A Particular Class Of Bigraphs

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ABSTRACT. A semi-complete bigraph G has adjacency matrix

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where $B + B^T = J - I$; so B is the adjacency matrix of a tournament T corresponding to G. We determine algebraic and structural properties of this class of graphs. In particular, we obtain a condition equivalent to the connectedness of a semi-complete bigraph; moreover we determine characterizations of semi-complete bigraphs having 4 distinct eigenvalues in the case of G regular or A irreducible. In particular a regular semi-complete bigraph has 4 distinct eigenvalues if and only if it corresponds to a doubly regular tournament.

1 Introduction

A *bigraph* G is a graph whose vertex set can be partitioned into two nonempty subsets V and W such that every edge of G joins V with W.

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If $|V| = r$ and $|W| = s$, we shall say that G is of *parts* (r, s).

G is regular of *degree* k if every vertex is incident with k edges.

We recall also that a *tournament* is an oriented complete graph; we shall say that the vertex v of a tournament T has *positive* [*negative*] *valence* k if there are k arcs *from* [*into*] v. Moreover if (v_i, v_j) is an arc of T, we shall say that v_i *dominates* v_j . For every vertex v, $od(v)$ [$id(v)$] denotes the number of vertices dominated by v [which dominate v]. A tournament T is *regular* of degree t if the positive valence of each of its vertices is t.

Let J denote the matrix all of whose entries are equal to 1. In the following we shall denote the (0,1)-adjacency matrix A of G in the form:

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}. \tag{1.1}$$

We shall say that the bigraph G is *semi-complete* (denoted by s-c) if it is of parts (n, n) and if the matrix B of (1.1) satisfies the relation:

$$B^T + B = J - I \tag{1.2}$$

From (1.2) it follows that B is the adjacency matrix of a tournament T of order n and we say that T *corresponds* to G. We also say that B corresponds to A and conversely.

We investigate the connections between the properties of the spectrum and the connectedness of the two classes of graphs, in particular when they are regular. So we obtain spectral properties of G and we prove that a connected s-c-bigraph has exactly 4 distinct eigenvalues if and only if it corresponds to a doubly regular tournament (Theorem 4.3). Moreover we obtain a condition equivalent to the connectedness of a s-c-bigraph (Theorem 2.6).

Recall that a square non negative matrix A is reducible provided there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix}$$

where A_1 and A_2 are square non vacuous matrices. If no such P exists, then the matrix is irreducible. Thus a primitive matrix is necessarily irreducible. The Perron-Frobenius Theorem (see Minc [11]) implies that the spectral radius, ρ, of an irreducible non negative primitive matrix is an algebraically simple eigenvalue and that the eigenspace corresponding to the Perron-value ρ is spanned by an eigenvector, the Perron-vector, each of whose entries is positive. It is well-known that A is irreducible if and only if its directed graph is strongly connected; in particular, if A is the adjacency matrix of an undirected graph G, A is irreducible if and only if G is connected.

We investigate the spectral properties of an s-c-bigraph with irreducible adjacency matrix having 4 distinct eigenvalues. In particular we obtain (Proposition 5.5) another characterization of doubly regular tournaments.

2 Regularity and Connectness

Throughout the paper G represents a s-c-bigraph with adjacency matrix A satisfying (1.1) and (1.2). $V = \{v_1, v_2, \cdots, v_n\}$ and $V' = \{v'_1, v'_2, \cdots, v'_n\}$ are the classes of vertices of G corresponding respectively to the rows and the columns of B.

So v_i and v'_j are adjacent if and only if (v_i, v_j) is an arc of T. We say that v_i *corresponds to* v'_i and conversely.

Remark 2.1. *If $V = \{v_1, v_2, \cdots, v_n\}$ and $V' = \{v'_1, v'_2, \cdots, v'_n\}$ are the classes of vertices of a s-c-bigraph G, then $\forall i, \in \{1, \cdots, n\}$, v_i and v'_i are not adjacent. Moreover, for every pair of vertices v_i, v_j of V, G contains one and only one of the edges (v_i, v'_j) and (v'_i, v_j). It implies that the number of edges in G is $\dfrac{n(n-1)}{2}$, the same of T.*

Proposition 2.2. *If G is a s-c-bigraph of parts (n, n) and T is the corresponding tournament, then G is regular with degree k and $n = 2k + 1$ if and only if T is regular with positive valence k.*

Proof: Let A be the adjacency matrix of G and J has order $2n$. G is regular, of degree k if and only if $AJ = kJ$, that is $BJ = kJ$ (and, from (1.2), $B^T J = kJ$). This relation means that T is regular of valence k. Moreover it is well known that, if T is a regular tournament of order n and of positive valence k, then $n = 2k + 1$. □

Proposition 2.3. *Let H be a connected component of order greater than 1 of a s-c-bigraph G such that $|H| \leq |G - H|$. Then $G \cong K_2$.*

Proof: Let V_H and W_H and $\bar{V}, \bar{W}$ be the sets of vertices of H and $G - H$ contained in V and V' respectively.

