Achromatic index of K_{12}

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ABSTRACT. Achromatic index of a graph G is the largest integer k admitting a proper colouring of edges of G in such a way that each pair of colours appears on some pair of adjacent edges. It is shown that the achromatic index of K_{12} is 32.

Let $f : E(G) \to [1,k] = \{1,\ldots,k\}$ be a proper colouring of edges of a graph G, i.e. $f(e_1) \neq f(e_2)$ whenever edges e_1, e_2 are adjacent. The least possible k, the chromatic index of G, is, according to the well-known theorem of Vizing, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$. If edges of G are properly coloured by $\chi'(G)$ colours, then evidently the colouring is *complete*—each pair of colours can be observed on some pair of adjacent edges (we shall say each of these edges realizes the join of corresponding two colours). Maximizing the number of colours in a proper complete colouring of edges of G we obtain the *achromatic index* of G. The achromatic index of K_n will be denoted by $A(n)$.

The problem of determining $A(n)$ has been raised probably by Bosák in 1972—see Ninčák [6], where a solution for $n \leq 7$ can be found; another solution for these values of n is disponible in Bories and Jolivet [1]. Bouchet [2] showed that to wait for a complete solution is practically hopeless. Indeed, he proved that, for a positive odd integer q and $n = q^2+q+1$, $A(n) = qn$ if and only if a projective plane of order q exists. This explains probably why only a modest development of the problem has been recorded, corresponding to solutions for $n = 8, 9$ (Bouchet), $n = 10$ (Turner et al. [7]) and $n = 11$ ([7] and Kundríková [6]). As concerns the first unknown value $A(12)$, Jamison [3] found a lower bound 31 and subsequently improved it to 32—see [2]. It is easy to see that $A(12) \le \frac{1}{2}\binom{12}{2} = 33$ [3,6]. The aim of this paper is to determine the exact value of $A(12)$.

Theorem. $A(12) = 32$.

Proof: Suppose there exists a proper complete colouring $f : E(K_{12}) \to [1,33]$. For $I \subseteq [1,33]$ let G_I be the graph induced by $f^{-1}(I)$; especially, for $I = \{j,k\}$ the graph $G_{jk} = G_I$ can be one of the following five graphs: $P_3 \cup 2K_2$ (a disjoint union of a path on three vertices and two K_2's), $P_4 \cup K_2$, $2K_{1,2}$, P_5, C_4. Further let $\mathcal{S}_i$ be the set of all i-element subsets I of $[1,33]$ with G_I decomposable into two (edge-disjoint) $K_{1,i}$'s and $\mathcal{S}_{i,2}$ the subset of $\mathcal{S}_i$ which contains exactly I's possessing a 2-element subset J with $G_J = P_5$.

(i) Any edge e yields at most 20 pairs of colours containing $f(e)$, hence all colour classes induced by f must contain two edges.

(ii) If $f(e_1) = f(e_2) = i$, $e_1 \neq e_2$, there are 36 edges adjacent to at least one of e_1, e_2, and consequently four pairs of them are coloured by the same colour; denote the set of corresponding four colours by $R(i)$. To distinguish between colours from $R(i)$ and from $[1,33] - R(i) - \{i\}$ note that for $j \in R(i)$ no component of G_{ij} containing an edge coloured by j is K_2, while for $k \in [1,33] - R(i) - \{i\}$ (exactly) one component of G_{ik} with an edge coloured by k is K_2.

(iii) For $I \subseteq [1,33]$ let $D(I)$ be the set of all colours from $[1,33] - I$ appearing on an edge which realizes at least $\lceil \frac{1}{2}|I| \rceil$ joins with colours from I. From the completeness of f it follows $D(I) = [1,33] - I$ and $|D(I)| = 33 - |I|$.

(iv) We have $G_I \neq 2K_{1,3}$ for every 3-element set $I \subseteq [1,33]$. Indeed, provided $G_I = 2K_{1,3}$ the set $D(I)$ can consist only of colours of edges incident to one of the centres of involved $K_{1,3}$'s (and their number is at most 15) and of colours of edges between leaves of these $K_{1,3}$'s hanging on differently coloured edges (at most 12 colours), which implies $|D(I)| \le 27 < 30 = 33 - |I|$ in contradiction with (iii).

(v) If $\mathcal{S}_i$ is empty, so is $\mathcal{S}_{i-2,2}$. To see this let I be an $(i-2)$-element subset of $[1,33]$ with G_I decomposable into two $K_{1,i-2}$'s and J its 2-element subset with $G_J = P_5$. If edges of G_J are $v_j v_{j+1}$, $j = 1,2,3,4$, any of $35 - i$ colours of $[1,33] - I$ must occur either on an edge incident to v_2 or v_4 (at most $25 - 2i$ colours) or on an edge incident to v_3 (at most 9 colours) or on the edge $v_1 v_5$ or on two independent edges, one incident to v_1 and the other to v_5. In such a case the number of colours of the last kind is at least i in contradiction with the emptiness of $\mathcal{S}_i$.

