On Smallest Maximally Nonhamiltonian Graphs

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ABSTRACT. Bollobas posed the problem of finding the least number of edges, $f(n)$, in a maximally nonhamiltonian graph of order n. Clark, Entringer and Shapiro showed $f(n) = \lceil 3n/2 \rceil$ for all even $n \geq 36$ and all odd $n \geq 53$. In this paper, we give the values of $f(n)$ for all $n \geq 3$ and show $f(n) = \lceil 3n/2 \rceil$ for all even $n \geq 20$ and odd $n \geq 17$.

1 Introduction

A graph G is maximally nonhamiltonian if G is not hamiltonian but $G+e$ is hamiltonian for any edge $e \notin G$. Bollobos [2; p167] posed the problem of finding the least number of edges, $f(n)$, in a maximally nonhamiltonian graph of order n. Bondy [3] has shown that any such graph with order $n \geq 7$ and containing m vertices of degree 2 has at least $(3n+m)/2$ edges. Thus, we have:

Lemma 1. *$f(n) \geq \lceil 3n/2 \rceil$ for all $n \geq 7$.*

A cubic graph is 3-edge-colorable if it is hamiltonian. Consequently 4-edge-chromatic cubics are candidates for smallest maximally nonhamiltonian graphs. Isaacs [5] was the first to construct an infinite family $\{J_k\}$ of such graphs. Clark, Entringer and Shapiro [4] have shown that the J_k and variations of them are maximally nonhamiltonian graphs which implies $f(n) = \lceil 3n/2 \rceil$ for all even $n \geq 36$ and all odd $n \geq 53$.

We show that further variations of the J_k are maximally nonhamiltonian graphs and hence, $f(n) = \lceil 3n/2 \rceil$ for all $n \geq 20$. A computer has been used in setting small cases. The values of $f(n)$ for $3 \leq n \leq 19$ are given in Table 1. The notation is that of [1].

ARS COMBINATORIA 45(1997), pp. 263-270

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(n)$	2	4	6	9	12	15	15	15	17	18	20	22	24	25	26	28	29

Table 1. The values of $f(n)$ for $3 \leq n \leq 19$

2 Smallest maximally nonhamiltonian graphs of order $n \geq 20$

We define Isaacs graph J_k for odd $k \geq 3$ as follows:

Let $V(J_k) = \{v_i : 0 \leq i \leq 4k-1\}$ and $E(J_k) = E_0 \cup E_1 \cup E_2 \cup E_3$ where

$$E_0 = \cup_{j=0}^{k-1} \{e_{4j,4j+1}, e_{4j,4j+2}, e_{4j,4j+3}\},$$
$$E_1 = \{e_{4j+1,4j+7} : 0 \leq j \leq k-1\},$$
$$E_2 = \{e_{4j+2,4j+6} : 0 \leq j \leq k-1\},$$
$$E_3 = \{e_{4j+3,4j+5} : 0 \leq j \leq k-1\}.$$

Subscripts should be read as modulo $4k$. We denote by P_j the subgraph of J_k induced by setting V_{4j}, V_{4j+1}, V_{4j+2} and V_{4j+3} for $0 \leq j \leq k-1$. Figure 1 shows J_5 and J_7 where we identify each vertex with its subscript.

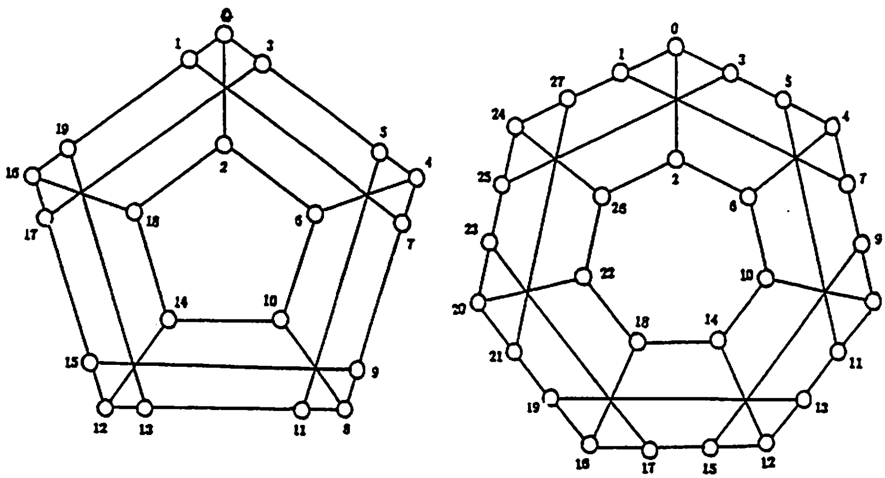

Figure 1. J_5 and J_7

To obtain additional maximally nonhamiltonian graphs we expand an edge to a triangle as follows: $e_{x,y} \in E(G)$ and $z \notin V(G)$, define $G(e_{x,y})$ by $V(G(e_{x,y})) = V(G) \cup \{z\}$ and $E(G(e_{x,y})) = E(G) \cup \{e_{x,z}, e_{y,z}\}$