Suppose $|V_H| \geq 2$. We have to distinguish the cases in which W_H contains or does not contain a vertex corresponding to one of the vertices of V_H.

In the first case we suppose that W_H contains a', the vertex corresponding to a vertex a of V_H. Suppose that $|\bar{W}| \geq |\bar{V}|$. As a is not adjacent to any vertex of $\bar{W}$, then a' is adjacent to all the corresponding ones of $\bar{W}$, which have to belong to V_H. Since $|H| \leq |G - H|$ and $|\bar{W}| \geq |\bar{V}|$, then $\bar{W}$ contains at least two elements, say b' and c'. Then $b, c \in V_H$. As either (b, c') or (b', c) belong to G, we have a contradiction. A similar argument applies if $|\bar{W}| \leq |\bar{V}|$.

In the second case we suppose W_H does not contain a vertex corresponding to a vertex of V_H. Let a, b vertices of V_H. Then a', b' belong to $\bar{W}$. As either (a, b') or (a', b) belong to G, again we have a contradiction.

So V_H and W_H have order 1. Because H is connected we obtain $G \cong K_2$. □

Proposition 2.4. *A s-c-bigraph G contains K_2 as component if and only if T contains an arc (u, v) such that $id(u) = n-2$ and $od(v) = n-2$.*

Proof: Let K_2 a component of G and u, v' its vertices. Then u is not adjacent to the remaining vertices of W. So in T u is dominated by all the vertices but v. Similarly v dominates all the vertices but u. The converse is immediate. □

Proposition 2.5. *A s-c-bigraph G contains an isolated vertex if and only if T contains a vertex u such that either $od(u) = n-1$ or $id(u) = n-1$.*

Proof: If the isolated vertex u belongs to V, then in T u does not dominate any vertex and $id(u) = n-1$. If $u \in V'$ and $u = v'$, then in T v dominates all the remaining vertices and $od(v) = n-1$. □

Theorem 2.6. *A s-c-bigraph G is connected if and only if T contains neither a vertex with $od(v)$ or $id(v)$ equal to $n-1$, nor an arc (u, v) with $id(u) = n-2$ and $od(v) = n-2$*

Proof: It follows from Propositions 2.4 and 2.5. □

Corollary 2.7. *If T is a regular tournament of order at least 5, then the corresponding s-c-bigraph is connected.*

Proof: If T is regular, clearly it does not contain a vertex v or an arc satisfying Theorem 2.6. □

3 Spectral Properties

In this section we prove some spectral properties of a s-c-bigraph, with particular interest in the case when G is regular.

Proposition 3.1. *Let T be a regular tournament of order n and let G be the regular corresponding semi-complete bigraph. Let ω be an eigenvalue of T of multiplicity m. Then $\pm|\omega|$ are eigenvalues of G of multiplicity m if ω is real and $2m$ if ω is complex. Moreover $\pm\dfrac{n-1}{2}$ are the unique simple eigenvalues of G and every other eigenvalue has even multiplicity.*

Proof: Since (1.1) is the adjacency matrix of G, we have $\phi(G, \lambda) = \det(\lambda^2 I - BB^T)$. As T is regular, B is normal [5]; thus there exists an unitary matrix U such that $UB\bar{U}^T$ is a diagonal matrix Δ. Since B is real, $B^T = U\bar{\Delta}\bar{U}^T$ and so $B^T B = BB^T = U\Delta\bar{\Delta}\bar{U}^T$.

If ω is a complex eigenvalue of T with multiplicity m, then $\omega \cdot \bar{\omega} = |\omega|^2$ is an eigenvalue of BB^T with multiplicity $2m$. In fact, since $Re(\omega) = -\frac{1}{2}$ [2], $\bar{\omega}$ is the only eigenvalue with the same modulus as ω. So $\pm|\omega|$ are eigenvalues of G with the same multiplicity $2m$.

Moreover, since B is regular, in [2] is proved that $\frac{n-1}{2}$ is the unique real eigenvalue of T and it is simple. Therefore $\pm\frac{n-1}{2}$ are the unique simple eigenvalues of A.

