Three Mutually Orthogonal Idempotent Latin Squares of Order 18

Xiafu Zhang
Department of Basic Science and Technology
Kunming Institute of Technology
Kunming, Yunnan 650093
P.R. China

Hangfu Zhang
Department of Mathematics
University of Southern California
Los Angeles, CA 90089–1113

ABSTRACT. Three mutually orthogonal idempotent Latin squares of order 18 are constructed, which can be used to obtain 3 HMOLS of type 5^{18} and type 23^{18} and to obtain a (90, 5,1)-PMD.

1 Introduction

A Latin square based on $\{1,2,\ldots,n\}$ is called idempotent if its (i,i) $(i = 1,2,\ldots,n)$ entries are i. It is known [1] that there exist three mutually orthogonal idempotent Latin squares of order n for any integer $n \geq 5$ and $n \neq 6$, 10,18, 22, 26. Recently, the authors [3] solved the cases $n = 22$ and 26. In this paper, we construct three mutually orthogonal idempotent Latin squares of order 18. As an application we use them to obtain 3 HMOLS of type 5^{18} and type 23^{18} and to obtain a (90, 5, 1)-PMD.

2 Method of Construction

Assume that $Z_{17} = (0,1,\ldots,16)$ is the cycle of residue of modules 17. Add a point ∞ to Z_{17} and define for every $a \in Z_{17}$

$$a + \infty = \infty + a = a - \infty = \infty - a = a \cdot \infty = \infty \cdot a = \infty.$$

Let $A = \|a_{ij}\|$, $i = 1, \ldots, 5$, $j = 1, \ldots, 19$ be a matrix over $Z_{17} \cup \{\infty\}$. Let $r_1 = (x_1, \ldots, x_{19})$, $r_2 = (y_1, \ldots y_{19})$ be rows of A, r_1 and r_2 are said to be orthogonal if

1) $(x_i, y_i) \neq (\infty, \infty)$, $i = 1, \ldots, 19$
2) there exist integers s and t, $1 \leq s \neq t \leq 19$, such that $x_s = \infty$, $y_t = \infty$.
3) for every $a \in Z_{17}$ there exists a pair (x_i, y_i) with $x_i - y_i \equiv a \pmod{17}$

The matrix A is said to be a DS-matrix if its rows are mutually orthogonal [4]. For $b \in Z_{17}$ define $A + b = \|a_{ij} + b\|$, $i = 1, \ldots, 5$, $j = 1, \ldots, 19$.

It is obvious that the matrix

$$\left\| \begin{array}{c|c|c|c|c} \infty & & & & \\ \infty & & & & \\ \infty & A & A+1 & \cdots & A+16 \\ \infty & & & & \\ \infty & & & & \end{array} \right\|$$

is an $OA(18, 5)$ orthogonal array over $Z_{17} \cup \{\infty\}$, which is equivalent to three mutually orthogonal Latin squares of order 18.

Clearly we can add an integer module 17 to any row, as well as to any column of A without losing the DS-property. So we assume that the first row of A is $(\infty, 0, \ldots, 0)$, and the first column is $(\infty, 0, 0, 0, 0)^T$.

Removing the first row and the first column we denote the remaining 4×18 matrix by AR. It has the properties [4]

(1) every row of AR is a permutation of $Z_{17} \cup \{\infty\}$
(2) every column of AR contains different elements.
(3) for every $a \in Z_{17}\{0\}$ there exists exactly one pair (x_i, y_i) with $x_i - y_i \equiv a \pmod{17}$ between any two rows $(x_1, \ldots, x_{18})$ and $(y_1, \ldots, y_{18})$ of AR.

We found, with aid of a computer, an AR as follows:

$$\left\| \begin{array}{cccccccccccccccccc} \infty & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 0 & \infty & 15 & 3 & 12 & 6 & 1 & 10 & 2 & 13 & 8 & 16 & 14 & 5 & 11 & 4 & 9 & 7 \\ 1 & 15 & \infty & 16 & 4 & 3 & 12 & 14 & 9 & 11 & 13 & 5 & 7 & 0 & 2 & 6 & 8 & 10 \\ 3 & 12 & 4 & 7 & \infty & 8 & 11 & 15 & 1 & 5 & 2 & 0 & 10 & 13 & 9 & 16 & 6 & 14 \end{array} \right\|$$

In the matrix A corresponding to the AR, add 1 to each element of the first row, 16 to the second, 15 to the third, 2 to the fourth, and 11 to the fifth. Replace 0 and ∞ by 17 and 18 respectively. Using the first and the second rows as row and column subscripts, using the other three rows of A as the first rows of three Latin squares respectively, we can construct three mutually orthogonal Latin squares of order 18 L_1, L_2, L_3 as shown below. It is clear they are all idempotent. We then have

Proposition. *There exist three mutually orthogonal idempotent Latin squares of order 18.*

1	10	4	16	8	17	11	6	14	12	3	9	2	7	5	18	13	15
14	2	11	5	17	9	1	12	7	15	13	4	10	3	8	6	18	16
18	15	3	12	6	1	10	2	13	8	16	14	5	11	4	9	7	17
8	18	16	4	13	7	2	11	3	14	9	17	15	6	12	5	10	1
11	9	18	17	5	14	8	3	12	4	15	10	1	16	7	13	6	2
7	12	10	18	1	6	15	9	4	13	5	16	11	2	17	8	14	3
15	8	13	11	18	2	7	16	10	5	14	6	17	12	3	1	9	4
10	16	9	14	12	18	3	8	17	11	6	15	7	1	13	4	2	5
3	11	17	10	15	13	18	4	9	1	12	7	16	8	2	14	5	6
6	4	12	1	11	16	14	18	5	10	2	13	8	17	9	3	15	7
16	7	5	13	2	12	17	15	18	6	11	3	14	9	1	10	4	8
5	17	8	6	14	3	13	1	16	18	7	12	4	15	10	2	11	9
12	6	1	9	7	15	4	14	2	17	18	8	13	5	16	11	3	10
4	13	7	2	10	8	16	5	15	3	1	18	9	14	6	17	12	11
13	5	14	8	3	11	9	17	6	16	4	2	18	10	15	7	1	12
2	14	6	15	9	4	12	10	1	7	17	5	3	18	11	16	8	13
9	3	15	7	16	10	5	13	11	2	8	1	6	4	18	12	17	14
17	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18

