $\alpha(H - C) = \alpha(H) - 1$ and $H - C$ is in W_2 . Let $C = acbd$ and let xy be a new line and $A = v_1v_2v_3v_4$ be a new 4-cycle. Form a new graph G with

$$V(G) = V(H) \cup V(A) \cup \{x, y\}, \text{ and}$$

$$E(G) = E(H) \cup E(A) \cup \{xy, v_1x, v_3y, v_2a, v_2b, v_4c, v_4d\}.$$

See Figure 3.

Then G is a W_2 graph of girth 4 with $\alpha(G) = \alpha(H) + 2$.

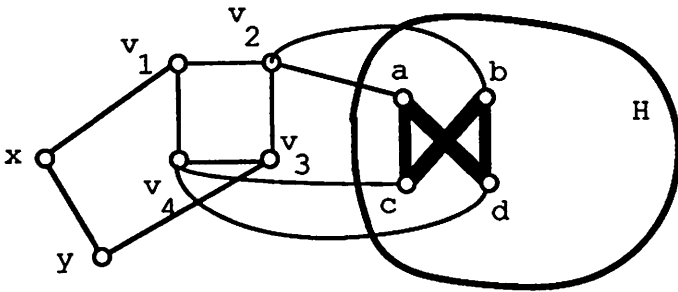


Figure 3

Suppose H_1 is the graph in Figure 4. If C is the 4-cycle in H_1 , then $H_1 - C$ is a W_2 graph. Also, $\alpha(H_1 - C) = 2 = \alpha(H_1) - 1$. Thus, we can use H_1 to construct a larger W_2 graph of girth 4 with independence number 5 via the construction in Construction 1. Call this graph G_5 .

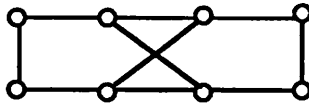


Figure 4

Let H_2 be the graph on 12 points in Figure 6. Let C be the 4-cycle $acbd$, as indicated in Figure 6. It can be checked that H_2 is a W_2 graph with $\alpha(H_2) = 4$, and $H_2 - C = H_1$. Thus, $\alpha(H_2 - C) = \alpha(H_2) - 1$, and $H_2 - C$ is a W_2 graph. Thus, we can build a larger W_2 graph of girth 4 (with

independence number 6) from H_2 via the construction in Construction 1. Call this graph G_6 .

Let $H_1 = G_3$. Note that G_5 satisfies the conditions in Construction 1, with the 4-cycle A that was used to build G_5 from G_3 satisfying $\alpha(G_5 - A) = \alpha(G_5) - 1$ and $G_5 - A \in W_2$. Hence, we can obtain a W_2 graph of girth 4 from G_5 , call it G_7 , via the construction in Construction 1. Therefore, by starting with $H_1 = G_3$ we can recursively use the construction in Construction 1 to generate an infinite family of W_2 graphs of girth 4, namely $G_3, G_5, G_7, G_9, \dots$, where $\alpha(G_n) = n$, for all odd n . Note that the "new" 4-cycle used to construct G_{2k+1} from G_{2k-1} is a 4-cycle in G_{2k+1} which satisfies the conditions in Construction 1. Thus, we "attach" to this 4-cycle to construct G_{2k+3} from G_{2k+1} via the construction in Construction 1. Similarly, by starting with $H_2 = G_4$, we can recursively generate W_2 graphs of girth 4, namely $G_4, G_6, G_8, G_{10}, \dots$, where $\alpha(G_n) = n$, for all even n .

By the nature of the construction in Construction 1, all graphs in the two infinite families just given are exactly 2-connected. In order to construct 3-connected and 4-connected W_2 graphs of girth 4, we develop a different construction in Construction 2.

Construction 2. Suppose H is a W_2 graph of girth 4 with disjoint 4-cycles C_1 and C_2 such that (i) $\alpha(H - C_i) = \alpha(H) - 1$, for $i = 1, 2$, and (ii) $H - C_i$ is a W_2 graph, for $i = 1, 2$. Also, H is either connected or has exactly two components. In the disconnected case, each component contains exactly one of the 4-cycles C_i .