As every other eigenvalue of T is complex, then every other eigenvalue of G has even multiplicity. □

Corollary 3.2. *Let T be a regular tournament of order n and let G be the regular corresponding s-c-bigraph. Then $\frac{n-1}{2}$ is a simple eigenvalue of T; moreover if $\alpha \neq \pm\frac{n-1}{2}$ is an eigenvalue of G of multiplicity $2m$, then $-\frac{1}{2} \pm \sqrt{\alpha^2 - \frac{1}{4}}$ are eigenvalues of T of multiplicity m.*

Proof: It follows from the condition that $Re(\omega) = -\frac{1}{2}$ and from Proposition 3.1. □

Proposition 3.3. *Let G be a connected semi-complete bigraph. Then G has at least 4 distinct eigenvalues.*

Proof: Let n be the order of G and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ their eigenvalues. By Perron-Frobenius theorem, λ_1 is simple and, as G is bipartite, $\lambda_n = -\lambda_1$. Every other eigenvalue λ_i satisfies $\lambda_i = -\lambda_{n+1-i}$. Suppose G has only three distinct eigenvalues. Then $\lambda_i = 0$ for every $i \neq 1, n$. In this case G has exactly one positive eigenvalue; then G is a complete multipartite graph, a contradiction. □

A question arises whether there exists a s-c-bigraph with exactly 4 distinct eigenvalues. In the next section we give characterizations of such bigraphs first in the case of G regular and A irreducible, then in the case of G connected.

4 Doubly Regular Tournaments

Recall that a tournament is *doubly regular with subdegree t* if all pairs of vertices jointly dominate precisely t vertices.

Such a tournament is regular of degree $k = 2t + 1$ and hence has $4t + 3$ vertices. If B is a doubly regular tournament matrix, then $BB^T = \frac{k+1}{2}I + \frac{k-1}{2}J$.

In [14] the following characterization is proved:

Theorem 4.1. *A regular tournament matrix A of order $n = 2k + 1$ is doubly regular if an only if it has exactly three distinct eigenvalues. In the case that A is doubly regular, its eigenvalues are k, $\frac{-1 \pm i\sqrt{n}}{2}$, the first being simple and the other two having multiplicities k.*

The problem of the existence of a doubly regular tournament in every admissible order is still open. Since Theorem 4.3 and the results of Section 5 provide new characterizations of doubly regular tournaments, they may have some bearing on the existence problem.

Proposition 4.2. *Let G be a regular s-c-bigraph with 4 distinct eigenvalues. Then the tournament corresponding to G is doubly regular.*

Proof: Let G a regular s-c bigraph of parts (n, n); we denote k the degree of G so that $n = 2k + 1$.

If G has 4 distinct eigenvalues, then these are $\pm k$, of multiplicity 1 and $\pm\rho$ of multiplicity $\frac{n-2}{2}$.

It is shown in [4, pg. 94] that if $\lambda_1 = r, \lambda_2, \cdots, \lambda_m$ is the spectrum of a graph G, then G is regular if and only if

$$\frac{1}{m}\sum \lambda_i^2 = r \tag{4.1}$$

r being the index of G.

If (4.1) is satisfied, r is the degree of G. In our case, we obtain

$$\frac{1}{2n}\left(2k^2 + (2n-2)\rho^2\right) = k;$$

then

$$\rho = \pm\sqrt{\frac{k+1}{2}}.$$

By corollary 3.2, we see that T has eigenvalues k, $\frac{-1 \pm i\sqrt{n}}{2}$. By theorem 4.1 T is doubly regular. □

Theorem 4.3. *A doubly regular tournament of order n exists if and only if there exists a regular s-c-bigraph of parts (n, n) with 4 distinct eigenvalues.*

Proof: Let T be a doubly regular tournament of order n. Then $n = 2k+1$, where k is the positive valence of T. Then the corresponding s-c-bigraph is regular of degree k.

By Theorem 4.1 T has three distinct eigenvalues, one real of multiplicity 1 and the others of multiplicity $\frac{n-1}{2}$. So by Propositions 3.1, G has 4 distinct eigenvalues. The converse follows from Proposition 4.2. □

Corollary 4.4. *Let T a doubly regular tournament of order $n = 2k+1$ and G the corresponding s-c-bigraph. Then the eigenvalues of T are k and $\frac{1}{2}(-1 \pm i\sqrt{n})$ of multiplicity 1 and k respectively, while the eigenvalues of the bigraph are $\pm k$, both of multiplicity 1, and $\pm\sqrt{\frac{k+1}{2}}$, both of multiplicity $2k$.*

Proof: It follows from Proposition 3.1 and Theorem 4.1. □

It is well known that a symmetric design $< \nu, k, \lambda >$ exists if and only if there exists a regular bigraph of order $2n$ with eigenvalues $\pm k$, $\pm(k-\lambda)^{\frac{1}{2}}$ of multiplicity 1 and $\nu - 1$ respectively ([4] pg. 190).