(vi) Evidently, $\mathcal{S}_5 = \emptyset$, since otherwise the colour i of the edge between the centres of two $K_{1,5}$'s forming G_I with $|I| = 5$ would contradict (ii) for $|R(i)| \ge 5$. Thus according to (v) $\mathcal{S}_{3,2} = \emptyset$.

(vii) If $I \in S_4$, using (iv) and $\mathcal{S}_{4,2} = \emptyset$ (a trivial consequence of $\mathcal{S}_{3,2} = \emptyset$) it is clear there exist disjoint 2-element subsets I^1, I^2 of I with $G_{I^1} =$

$C_4 = G_{I^2}$. Let x_1, x_2 be vertices of G_I of degree 4 and let $Y^i = \{y_1^i, y_2^i\} = V(G_{I^i}) - \{x_1, x_2\}$, $i = 1,2$. Three cases are to be distinguished according to the position of the edge $z_1z_2 \neq x_1x_2$ coloured by the colour $k = f(x_1x_2)$.

(vii.i) If $z_1, z_2 \notin Y^1 \cup Y^2$, the cardinality of $D(I \cup \{k\})$ can be bounded from above by 12 (colours of edges incident to x_1 or x_2) + 8 (of edges between z_1, z_2 and $Y^1 \cup Y^2$) + 4 (of edges joining Y^1 to Y^2) = 24 which is insufficient since $|[1,33] - (I \cup \{k\})| = 28$.

(vii.ii) Provided $|\{z_1, z_2\} \cap (Y^1 \cup Y^2)| = 1$ it is easy to see analogously as in (vii.i) that $|D(I \cup \{k\})| \leq 25$, still a contradiction with (iii).

(vii.iii) In the last case there is $i \in \{1,2\}$ such that $z_1z_2 = y_1^iy_2^i$. The cardinality of $D(I \cup \{k\})$ is at most 12 (colours of edges incident to x_1 or x_2) + 12 (of edges from Y^i to $V(K_{12}) - (\{x_1, x_2\} \cup Y^1 \cup Y^2)$) + 4 (of edges connecting Y^1 to Y^2); with respect to (iii) the inequality turns into equality. Let J be the set of 24 colours of the first two kinds, $J(v)$ the subset of J formed by colours of edges incident to a vertex v and let $J(v_1, v_2) = J(v_1) \cup J(v_2)$ for $v_1, v_2 \in V(K_{12})$. Then $|J(x_1)| = |J(x_2)| = |J(z_1)| = |J(z_2)| = 6$, $J(x_1) \cap J(x_2) = \emptyset = J(x_1, x_2) \cap J(z_1, z_2)$, $J(z_1, z_2) \subseteq J(y_1^{3-i}, y_2^{3-i})$ (for joins of colours from $J(z_1, z_2)$ and those from I^{3-i}) and $|J(x_1, x_2) \cap J(y_1^{3-i}, y_2^{3-i})| \leq 1$ (there are exactly 13 edges incident to y_1^{3-i} or y_2^{3-i} which are "free" for colours from J).

Now the colour $l = f(y_1^{3-i}y_2^{3-i})$ can realize its join with k in two manners.

If $f(x_jv) = l$ for some $j \in \{1,2\}$ and consequently $l \in J(x_j) \cap J(y_1^{3-i}, y_2^{3-i})$, then the colour joins between colours from $J(x_{3-j})$ and the colour l can be realized only at the vertex v, hence $J(x_{3-j}) - \{f(x_{3-j}v)\} \in S_5$ in contradiction with (vi).

If $f(y_j^iv) = l$ for some $j \in \{1,2\}$, then at least 11 colours from $J(x_1, x_2)$ must appear on edges incident to v (due to the join with l) which is impossible since one of 11 edges incident to v is coloured by $l \notin J(x_1, x_2)$.

Thus all possibilities lead to a contradiction and we have $\mathcal{S}_4 = \emptyset = S_{2,2}$.

(viii) From the assumption $I \in S_3$ by help of (iv) and $\mathcal{S}_{3,2} = \emptyset$ it follows there exists a 2-element set $J \subseteq I$ such that $G_J = C_4$. The graph G_I has two vertices x_1, x_2 of degree 3, two vertices y_1^i, y_2^i of degree i, $i = 1,2$, and the set I consists of colours $f(x_1y_1^2) = f(x_2y_2^2)$, $f(x_1y_2^2) = f(x_2y_1^2)$ and $f(x_1y_1^1) = f(x_2y_2^1)$.

Any of 30 colours from $[1,33] - I$ must occur either on an edge incident to x_1 or x_2 (maximum 15 colours) or on one of edges $y_i^1y_j^2$, $i,j = 1,2$, or simultaneously on an edge incident to y_1^1 or y_2^1 and on an (other) edge incident to y_1^2 or y_2^2. There exists a set J of 11 colours of the last kind. Then $J(y_1^i, y_2^i) = J$, $|J(y_1^i) \cap J(y_2^i)| \leq 1$, $i = 1,2$, and $\max\{|J(y_j^i)|; i,j = 1,2\} \leq 6$ – if $|J(y_j^i)| \geq 7$, there is $k \in \{1,2\}$ with $|J(y_j^i) \cap J(y_k^{3-i})| \geq 4$ and any 4-element subset of $J(y_j^i) \cap J(y_k^{3-i})$ belongs to $\mathcal{S}_4$ which is forbidden by (vi). From analogous reasons we have $2 \le |J(y_i^1) \cap J(y_j^2)| \le 3$ and $8 \le |J(y_i^1, y_j^2)| \le 9$ for $i, j = 1, 2$.