Let $C_1 = u_1y_1v_1x_1$, $C_2 = u_2y_2v_2x_2$, and let $A = abcd$ be a new 4-cycle. Form a new graph G with

$$V(G) = V(H) \cup V(A), \text{ and}$$

$$E(G) = E(H) \cup E(A) \cup \{au_1, av_1, cx_1, cy_1, bx_2, by_2, du_2, dv_2\}.$$

See Figure 5.

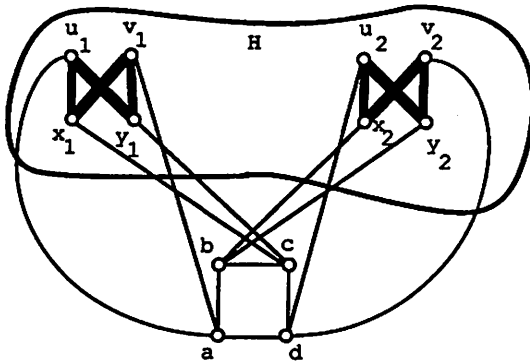


Figure 5

is a W_2 graph and $\alpha(F_1 - A_1) = \alpha(F_1) - 1$. So we form a new graph $F_{1,1}$ from F_1 by adjoining a new 4-cycle A_2 to $C_{1,2}$ and A_1 by the construction in Construction 2. By Construction 2, graph $F_{1,1}$ is a W_2 graph of girth 4 with $\alpha(F_{1,1}) = \alpha(F_1) + 1$.

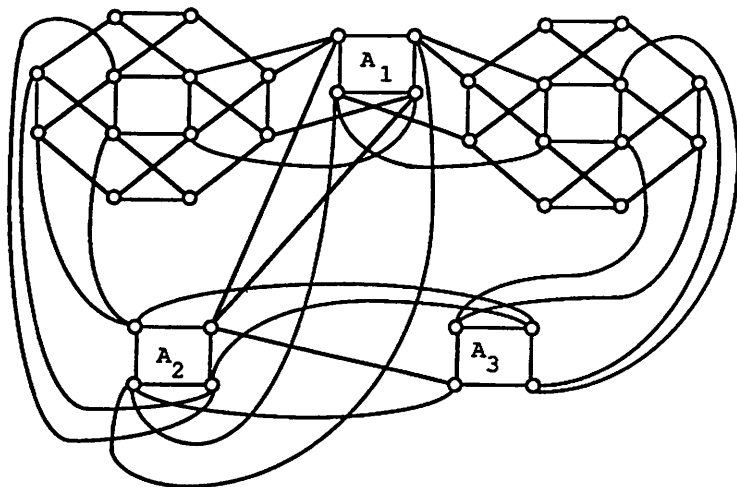


Figure 8

Clearly, $F_{1,1} - A_2$ is in W_2 and $\alpha(F_{1,1} - A_2) = \alpha(F_{1,1}) - 1$. Since $H_2 - C_{2,2}$ is in W_2 , then by Construction 2 the graph $F_{1,1} - C_{2,2}$ is in W_2 . Also, $\alpha(F_{1,1} - C_{2,2}) = \alpha(F_1 - C_{2,2}) + 1 = (\alpha(F_1) - 1) + 1 = \alpha(F_1) = \alpha(F_{1,1}) - 1$. So we form a new graph $F_{1,2}$ by adjoining a new 4-cycle A_3 to A_2 and $C_{2,2}$ by the construction given in Construction 2. Let $G_1 = F_{1,2}$. G_1 is shown in Figure 8. Then G_1 is a W_2 graph of girth 4 by Construction 2. Also, G_1 is 3-connected, $|V(G_1)| = 36$ and $\alpha(G_1) = 2\alpha(H_i) + 3 = 11$.

We conjecture that it is possible to construct an infinite family of 3-connected W_2 graphs of girth 4 by using Construction 2 and a technique generalized from that used to construct the graph G_1 given above.