In the case of skew-Hadamard design, from Corollary 3.3 of [14] and from Proposition 3.6 of [13] and from Theorem 4.3 we obtain the following

Corollary 4.5. *A symmetric skew-Hadamard design of order n exists if and only if there exists a skew symmetric regular bigraph of order $2n$ with four distinct eigenvalues.*

Theorem 4.6. *A regular s-c-bigraph G has four distinct eigenvalues if and only if the number of vertices adjacent to any two vertices of the same class is constant.*

Proof: Let G be a s-c-bigraph. If the vertices v_i, v_j in V are adjacent in G to the vertex v'_h in V', then in T v_i, v_j dominate v_h. If the vertices v'_i, v'_j in V' are adjacent to the vertex v_k in V, then in T v_i, v_j are dominated by v_k.

If G is regular with 4 distinct eigenvalues, then by Theorem 4.3 the tournament T corresponding to G is doubly regular. Recall that a tournament T of order $n = 2k+1$ is doubly regular if and only if for any pair of vertices the number of vertices that dominate both of them is constant and equal to t. The number of vertices that are dominated by both of them is also equal to t. The converse follows easily. □

Proposition 4.7. *Let u, v be vertices of a s-c-bigraph corresponding to a doubly regular tournament. If they belong to the same class of vertices, their distance is 2, while, if they belong to different classes and are not adjacent, their distance is 3.*

Proof: By Theorem 2.6, G is connected. It is proved [4, pg. 88] that if a connected graph has exactly m distinct eigenvalues, then its diameter D satisfies the inequality $D \leq m - 1$. So because G has 4 distinct eigenvalues then $D \leq 3$. As G is a bipartite graph, the property follows. □

Recall that the *complexity* of a graph G is the number of spanning trees of G. Now we determine the complexity $\kappa(G)$ of a s-c-bigraph G associated with a doubly-regular tournament T. We recall [1] that if the eigenvalues of an undirected regular graph of valence k and order n are $k, \lambda_1 \cdots \lambda_{s-1}$ of multiplicities $1, m_1, ..., m_{s-1}$ respectively, then the complexity is

$$\kappa(G) = n^{-1} \prod_{r=1}^{s-1} (k - \lambda_r)^{m_r}$$

In this case, since G has order $2n = 2(2k+1)$ we obtain

$$\kappa(G) = k(2k+1)^{2k-1} \cdot \left(\frac{k-1}{2}\right)^{2k}$$

Let G be a s-c-bigraph of parts (n, n). Denote by $\bar{G}$ the graph complementary to G with respect to the complete bipartite graph $K_{n,n}$, which we call bi-complementary. So $\bar{G}$ is bipartite with the same classes of vertices of G and a vertex of V and one of V' are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Proposition 4.8. *Let $\bar{G}$ the s-c-bigraph bi-complementary of the bigraph G corresponding to a doubly regular tournament T of order $n = 2k+1$. Then the eigenvalues of $\bar{G}$ are $\pm(k+1)$ of multiplicities 1 and $\pm\sqrt{\frac{k+1}{2}}$ of multiplicities $n-1$.*

Proof: If (1.1) represents the graph G, then the adjacency matrix $\bar{A}$ of $\bar{G}$ is

$$\bar{A} = \begin{bmatrix} 0 & J-B \\ J-B^T & 0 \end{bmatrix}$$

where J has order n. Then

$$\begin{aligned}\phi(\bar{G}, \lambda) &= \begin{vmatrix} \lambda I & -J+B \\ -J+B^T & \lambda I \end{vmatrix} \\ &= \left|\lambda^2 I - (J-B)(J-B^T)\right| \\ &= \left|\left(\lambda^2 - \frac{k+1}{2}\right) I - \frac{k+1}{2} J\right| \\ &= (\lambda^2 - (k+1)^2) \cdot \left(\lambda^2 - \frac{k+1}{2}\right)^{n-1}.\end{aligned}$$

Last equality follows from the property that $\det(aI - bJ) = (a+nb)^{n-1}$. This implies the proposition. □

Corollary 4.9. *Let T a doubly regular tournament of order $n = 4t+3$, G the corresponding s-c-bigraph and A, $\bar{A}$ the adjacency matrices of G and $\bar{G}$. Then $|A| = (t+1)(2t+1)^2$ and $|\bar{A}| = (t+1)(2t+2)^2$. In both the cases they are positive integers.*

Proof: It follows from Corollary 4.4 and Proposition 4.8. □

5 Semi-complete bigraphs having four distinct eigenvalues

In the preceding section we saw that a doubly regular tournament corresponds to a s-c-bigraph with exactly four eigenvalues. The question arises as to whether every connected s-c-bigraph having four eigenvalues necessarily corresponds to a doubly regular tournament. In this section, we answer that question in the affirmative.