Furthermore, we expand vertices to triangles. For $v \in V(G)$ with neighbors v_1, v_2, v_3 and w_1, w_2, $w_3 \notin V(G)$, define $G(v)$ by $V(G(v)) = V(G-v) \cup \{w_1, w_2, w_3\}$ and $E(G) = E(G-v) \cup \{e_{v1,w1}, e_{v2,w2}, e_{v3,w3}, e_{w1,w2}, e_{w2,w3},$

$e_{w1,w3}\}$. Let $J_k\ (v_1,\ldots,v_s)$ denote the graph obtained from J_k by expanding $v_1,\ldots,v_s$ to triangles. We abuse notation slightly by denoting by P_i the subgraph induced by the original vertices of P_i, together with the vertices of the expansion.

Define G_n $(20 \le n \le 59)$ as in Table 2.

m	$G_{4k+m}(k=5)$	$G_{4k+m}(\text{odd } k \ge 7)$
0	J_5	J_k
1	$J_5(e_{11,13})$	$J_k(e_{16,18})$
2	$J_5(v_2)$	$J_k(v_0)$
3	$J_5(v_2, e_{11,13})$	$J_k(v_0, e_{16,18})$
4	$J_5(v_2, v_7)$	$J_k(v_0, v_4)$
5	$J_5(v_2, v_7, e_{11,13})$	$J_k(v_0, v_4, e_{16,18})$
6	$J_5(v_2, v_7, v_{17})$	$J_k(v_0, v_4, v_8)$
7	$J_5(v_2, v_7, v_{17}, e_{11,13})$	$J_k(v_0, v_4, v_8, e_{16,18})$

Table 2

It is easily seen that G is hamiltonian if $G(e)$ is hamiltonian, and G is hamiltonian if and only if $G\ (v_1,\ldots,v_s)$ is hamiltonian. Since J_k is nonhamiltonian, we have:

Lemma 2. *The graphs G_n are nonhamiltonian graphs for all $n \ge 20$.* □

We observe that a nonhamiltonian graph is maximally nonhamiltonian if and only if every two nonadjacent vertices are joined by a hamiltonian path.

With the help of a computer,we have verified that G_n $(20 \le n \le 59)$ are maximally nonhamiltonian graphs. Since $|E(G_n)| = \lceil 3n/2 \rceil$, we have $f(n) \le \lceil 3n/2 \rceil$. By lemma 1, $f(n) \ge \lceil 3n/2 \rceil$, hence, we have:

Lemma 3. $f(n) = \lceil 3n/2 \rceil$ *for* $20 \le n \le 59$. □

For a hamiltonian (u,v)-path P in J_k, let $P(i,j)$ denote the set of edges of P joining $V(P_i)$ to $V(P_j)$. Note that $P(i,i+1) \ge 1$ while $|P(i+1,i+2)| = 3$ when $|P(i,i+1)| = 1$ provided $u, v \notin V(P_i) \cup V(P_{i+1}) \cup V(P_{i+2})$.

We say P_i is ordinary if there are at most 5 vertices in it.

Lemma 4. *Assume odd $k \ge 11$ and $0 \le m \le 7$ and let P be a hamiltonian (u,v)-path of G_{4k+m} with $|P(i,i+1)| \ge 2$, where $u, v \in V(P_i) \cup V(P_{i+1})$ and P_i, P_{i+1} are ordinary. Then P can be extended to a hamiltonian path P' of G_{4k+8+m} connecting two vertices which correspond to u and v in the natural way so that: $E(P') \supseteq \{e_{s,t} : e_{s,t} \in E(P), 0 \le s,t \le 4i+3\} \cup \{e_{s+8,t+8} : e_{s,t} \in E(P), 4i+4 \le s,t \le 4k+3\}$.*

Proof: The various instances of the path P' to be described can be examined using Figure 2.