Beginning with H given above in Figure 6, we can obtain the graph H' given in Figure 9 by two successive applications of Construction 2. Thus, H' is a W_2 graph of girth 4. Note that H' is 4-connected. We conjecture that it is possible to construct an infinite family of 4-connected W_2 graphs of girth 4 by using Construction 2 and by having H' play the role of H above in constructing G_1 .

Not all W_2 graphs of girth 4 arise from the constructions given above. Neither of the graphs given in Figure 10 can be built using our constructions. The graph on 13 points is 4-regular and was found by Royle [15]

using a computer program. Note that neither of the graphs has any 4-cycle that satisfies the conditions in Construction 1 or Construction 2.

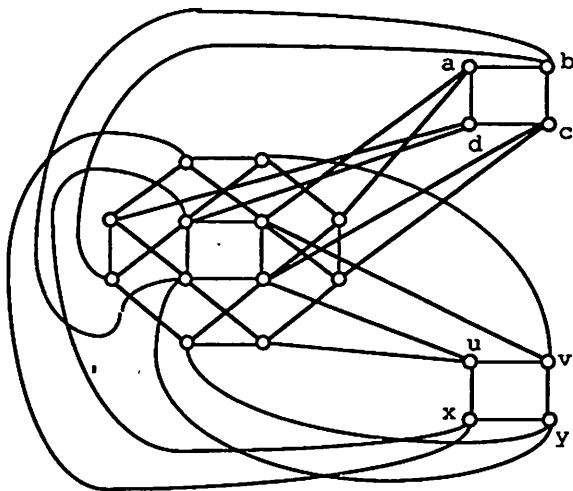


Figure 9

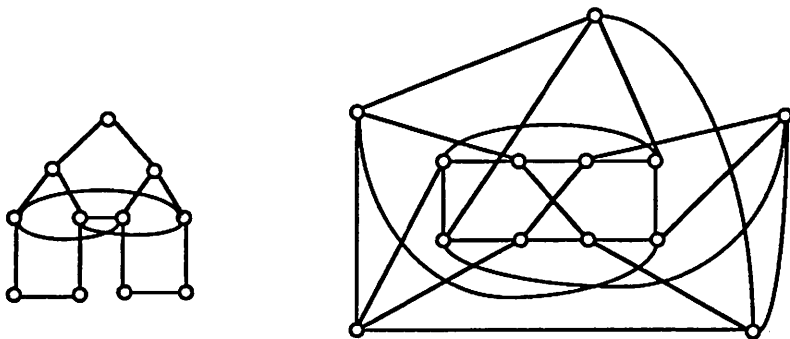


Figure 10

Cutsets

Now that we have constructed some W_2 graphs of girth 4, we look at minimum point cutsets for such graphs.

Theorem 8. *If G is a W_2 graph of girth 4, then G is 2-connected.*

Proof: Assume to the contrary that G has a cutpoint v . Let G_1, G_2, \dots, G_n be the components of $G - v$. By Theorem 4, graphs G_1, \dots, G_n are W_2 graphs. Let $N_i = N(v) \cap G_i$, for $i = 1, \dots, n$. Since G has girth 4, then N_i

is independent for all i . Since $G_i \in W_2$, there exists maximum independent sets J_i in G_i such that $J_i \cap N_i = \emptyset$, for all i . Clearly, $J = J_1 \cup \dots \cup J_n$ is an independent set in G . Consequently, J and $\{v\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G . This is a contradiction since $G \in W_2$. Hence, G is 2-connected. \square

Lemma 9. *Suppose G is a W_2 graph of girth 4 and $\{u, v\}$ is a cutset of G . If $u \sim v$, then every component of $G - \{u, v\}$, except possibly one, is a W_2 graph.*

Proof: Let G_1, \dots, G_n be the components of $G - \{u, v\}$. Let $U_i = N(u) \cap G_i$ and $V_i = N(v) \cap G_i$, for all i . Since G has girth 4, then $x \in U_i$ implies x is not adjacent to v , and $y \in V_i$ implies y is not adjacent to u , for all i . Also, U_i and V_i are independent sets, for all i .