Proposition 5.1. *Suppose that B is an $n \times n$ tournament matrix and that $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ is an irreducible matrix with four distinct eigenvalues. Then B has at most two distinct eigenvalues whose real part is not $-\frac{1}{2}$, and B has a single complex conjugate pair of eigenvalues with real part equal to $-\frac{1}{2}$.*

Proof: Recall that the nonzero eigenvalues of G are positive and negative square roots of the eigenvalues of BB^T. If G were singular and had spectral radius r, then the eigenvalues of G would include $\pm r$, 0, and at least one other nonzero eigenvalue, s say, the last since G must have at least four eigenvalues. But then $-s$ would also be an eigenvalue of G, contradicting the fact that G has exactly four eigenvalues. Consequently, G must be nonsingular, and so its eigenvalues are precisely the square roots of the eigenvalues of BB^T.

Since G is irreducible, its perron value has algebraic multiplicity one, and it follows that the perron values of both BB^T and B^TB also have algebraic multiplicity one. Further, since G has four distinct eigenvalues, we find that both BB^T and B^TB have exactly two distinct eigenvalues: the perron value, a say,and another eigenvalue $b < a$, with algebraic multiplicity $n-1$.

Let u be a perron vector for BB^T, normalized so that $u^Tu = 1$; then the eigenprojection matrix corresponding to a for BB^T is uu^T. Let E be the eigenprojection corresponding to b for BB^T. Using the spectral resolution, we have $BB^T = auu^T + bE$, and $uu^T + E = I$. Hence we find that $BB^T = (a-b)uu^T + bI$. A similar argument shows that $B^TB = (a-b)vv^T + bI$, where v is the perron vector for B^TB normalized so that $v^Tv = 1$.

Let $\mathbf{s} = B\mathbf{1}$, and let $\mathbf{r}^T = \mathbf{1}^TB$. Since $B^T = J - I - B$, we find that

$$\mathbf{s}\mathbf{1}^T - B - B^2 = (a-b)uu^T + bI \tag{5.1}$$

and that

$$\mathbf{1}\mathbf{r}^T - B - B^2 = (a-b)vv^T + bI \tag{5.2}$$

In particular, (5.1) and (5.2) imply that

$$\frac{n^2(n-1)}{2} - \mathbf{1}^TB\mathbf{1} - \mathbf{1}^TB^2\mathbf{1} = \mathbf{1}^T[\mathbf{s}\mathbf{1}^T - B - B^2]\mathbf{1} = (a-b)(u^T\mathbf{1})^2 - bn$$

and

$$\frac{n^2(n-1)}{2} - \mathbf{1}^T B\mathbf{1} - \mathbf{1}^T B^2\mathbf{1} = \mathbf{1}^T[\mathbf{1}\mathbf{r}^T - B - B^2]\mathbf{1} = (a-b)(v^T\mathbf{1})^2 - bn,$$

from wihch it follows that $u^T\mathbf{1} = v^T\mathbf{1}$. Further,

$$[B^2 + B + bI]\mathbf{1} = [\mathbf{s}\mathbf{1}^T - (a-b)uu^T]\mathbf{1} = [\mathbf{1}\mathbf{r}^T - (a-b)vv^T]\mathbf{1}$$

so that

$$n\mathbf{s} - (a-b)(u^T\mathbf{1})u\left(\frac{n(n-1)}{2}\right)\mathbf{1} + (a-b)(v^T\mathbf{1})v = 0.$$

Hence we see that

$$(a-b)(u^T\mathbf{1})[u-v] = n\left[\mathbf{s} - \left(\frac{n-1}{2}\right)\right]\mathbf{1} \tag{5.3}$$

so that $u = v + \varepsilon\left[s - \left(\frac{n-1}{2}\right)\mathbf{1}\right]$, where $\varepsilon = \frac{n}{(a-b)u^T\mathbf{1}} > 0$.