L_1

1	6	5	14	16	11	13	15	7	9	2	4	8	10	12	17	18	3
18	2	7	6	15	17	12	14	16	8	10	3	5	9	11	13	1	4
2	18	3	8	7	16	1	13	15	17	9	11	4	6	10	12	14	5
15	3	18	4	9	8	17	2	14	16	1	10	12	5	7	11	13	6
14	16	4	18	5	10	9	1	3	15	17	2	11	13	6	8	12	7
13	15	17	5	18	6	11	10	2	4	16	1	3	12	14	7	9	8
10	14	16	1	6	18	7	12	11	3	5	17	2	4	13	15	8	9
9	11	15	17	2	7	18	8	13	12	4	6	1	3	5	14	16	10
17	10	12	16	1	3	8	18	9	14	13	5	7	2	4	6	15	11
16	1	11	13	17	2	4	9	18	10	15	14	6	8	3	5	7	12
8	17	2	12	14	1	3	5	10	18	11	16	15	7	9	4	6	13
7	9	1	3	13	15	2	4	6	11	18	12	17	16	8	10	5	14
6	8	10	2	4	14	16	3	5	7	12	18	13	1	17	9	11	15
12	7	9	11	3	5	15	17	4	6	8	13	18	14	2	1	10	16
11	13	8	10	12	4	6	16	1	5	7	9	14	18	15	3	2	17
3	12	14	9	11	13	5	7	17	2	6	8	10	15	18	16	4	1
5	4	13	15	10	12	14	6	8	1	3	7	9	11	16	18	17	2
4	5	6	7	8	9	10	11	12	13	14	15	16	17	1	2	3	18

L_2

1	18	2	5	9	12	16	13	11	4	7	3	10	17	8	6	15	14
16	2	18	3	6	10	13	17	14	12	5	8	4	11	1	9	7	15
8	17	3	18	4	7	11	14	1	15	13	6	9	5	12	2	10	16
11	9	1	4	18	5	8	12	15	2	16	14	7	10	6	13	3	17
4	12	10	2	5	18	6	9	13	16	3	17	15	8	11	7	14	1
15	5	13	11	3	6	18	7	10	14	17	4	1	16	9	12	8	2
9	16	6	14	12	4	7	18	8	11	15	1	5	2	17	10	13	3
14	10	17	7	15	13	5	8	18	9	12	16	2	6	3	1	11	4
12	15	11	1	8	16	14	6	9	18	10	13	17	3	7	4	2	5
3	13	16	12	2	9	17	15	7	10	18	11	14	1	4	8	5	6
6	4	14	17	13	3	10	1	16	8	11	18	12	15	2	5	9	7
10	7	5	15	1	14	4	11	2	17	9	12	18	13	16	3	6	8
7	11	8	6	16	2	15	5	12	3	1	10	13	18	14	17	4	9
5	8	12	9	7	17	3	16	6	13	4	2	11	14	18	15	1	10
2	6	9	13	10	8	1	4	17	7	14	5	3	12	15	18	16	11
17	3	7	10	14	11	9	2	5	1	8	15	6	4	13	16	18	12
18	1	4	8	11	15	12	10	3	6	2	9	16	7	5	14	17	13
13	14	15	16	17	1	2	3	4	5	6	7	8	9	10	11	12	18

L_3

3 Some Applications

This new result leads to some applications. Since three mutually orthogonal idempotent Latin squares of order n are equivalent to three holey mutually orthogonal Latin square (3 HMOLS) of type 1^n, we may combine the known results in [2, Theorem 1.1] and [3, Corollary 1] with the new 3 HMOLS of type 1^{18} to obtain the following

Corollary 1. *There exist 3 HMOLS of type* 1^n *if* $n \geq 5$ *and* $n \neq 6, 10$.

From Corollary 1 we further obtained 3 HMOLS of type 5^{18} and 23^{18} which were listed as unknown cases in [2, Theorem 5, 10].

Corollary 2. *There exist 3 HMOLS of type* 5^{18} *and* 23^{18}.

3 HMOLS of type 1^n can be used to construct a $(5n, 5, 1)$-PMD [1]. The new 3 HMOLS of type 1^{18} can give a $(90, 5, 1)$-PMD which is previously unknown.

Corollary 3. *There exists a* $(90, 5, 1)$*-PMD.*

Acknowledgement

The authors wish to thank Professor L. Zhu for his help during the preparation of this note.

References

[1] F.E. Bennett, K.T. Phelps, C.A. Rodger and L. Zhu, Constructions of Perfect Mendelsohn Designs. *Discrete Math* **103** (1992), 139–151.

[2] D.R. Stinson and L.Zhu, On the Existence of Three MOLS with Equal-sized Holes. *Australia J. Combin.* **4** (1991), 33–47.

[3] J. Abel, Xiafu Zhang and Hangfu Zhang, Three Mutually Orthogonal Idempotent Latin Squares of Orders 22, and 26, preprint.

[4] D.T. Todorow, Three Mutually Orthogonal Latin Squares of Order 14, *Ars Combinatoria* **20** (1985), 45–48.