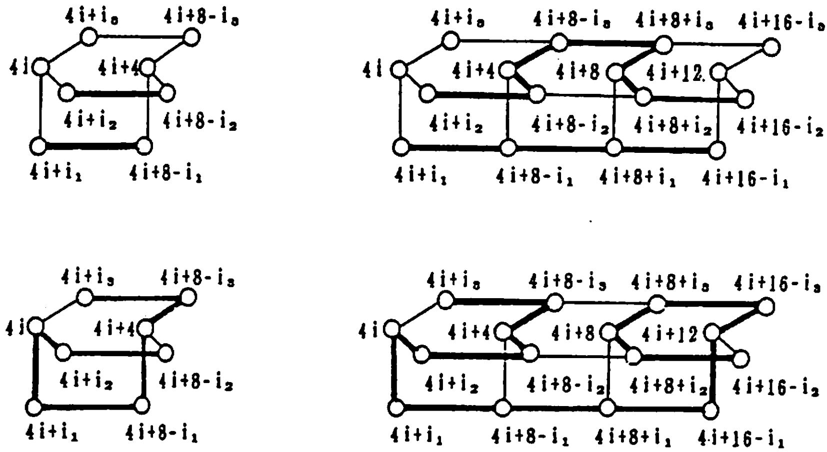

Figure 2. P can be extended to a hamiltonian path P'

First, assume $|P(i, i+1)| = 2$ and let $e_{4i+i_1,4i+8-i_1}$ and $e_{4i+i_2,4i+8-i_2}$ be those two edges contained in P and let $\{i_1, i_2, i_3\} = \{1, 2, 3\}$, then if we replace those edges by the following paths: $(4i+i_1, 4i+8-i_1, 4i+8+i_1, 4i+16-i_1)$ and $(4i+i_2, 4i+8-i_2, 4i+4, 4i+4-i_3, 4i+12+i_3, 4i+8, 4i+8+i_2, 4i+16-i_2)$.) We get a desired path P'.

Next, assume $|P(i, i+1)| = 3$ and let $4i+i_1$ and $4i+i_2$ be the vertices neighboring $4i$ on P, and $4i+8-i_1$ and $4i+8-i_3$ be those neighboring $4i+4$ on P. In this case, we get a desired path P' by replacing the following subpath: $(4i+i_3, 4i+8-i_3, 4i+4, 4i+8-i_1, 4i+i_1, 4i, 4i+i_2, 4i+8-i_2)$ by $(4i+i_3, 4i+8-i_3, 4i+4, 4i+8-i_2, 4i+i_2, 4i, 4i+i_1, 4i+8-i_1, 4i+8+i_1, 4i+16-i_1, 4i+12, 4i+16-i_3, 4i+8+i_3, 4i+8, 4i+8+i_2, 4i+16-i_2)$ □

When we replace a hamiltonian path P in G_{4k+m} by the hamiltonian path P' in G_{4k+8+m} in the manner just described we say that P is expanded at $(i, i+1)$.

For G_{4k+m} form an isomorphic copy $G_{4k+m}(i, i+1)$ $(6 \leq i \leq k)$ of G_{4k-8+m} by deleting vertices $V(P_i) \cup V(P_{i+1})$ and adding edges $\{e_{4i-3,4i+11}, e_{4i-2,4i+10}, e_{4i-1,4i+9}\}$ if $u, v \notin V(P_i) \cup V(P_{i+1})$.

Lemma 5. *The graphs G_{4k+m} are maximally hamiltonian graphs for all $k \geq 15$ and $0 \leq m \leq 7$.*

Proof: From Lemma 1, G_{4k+m} are nonhamiltonian, we need only verify that for any two nonadjacent vertices $u, v \in V(G_{4k+m})$, u, v are joined by a hamiltonian path.

Case 1: $u, v \notin V(P_5) \cup V(P_6) \cup V(P_7)$.