Since $[B^2 + B + bI]\mathbf{s} = [\mathbf{s}\mathbf{1}^T - (a-b)uu^T]\mathbf{s} = [\mathbf{1}r^T - (a-b)vv^T]\mathbf{s}$, we have

$$\begin{aligned}
&\frac{n(n-1)}{2}\mathbf{s} - (a-b)(v^T s)\varepsilon\left[\mathbf{s} - \frac{n-1}{2}\mathbf{1}\right] \\
&- (a-b)\varepsilon\left[\mathbf{s}^T\mathbf{s} - \frac{n(n-1)^2}{4}\right]v \\
&- (a-b)\varepsilon^2\left[\mathbf{s}^T\mathbf{s} - \frac{n(n-1)^2}{4}\right]\left[\mathbf{s} - \frac{n-1}{2}\right] \\
&= (\mathbf{r}^T\mathbf{s})\mathbf{1}
\end{aligned} \tag{5.4}$$

If $\mathbf{s}\mathbf{s}^T - \frac{n(n-1)^2}{4} = 0$, it follows that each entry in $\mathbf{s}$ must equal $\frac{n-1}{2}$, so that B is regular. If $\mathbf{s}^T\mathbf{s} - \frac{n(n-1)^2}{4} \neq 0$, then (5.4) implies that v is a linear combination of $\mathbf{1}$ and s, and hence we see from (5.3) that u is also a linear combination of $\mathbf{1}$ and s. In either case, we find from (5.1) that $B^2\mathbf{1}$ is a linear combination of $\mathbf{1}$ and s. Consequently, a result of Kirkland and Shader [5] implies that B has at most two eigenvalues whose real part is not equal to $-\frac{1}{2}$. Moreover, since G is irreducible, it follows that $n \geq 3$, and hence B must have at least one eigenvalue whose real part is $-\frac{1}{2}$.

Let w be an eigenvector of B corresponding to an eigenvalue $\lambda = -\frac{1}{2} + i\gamma$. Then w is orthogonal to both $\mathbf{1}$ and s (see [13], for example), and hence it is orthogonal to u. Since $[B^2 + B + bI]w = [s\mathbf{1}^T - (a-b)uu^T]w = 0$, we find that $\lambda^2 + \lambda + b = 0$. Substituting $\lambda = -\frac{1}{2} + i\gamma$ into this last equation yields $b = \frac{1}{4} + \gamma^2 = |\lambda|^2$. Hence any eigenvalue of B whose real part is $-\frac{1}{2}$ is equal to either $-\frac{1}{2} + i\gamma$ or $-\frac{1}{2} - i\gamma$, where $\gamma = \sqrt{b - \frac{1}{4}}$. Thus, B has a single complex conjugate pair of eigenvalues with real part equal to $-\frac{1}{2}$. □

Theorem 5.2. *Suppose that B is an $n \times n$ tournament matrix and that n is odd. If $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ is an irreducible matrix with four distinct eigenvalues, then B is doubly regular.*

Proof: By Proposition 5.1, B has at most two eigenvalues whose real part is not equal to $-\frac{1}{2}$. Now $-\frac{1}{2}$ is not an eigenvalue of B (indeed of any integral matrix) since it is not an algebraic integer. Consequently, all eigenvalues of B with real part equal to $-\frac{1}{2}$ must come in complex conjugate pairs, and we find that the number of eigenvalues with real part $-\frac{1}{2}$ is even. Since n is odd, it follows that the number of eigenvalues of B with real part $-\frac{1}{2}$ must be $n-1$. Since $trace(B) = 0$, we find that the perron value of B is $\frac{n-1}{2}$, so that B must be regular (see [6], for example). Further, Proposition 5.1 also implies that B has, apart from its perron value $\frac{n-1}{2}$, just two other eigenvalues which form a complex conjugate pair. It now follows that B must be a doubly regular matrix (see theorem 4.1). □

Theorem 5.3. *Suppose that B is an $n \times n$ tournament matrix and that n is even. If $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ is an irreducible matrix with four distinct eigenvalues, then the eigenvalues of B are*

$$\frac{n - 2 \pm \sqrt{n^2 - 4n + 4 + 4(n-2)\sqrt{n-1}}}{4}$$

and

$$-\frac{1}{2} \pm \frac{i\sqrt{n - 1 - 2\sqrt{n-1}}}{2}.$$

Further, half of the row sums of B are equal to

$$\frac{n-1}{2} + \frac{\sqrt{n - 1 + (n-2)\sqrt{n-1}}}{2},$$

and the other half are equal to $\dfrac{n-1}{2} - \dfrac{\sqrt{n-1+(n-2)\sqrt{n-1}}}{2}$.