Now consider the colour $k = f(x_1x_2)$; the second occurrence of k must be on an edge joining two from among vertices of the set $Y = \{y_1^1, y_2^1, y_1^2, y_2^2\}$ – we have evidently $|J(v)| \le 4$ for each $v \in V(K_{12}) - Y$ and as a consequence $|J(v_1)| + |J(v_2)| \le 10$ for any edge v_1v_2 fulfilling $\{v_1, v_2\} \not\subseteq Y$. We have seen above that also edges $y_i^1 y_j^2$, $i, j = 1, 2$, are insufficient for the colour k. Thus we know there is $i \in \{1, 2\}$ with $f(y_1^i y_2^i) = k$. The colour $l = f(y_1^{3-i} y_2^{3-i}) \ne k$ due to its join with k must then be used also for an edge incident to some of vertices x_1, x_2, y_1^i, y_2^i. However, the incidence to x_1 or x_2 must be excluded for otherwise the set $\{f(x_1 y_1^{3-i}), l\}$ would be in $\mathcal{S}_{2,2}$ in contradiction with (vii). Finally, there exists $j \in \{1, 2\}$ and an edge incident to y_j^i coloured by l.

The colour l has all properties for being able to be chosen to the set J, hence without loss of generality $l \in J$. In such a case $J(y_1^{3-i}) \cap J(y_2^{3-i}) = \{l\}$ and analogously as above $|J(y_1^{3-i})| = |J(y_2^{3-i})| = 6$ and $|J(y_j^i) \cap J(y_1^{3-i})| = |J(y_j^i) \cap J(y_2^{3-i})| = 3$. As the 4-element set $(J(y_j^i) \cap J(y_1^{3-i})) \cup (J(y_j^i) \cap J(y_2^{3-i})) - \{l\}$ belongs to $R(l)$ and $R(l)$ contains also $3 - i$ colours from I, we get $|R(l)| \ge 7 - i \ge 5$, a contradiction with (ii). A conclusion is that the set $\mathcal{S}_3$ must be empty.

(ix) Consider two edges x_1x_2 and y_1y_2 coloured by 1. In view of $\mathcal{S}_{2,2} = \emptyset$ at most two colours of $R(1)$ are present on the edges x_iy_j, $i, j = 1, 2$ (for each such colour k necessarily $G_{1k} = C_4$). That is why there exists a colour of $R(1)$, say 2, occurring on edges w_1z_1, w_2z_2 between two vertices $w_1, w_2 \in W = \{x_1, x_2, y_1, y_2\}$ and two vertices $z_1, z_2 \in V(K_{12}) - W$. From among colours of 12 edges joining $V(K_{12}) - W - \{z_1, z_2\}$ to $W - \{w_1, w_2\}$ (they are pairwise different due to the necessity of their joins with the colour 2) at most three occur on an edge incident to w_1 or w_2, since in such a case they are in $R(1)$. Thus there exists a set I of at least nine of these colours which realize their join with 2 on edges incident to z_1 or z_2. Then we can find $w \in W - \{w_1, w_2\}$ with $|I(w)| \ge 5$ and $i \in \{1, 2\}$ such that $|I(z_i)| \ge 3$. As $|I(w) \cap I(z_i)| \ge 3$ and $f(wz_i) \notin I$, any 3-element subset of $I(w) \cap (z_i)$ belongs to $\mathcal{S}_3$ and we have obtained a contradiction with (viii). □

References

[1] F. Bories and J.-L. Jolivet, On complete colorings of graphs, *Recent Advances in Graph Theory* (M. Fiedler, ed.) Academia, Prague, 1975, pp. 75–87.

[2] A. Bouchet, Indice achromatique des graphes multiparti complets et réguliers, *Cahiers Centre Études Rech. Opér.* 20 (1978), 331–340.

[3] R. E. Jamison, On the edge achromatic numbers of complete graphs, *Discrete Math.* 74 (1989), 99–115.

[4] R. E. Jamison, On the achromatic index of K_{12}, *Congr. Numer.* **81** (1991), 143–148.

[5] J. Ninčák, Colourings and Hamiltonian Cycles in Regular Graphs (*Dissertation*), Minsk, 1973. (Russian)

[6] M. Kundríková, Complete Colourings of Graphs (*Diploma Work*), P. J. Šafárik University, Košice, 1985. (Slovak)

[7] C. A. Turner, R. Rowley, R. E. Jamison and R. Laskar, The edge achromatic number of small complete graphs, *Congr. Numer.* **62** (1988), 21–36.