Suppose that $x \in U_i, y \in V_i$ implies $x \sim y$, for all i . Let $U = U_1 \cup \dots \cup U_n$. Then U and $\{v\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G , contradicting $G \in W_2$. Thus, there exists $j \in \{1, \dots, n\}$ such that x and y are points in $G_j, x \in U_j, y \in V_j$ and x is not adjacent to y .

Consider the graph $G_x = G - N[x]$. Since x is not adjacent to y , then $y \in G_x$. Since $G \in W_2$, then by Theorem 3 so is G_x . Then v is a cutpoint for G_x , and by Theorem 4, the graph $G_x - v$ is a W_2 graph. Since G_i is a component of $G_x - v$, for $i \neq j$, then G_i is a W_2 graph, $i \neq j$. \square

Theorem 10. *Suppose G is a W_2 graph of girth 4 and $\{u, v\}$ is a cutset for G . Then $\{u, v\}$ is independent.*

Proof: Suppose $u \sim v$. Let G_1, \dots, G_n be the components of $G - \{u, v\}$. Let $U_i = N(u) \cap G_i$ and $V_i = N(v) \cap G_i$, for all i . Since G has girth 4, then U_i and V_i are disjoint independent sets, for all i .

Case 1. Suppose G_i is a W_2 graph, for all i . Then there exist maximum independent sets J_i in G_i such that $J_i \supseteq V_i$ and $J_i \cap U_i = \emptyset$, for all i . Let $J = J_1 \cup \dots \cup J_n$. Then J and $\{u\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G , contradicting $G \in W_2$.

Case 2. So G_j is not a W_2 graph, for some j . By Lemma 9, graph G_i is a W_2 graph for $i \neq j$. So let $J_i \supseteq V_i$ be a maximum independent set in G_i such that $J_i \cap U_i = \emptyset$, for all $i \neq j$. For each $i \neq j$, pick $x_i \in U_i$. Let $X = \{x_i : i \neq j\}$. Clearly X is an independent set. By Theorem 3, the graph $G_X = G - \{X \cup N(X)\}$ is a W_2 graph.

Suppose there exists some $y \in V_i$ such that y is not adjacent to x_i , for some $i \neq j$. Then v is a cutpoint for G_X . By Theorem 4, the graph $G_X - v$ is in W_2 . Since $G_X - v$ contains G_j as a component and G_j is not a W_2 graph, we obtain a contradiction. Thus, $y \in V_i$ implies $y \sim x_i$, for all $i \neq j$.

Let H be the subgraph of G induced by $G_j \cup \{v\}$. Since $y \in V_i$ implies $y \sim x_i$, for all $i \neq j$, then H is a component of G_X . Since $G_X \in W_2$, then $H \in W_2$. Hence, there exists maximum independent set J_j in H such that $J_j \supseteq V_j$ and $J_j \cap U_j = \emptyset$. Let $J = J_1 \cup \dots \cup J_n$. Then J and $\{u\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G . This contradicts $G \in W_2$.

Therefore $\{u, v\}$ must be independent. □

Since a cutset of size two in a W_2 graph of girth 4 is independent, we are led to ask if the same is true for minimum cutsets of size three or more. The next two lemmas help to answer the question for minimum cutsets of size three in W_2 graphs of girth 4.

Lemma 11. *Suppose G is 3-connected W_2 graph of girth 4 and $\{u, v, t\}$ is a cutset for G . Then $\{u, v, t\}$ does not induce exactly one line.*

Proof: Assume to the contrary that $\{u, v, t\}$ induces precisely the line uv . Let G_1, \dots, G_n be the components of $G - \{u, v, t\}$. Let $U_i = N(u) \cap G_i$, $V_i = N(v) \cap G_i$, and $T_i = N(t) \cap G_i$, for all i .