By induction there exists special (u, v)-paths P in $G_{4k-8+m} \cong G_{4k+m}$ $(6, 7)$. By Lemma 4, expand P at $(5, 6)$ if $P(5, 6) \geq 2$, otherwise expand P

at $(6,7)$ to obtain special (u,v)-path in G_{4k+m}.
Case 2:

$(u, v \in V(P_5) \cup V(P_6) \cup V(P_7))$ or
$(u \in V(P_5) \cup V(P_6) \cup V(P_7)$ and $v \notin V(P_8) \cup V(P_9) \cup V(P_{10}))$ or
$(v \in V(P_5) \cup V(P_6) \cup V(P_7)$ and $u \notin V(P_8) \cup V(P_9) \cup V(P_{10}))$

By induction, there exists a special (u,v)-path P in $G_{4k-8+m} \cong G_{4k+m}$ $(9,10)$. By Lemma 4, expand P at $(8,9)$ if $P(8,9) \geq 2$, otherwise expand P at $(9,10)$ to obtain a special (u,v)-path in G_{4k+m}.
Case 3:

$(u \in V(P_5) \cup V(P_6) \cup V(P_7)$ and $v \in V(P_8) \cup V(P_9) \cup V(P_{10}))$ or
$(v \in V(P_5) \cup V(P_6) \cup V(P_7)$ and $u \in V(P_8) \cup V(P_9) \cup V(P_{10}))$.

By induction, there exists a special (u,v)-path in $G_{4k-8+m} \cong G_{4k+m}$ $(12,13)$. By Lemma 4, expand P at $(11,12)$ if $P(11,12) \geq 2$, otherwise expand P at $(12,13)$ to obtain a special (u,v)-path in G_{4k+m}. □

From Lemma 3 and Lemma 5, we have:

Theorem 1. *$f(n) = \lceil 3n/2 \rceil$ for all $n \geq 20$.*

3 Smallest maximally nonhamiltonian graphs with order $n \leq 19$

We denote the values for $f(n)$ in Table 1 as f_n for the upper bounds on $f(n)$. We have:

Lemma 6. *$f(n) \leq f_n$ for all $3 \leq n \leq 19$.*

Proof: It is easily verified that the graph G_n shown in Figure 3 are all maximally nonhamiltonian graphs with order n, $(3 \leq n \leq 19)$. Since $|E(G_n)| = f_n$, we have $f(n) \leq f_n$ for all $3 \leq n \leq 19$. □

For $n = 10, 11, 12, 13, 17, 19$, $f_n = \lceil 3n/2 \rceil$. By Lemma 1 and Lemma 6, we have:

Lemma 7. *$f(n) = f_n$ for all $n = 10, 11, 12, 13, 17, 19$.* □

Lemma 8. *$f(n) = f_n$ for all $n = 14, 16, 18$.*

Proof: By Lemma 1, $f(14) \geq \lceil 3n/2 \rceil$. By Lemma 6, $f(14) \leq f_{14} = 22$. Hence, we need only show that $f(14) \neq 21$. Now, we show it by contradiction. Suppose there is a maximally nonhamiltonian graph H_{14} with order $n = 14$ and $|E(H_{14})| = 21$. By Bondy [3], H_{14} is a 3-regular graph. For $v \in V(H_{14})$ with neighbours v_1, v_2, v_3 and v_1 is not joined to v_2, there is a cycle with 13 vertices in H_{14} because there is a hamiltonian (v_1, v_2)-path in H_{14} (see Figure 4).