Proof: From Proposition 5.1, B has at most two eigenvalues with real part not equal to $-\dfrac{1}{2}$. A result of Kirkland [8] implies that those eigenvalues are: the perron value $\rho = \dfrac{n-2+\sqrt{n^2-\frac{16\delta^2}{n}}}{4}$ and another eigenvalue $\sigma = \dfrac{n-2-\sqrt{n^2-\frac{16\delta^2}{n}}}{4}$, where $\delta^2 = \mathbf{s}^T\mathbf{s} - \dfrac{n(n-1)^2}{4}$ and $s = B\mathbf{1}$. From Proposition 5.1, B has just two other eigenvalues — a complex conjugate pair of the form $-\frac{1}{2} \pm i\gamma$, each of multiplicity $\frac{n-2}{2}$. Since $trace(B^2) = 0$, we find that $2\left(\dfrac{(n-2)^2+n^2-\frac{16\delta^2}{n}}{16}\right) + (n-2)\left(\frac{1}{4}-\gamma^2\right) = 0$, which yields $\gamma^2 = \dfrac{n^2-n-\frac{8\delta^2}{n}}{4(n-2)}$. It remains only to find δ in order to sort out the eigenvalues of B. As in Proposition 5.1, we have $BB^T = (a-b)uu^T + bI$ and $B^TB = (a-b)vv^T + bI$ (note that $b = \frac{1}{4} + \gamma^2 = \dfrac{n^2-2-\frac{8\delta^2}{n}}{4(n-2)}$). Considering the diagonal entries of BB^T, we see that for $1 \le j \le n$ $s_j = (a-b)u_j^2 + b$, whence $u_j = \sqrt{\dfrac{s_j-b}{a-b}}$ for $1 \le j \le n$; a similar argument yields $v_j = \sqrt{\dfrac{n-1-s_j-b}{a-b}}$ for $1 \le j \le n$. Moreover, since $u-v$ is a multiple of $s - \left(\frac{n-1}{2}\right)\mathbf{1}$, we have for some ϑ, $\sqrt{s_j-b} - \sqrt{n-1-s_j-b} = \vartheta\left(s_j - \frac{n-1}{2}\right)$.

Hence

$$2\left(s_j - \frac{n-1}{2}\right) = \vartheta\left(s_j - \frac{n-1}{2}\right)\left(\sqrt{s_j-b} - \sqrt{n-1-s_j-b}\right)$$

for $1 \le j \le n$. Now $s_j \ne \frac{n-1}{2}$ for $1 \le j \le n$; in fact the left side is an integer while the right side is not. So it follows that $s_j = \frac{n-1}{2} \pm x$ for some x. From the fact that $\mathbf{1}^Ts = \frac{n(n-1)}{2}$, we deduce that $\frac{n}{2}$ of the $s_j's$ are equal to $\frac{n-1}{2} + x$ and $\frac{n}{2}$ of the $s_j's$ are equal to $\frac{n-1}{2} - x$. Since $\frac{\delta^2}{n} = x^2$, it remains only to find x in order to complete the proof.

Let $y = \frac{1}{\sqrt{n}}\mathbf{1}$ and let $z = \frac{1}{\delta}\left(s - \frac{n-1}{2}\mathbf{1}\right)$ (z is unit vector). Since $B^2\mathbf{1}$ is a linear combination of $\mathbf{1}$ and s, a result in [9] implies that a perron vector p of B is a linear combination of y and z, namely $p = (p^Ty)y + (p^Tz)z$. Since $p^Tz = \left(\frac{n-1}{2} - \rho\right)\frac{p^T\mathbf{1}}{\delta}$, we see that $p_1 = \frac{1}{\sqrt{n}}y + \dfrac{\frac{n-1}{2}-\rho}{\delta}z$ is a perron vector for B. Now $By = \frac{1}{\sqrt{n}}s = \frac{n-1}{2}y + \frac{\delta}{\sqrt{n}}z$, and from the fact that $Bp_1 = \rho p_1$ we find that $Bz = -\frac{\delta}{\sqrt{n}}y + \left(\rho - \frac{\delta^2}{n\left(\frac{n-1}{2} - \rho\right)}\right)z = -\frac{\delta}{\sqrt{n}}y - \frac{1}{2}z.$ It now follows that

$$BB^T y = \left(\frac{(n-1)^2}{4} + \frac{\delta^2}{n}\right)y + \frac{\delta\sqrt{n}}{2}z$$

and

$$BB^T z = \frac{\delta\sqrt{n}}{2}y + \left(\frac{\delta^2}{n} + \frac{1}{4}\right)z.$$

Letting $v_1, \cdots, v_{\frac{n-2}{2}}$ be orthonormal eigenvectors of B which (collectively) span the eigenspaces corresponding to $-\frac{1}{2} \pm i\gamma$. (It can be shown that any eigenvector of a tournament matrix corresponding to an eigenvalue whose real part is $-\frac{1}{2}$ is always orthogonal to $\mathbf{1}$, s, and to the eigenspace corresponding to any other eigenvalue, so such v_j's always exist).