Since t is adjacent to neither u nor v , then we must have $t \sim x$ for all $x \in U_i$ and $t \sim y$ for all $y \in V_i$, for all values of i except possibly one. Otherwise, the graph G_i is a W_2 graph with cutset $\{u, v\}$, contradicting Theorem 10. Without loss of generality, we assume $t \sim x$ for all $x \in U_i$ and $t \sim y$ for all $y \in V_i$, for all $i \neq 1$. Since G has no triangles, it follows that the sets $U_i \cup V_i$ are independent, for $i \neq 1$.

Consider any component different from G_1 , say G_2 . Choose $s \sim t$ such that $s \in V_2$. Then the graph G_s has u as a cutpoint. So by Theorem 4, graph $G_s - u$ is a W_2 graph. Since G_1 is a component of $G_s - u$, then G_1 is a W_2 graph.

Case 1. Suppose there exists $a \in U_1$ and $b \in V_1$ such that $a \sim t$ and $b \sim t$. Since G has no triangles, then a is not adjacent to b . Thus, G_a is a W_2 graph which has v as a cutpoint. By Theorem 4, the graph $G_a - v$ is a W_2 graph. Since G_i , $i \neq 1$, is a component of $G_a - v$, then G_i , $i \neq 1$, is a W_2 graph. Thus, there exist maximum independent sets J_i in G_i such that $J_i \cap V_i = \emptyset$ ($i \neq 1$), and there exists a maximum independent set J_1 in G_1 such that $a \in J_1$ and $J_1 \cap V_1 = \emptyset$. Let $J = J_1 \cup \dots \cup J_n$. Then J and $\{v\}$ are independent sets in G which don't extend to disjoint maximum independent sets in G , contradicting $G \in W_2$.

Case 2. So either t is not adjacent to a for all $a \in U_1$, or t is not adjacent to b for all $b \in V_1$. Without loss of generality, assume t is not adjacent to a for all $a \in U_1$.

Case 2.1. Suppose $T_1 - V_1 \neq \emptyset$. Let $x \in T_1 - V_1$; that is, $t \sim x$ and $x \in V_1$. From the assumption t is not adjacent to a for all $a \in U_1$, we see also that $x \notin U_1$. If there exists $a \in U_1$ such that x is not adjacent to a , or

$b \in V_1$ such that x is not adjacent to b , then the W_2 graph G_x has $\{u, v\}$ as a cutset. This contradicts Theorem 10. Thus, $x \sim a$ for all $a \in U_1$ and $x \sim b$ for all $b \in V_1$. Since G has girth 4, then $a \in U_1$ and $b \in V_1$ imply that a is not adjacent to b . Similarly, $b \in V_1$ implies b is not adjacent to t . Hence, $T_1 \cap V_1 = \emptyset = T_1 \cap U_1$. Therefore, $T_1 - V_1 = T_1$. Thus, if $y \in T_1$, it follows that $y \sim a$ for all $a \in U_1$.

Fix $z \in U_1$. From above, $z \sim y$ for all $y \in T_1$. But then G_z has v as a cutpoint and so by Theorem 4 the graph $G_z - v$ is a W_2 graph.

Case 2.1.1. Suppose $n \geq 3$. Then t is a cutpoint for $G_z - v$. By Theorem 4, graph $G_z - v - t$ is a W_2 graph. Since $G_i, i \neq 1$, is a component of $G_z - v - t$, then $G_i, i \neq 1$, is a W_2 graph. Thus, there exist maximum independent sets J_i in G_i such that $V_i \cap J_i = \emptyset$ ($i \neq 1$), and there exists maximum independent set J_1 in G_1 such that $z \in J_1$ and $V_1 \cap J_1 = \emptyset$. Let $J = J_1 \cup \dots \cup J_n$. Then J and $\{v\}$ don't extend to disjoint maximum independent sets in G , contradicting $G \in W_2$.