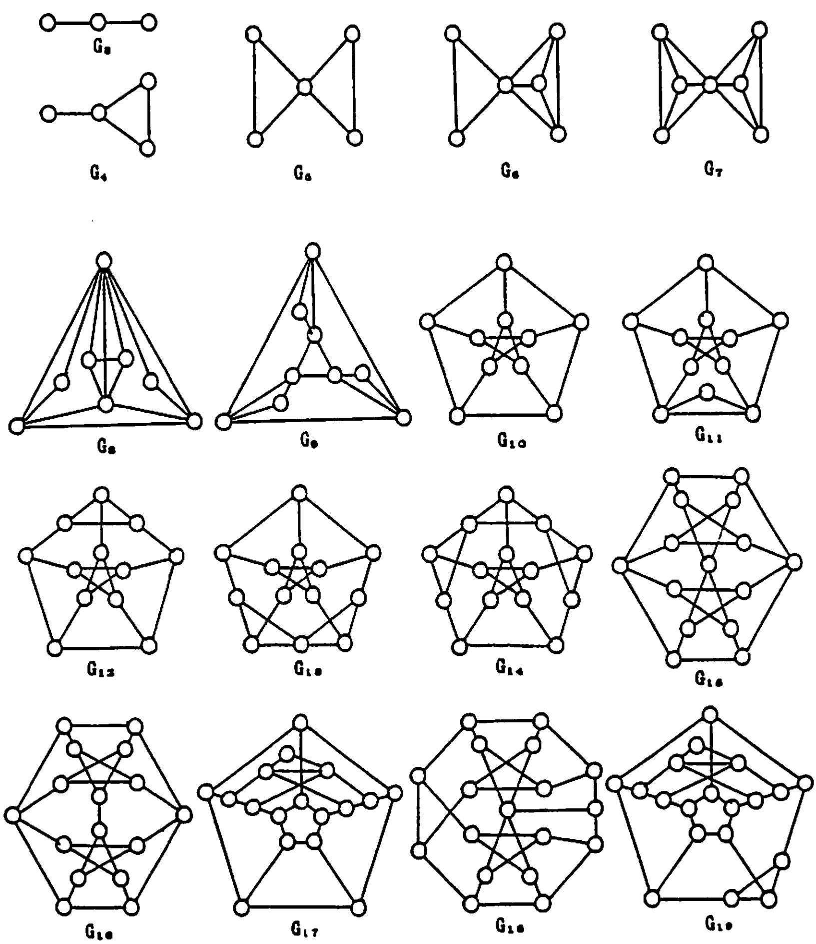

Figure 3. Maximally nonhamiltonian graphs G_n with order n $(3 \le n \le 19)$

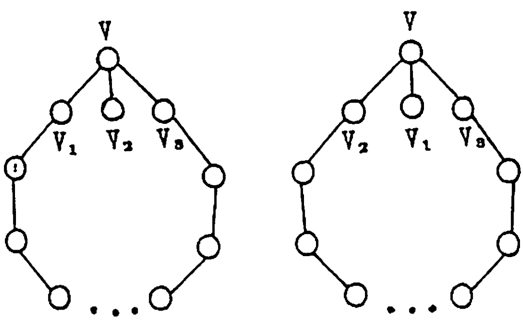

Figure 4. There is a cycle with 13 vertices in H_{14}

Now we begin with $H = C_{13} \cup \{v_{14}\} \cup \{e_{14,13}\}$, repeatly, add edges one by one into H until H is 3-regular. Let

$$S_{14} = \{H_{14} \colon H_{14} \text{ is maximally nonhamiltonian and } |E(H_{14})| = 21\}$$

Following algorithm will construct all the graphs in S_{14}.

Algorithm 1

```
1   Procedure construct-graph
2   begin
3   H: = C13 ∪ {v14} ∪ {e14,13};
4   S: = {H}; S2: = Φ; Q: = 14; S: = Φ;
5   while Q < (3n + 1)/2 do
6     begin
7     for every graph H ∈ S1 do
8         for every ex,y ∉ E(H) and degree (vx) ≤ 2 and degree (vy) ≤ 2 do
9         begin
10        H1: = H + ex,y;
11        if H1 is nonhamiltonian then S2: = S2 ∪ {H1}
12        end;
13    S1: = S2; S2: = Φ; Q: = Q + 1
14    end;
15  for every graph H ∈ S1 do
16    if every two nonadjacent vertices of H are joined by
          a hamiltonian path then S: = S ∪ {H};
17  output S;
18  end
```

With the help of a computer, we get $|S| = 0$, a contradiction to the supposition, hence we have $f(14) = 22$.

In a similar way, we get $f(16) = 25$ and $f(18) = 28$. □

Lemma 9. *$f(n) = f_n$ for $n = 7, 8, 9, 15$.*

Proof: Suppose there is a maximally nonhamiltonian graph H_{15} with order $n = 15$ and $|E(H_{15})| = 23$. By Bondy [3], there are at most two vertices with degree 4 or one vertex with degree 5. Let

$$m = \sum_{\deg(v_i) \geq 3} (\text{degree } (v_i) - 3).$$

We change Algorithm 1 into Algorithm 2 by replacing the sentences 3,8 with the following sentences accordingly:

3* $H: = C_{14} \cup \{v_{15}\} \cup \{e_{15,14}\}$;
8* for every $e_{x,y} \in E(H)$ and ($m < 2$ or degree $(v_x) \leq 2$ and degree $(v_y) \leq 2$) do

Let $S_{15} = \{H_{15} : H_{15}$ is maximally nonhamiltonian and $|E(H_{15})| = 23\}$. Algorithm 2 will construct all the graphs in S_{15}. With the help of a computer, we get $|S| = 0$, a contradiction to the supposition, hence, we have $f(15) = 24$.

In a similar way, we get $f(7) = 12$, $f(8) = 15$, $f(9) = 15$. □

For the $3 \leq n \leq 6$, it is easy to verify $f(n) = f_n$. So we have:

Theorem 2. *$f(n) = f_n$ for all $3 \leq n \leq 19$.*

4 Acknowledgement

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