We find that the matrix $M = \left[y|z|v_1|\cdots|v_{\frac{n-2}{2}}\right]$ is unitary, and that

$$M^*BB^TM = \begin{bmatrix} \frac{(n-1)^2}{4} + \frac{\delta^2}{n} & \frac{\delta\sqrt{n}}{2} & & & \\ \frac{\delta\sqrt{n}}{2} & \frac{\delta^2}{n} + \frac{1}{4} & & 0 & \\ & & b & & \\ & 0 & & \ddots & \\ & & & & b \end{bmatrix}.$$

Since BB^T has a as an eigenvalue of multiplicity one and b as an eigenvalue of multiplicity $n-1$, it follows that the eigenvalues of the matrix $C = \begin{bmatrix} \frac{(n-1)^2}{4} + \frac{\delta^2}{n} & \frac{\delta\sqrt{n}}{2} \\ \frac{\delta\sqrt{n}}{2} & \frac{\delta^2}{n} + \frac{1}{4} \end{bmatrix}$ are a and b.

Calculating the eigenvalues of C and using the fact that $\frac{\delta^2}{n} = x$, we find $a = \frac{1}{2}\left(\frac{n^2-2n+2}{4} + 2x^2 + n\sqrt{\frac{(n-2)^2}{4} + x^2}\right)$, while $b = \frac{1}{2}\left(\frac{n^2-2n+2}{4} + 2x^2 - n\sqrt{\frac{(n-2)^2}{4} + x^2}\right)$. But b is also equal to

$\frac{n^2-2-8x^2}{4(n-2)}$, so equating the two expressions for b yields, after some simplifications, $x^4 + \left(\frac{n-1}{2}\right)x^2 - \frac{(n-1)(n^2-5n+5)}{16} = 0$, so that $x^2 = \frac{n-1}{4} + \frac{n-2}{4}\sqrt{n-1}$. The formulae for the eigenvalues of B and for $B1$ now follow. □

Lemma 5.4. *If n is even, then $\sqrt{n-1+(n-2)\sqrt{n-1}}$ is not an integer, unless $n=2$.*

Proof: Suppose that it is an integer, j say, and let $n=2k$. Then $2k-1+2(k-1)\sqrt{2k-1} \in \mathbf{Z}$, and hence $\sqrt{2k-1} \in \mathbf{Q}$. Let $\sqrt{2k-1} = \frac{p}{q}$ with $gcd(p,q)=1$. It follows that $2k-1$ must be a square (necessarilly odd), say $2k-1=l^2$. Thus $l(l^2+l-1)=j^2$. Let α be a prime factor of l^2+l-1, say with $\alpha^i | l^2+l-1$, $\quad \alpha^{i+1} \nmid l^2+l-1$. Then $\alpha | j$, so $j=\alpha^m\beta$ for some m, where $\alpha \nmid \beta$. Hence $2m \geq i$.

If $2m > i$, we have, since $l^2+(l^2-1)l = j^2$, that α must divide l, a contradiction.

Thus we see that $2m=i$, so that every prime factor of l^2+l-1 has even multiplicity. Thus l^2+l-1 is the square of some integer, say $l^2+l+1=\mu^2$. Certainly it is not possible $l \geq \mu$, unless $l=\mu=1$. So $l<\mu$, then $l+1 \leq \mu$ and $l(l+1) \leq \mu^2$. This implies $l(l+1)-1 < \mu^2$, a contradiction. □

Proposition 5.5. *Let G be a connected s-c-bigraph, corresponding to a tournament T and having an irreducible adjacency matrix. Then T is doubly regular if and only if G has 4 distinct eigenvalues.*

Proof: Let T doubly regular. By Theorem 4.3 G has 4 distinct eigenvalues. Conversely, let G have 4 distinct eigenvalues. If n is odd, by Theorem 5.2 T is doubly regular. Suppose that n is even. Then by Theorem 5.3 half of the row-sums of B are $\frac{n-1}{2} + \frac{\sqrt{n-1+(n-2)\sqrt{n-1}}}{2}$ and the other half are $\frac{n-1}{2} - \frac{\sqrt{n-1+(n-2)\sqrt{n-1}}}{2}$. Since the row-sums of B are integers, then so is

$$\left(\frac{n-1}{2}+\frac{\sqrt{n-1+(n-2)\sqrt{n-1}}}{2}\right)-$$
$$-\left(\frac{n-1}{2}-\frac{\sqrt{n-1+(n-2)\sqrt{n-1}}}{2}\right)$$
$$=\sqrt{n-1+(n-2)\sqrt{n-1}}.$$

But by the lemma above, we must have $n = 2$, and it follows that A is not irreducible, a contradiction. □

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