Case 2.1.2. So assume $n = 2$. Let H be the graph induced by $G_2 \cup t$. Then H is a component of $G_z - v$; hence, H is a W_2 graph. From earlier, U_2 and V_2 are disjoint and independent, and $t \sim x$ for all $x \in U_2$. Since H is a W_2 graph, there exists maximum independent set J_H in H such that $J_H \supseteq U_2$ and $J_H \cap V_2 = \emptyset$. Note that $t \notin J_H$. Since G_1 is a W_2 graph, there exists maximum independent set J_1 in G_1 such that $z \in J_1$ and $J_1 \cap V_1 = \emptyset$. Then $J = J_1 \cup J_H$ is independent in G . So J and $\{v\}$ don't extend to disjoint maximum independent sets in G , a contradiction.

Case 2.2. Thus, $T_1 - V_1 = \emptyset$. Hence, $V_1 \supseteq T_1$. Since U_1 and V_1 are disjoint independent sets in G_1 , then there exists maximum independent set $J_1 \supseteq U_1$ in G_1 such that $J_1 \cap V_1 = \emptyset$. But then $J_1 \cup \{t\}$ and $\{v\}$ are disjoint independent sets in G which don't extend to disjoint maximum independent sets in G , contradicting $G \in W_2$.

Therefore, $\{u, v, t\}$ does not induce exactly one line in G . □

Lemma 12. *Suppose G is a 3-connected W_2 graph of girth 4 with cutset $\{u, v, t\}$. Then $\{u, v, t\}$ induces at most one line.*

Proof: Assume to the contrary that $\{u, v, t\}$ induces two lines, say uv and vt (since G has girth 4, then $\{u, v, t\}$ cannot induce three lines). Let G_1, \dots, G_n be the components of $G - \{u, v, t\}$. Let $U_i = N(u) \cap G_i$, $V_i = N(v) \cap G_i$ and $T_i = N(t) \cap G_i$, for all i . Note that $U_i \cap V_i = \emptyset = V_i \cap T_i$, for all i ; however, we do not know that $U_i \cap T_i = \emptyset$.

Suppose for all $x \in U_i$, for all i , that $t \sim x$. Then $\{t\}$ and $\{u\}$ don't extend to disjoint maximum independent sets in G , contradicting $G \in W_2$. So, without loss of generality, assume there exists some $x \in U_1$ such that t is not adjacent to x .

Suppose there exists some $z \in V_1$ such that z is not adjacent to x . Then

G_x has $\{v, t\}$ as a cutset, contradicting Theorem 10. We are implicitly using Theorem 13 here, which states that a W_2 graph of girth 4 is 2-connected. Hence, $z \in V_1$ implies $x \sim z$.

Since G is 3-connected, $T_1 \neq \emptyset$. If there exists some $y \in T_1$ such that x is not adjacent to y , then G_x will have $\{v, t\}$ as a cutset, again contradicting Theorem 10. So $y \in T_1$ implies $y \sim x$. Since G has girth 4 and $x \sim y$ for all $y \in T_1$, then $U_1 \cap T_1 = \emptyset$. Since $x \sim y$ for all $y \in T_1$ and $x \sim z$ for all $z \in V_1$, then $y \in T_1$ and $z \in V_1$ implies y is not adjacent to z . Thus, if $y \in T_1$, then G_y has $\{u, v\}$ as a cutset. This contradicts Theorem 10.

Therefore, $\{u, v, t\}$ cannot induce two lines in G . Since G has girth 4, it follows that $\{u, v, t\}$ induces at most one line. \square

With the two preceding lemmas, it is a simple matter to prove in the next theorem that a minimum cutset of size three in a W_2 graph of girth 4 must be independent.

Theorem 13. *If G is a 3-connected W_2 graph of girth 4 with cutset $\{u, v, t\}$, then $\{u, v, t\}$ is independent.*

Proof: By Lemma 12, the set $\{u, v, t\}$ induces at most one line. By Lemma 11, the set $\{u, v, t\}$ does not induce exactly one line. Hence, $\{u, v, t\}$ induces no lines in G ; that is, $\{u, v, t\}$ is independent in G . \square

For minimum cutsets of size four or greater, we do not know if the cutsets must be independent.

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