Some Games related to Permutation Group Statistics and their type B Analogues*

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ABSTRACT. A Coin tossing game – with a biased coin with probability q for the tail – for n persons was discussed by Moritz and Williams in 1987, in which the probability for players to go out in a prescribed order is described by what is commonly called the "major index" (due to Major MacMahon), which is an important statistic for the permutation group S_n. We first describe a variation on this game, for which the same question is answered in terms of the better known statistic "length function" in the sense of Coxeter group theory (also called "inversion number" in combinatorial literature). This entails a new bijection implying the old equality (due to MacMahon) of the generating functions for these two statistics.

Next we describe a game for $2n$ persons where the 'same' question is answered in terms of the Coxeter length function for the reflection group of type B_n. We conclude with some miscellaneous results and question.

Introduction

In a 1987 article [8], Moritz and Williams described a coin-tossing game discussed a combinatorial problem based on the game. The authors, Moritz and williams, raised a few natural questions but left them unanswered. Their questions were related with the elements of the permutation group

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$\mathcal{S}_n$. For any element $\sigma \in \mathcal{S}_n$, the statistics "descent set", "major index", and "inversion number (length of the permutation)" are well known. The equality of the generating functions for the two statistics major index and inversion number is originally due to MacMahon [6], [7], (see also page 16, in [5]). Later in 1968 Foata (cf., [3]) constructed a bijection implying this equality. To answer the questions put up by Moritz and Williams, we interpret their statistic in terms of descent set, which readily relates their statistic with the major index.

It is well-known in Coxeter group theory that the permutation group $\mathcal{S}_n$ corrresponds to the Coxeter group of type A_{n-1}. Using the "root system" of the Coxeter group $\mathcal{S}_n$, we modify the game in such a way that the problem discussed in [8] actually boils down to computing the length (inversion number) of the element $\sigma \in \mathcal{S}_n$. In the process we show that the generating function for the statistics considered by Moritz and Williams in [8] and the new statistics (inversion numbers) is the same. Using the two games, we give a bijection between the statistics major index and inversion number distinct from that given by Foata. In Section 2, we define another game which consists of $2n$ players out of which n players are dummy (corresponding to each of the n players we have dummy players playing the game as well). The player and its dummy go out of the game as soon as one of them tosses a head. For this game, the question regarding the probability of players going out according to a prescribed order is given by the length function for the Coxeter group of type B. Finally, we conclude this paper by giving some results and asking a question to the interested readers.

Section 1

In [8], Moritz and Williams considered the following coin-tossing problem: Players $P_1, P_2, \ldots, P_n$ each toss a (possibly biased) coin, in turn (in cyclic order). If a player tosses a head, he goes out of the game and doesn't toss again. The remaining players continue to toss until all go out. For any permutation σ of the n players, find the probability that the players will go out in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$. We denote this game by **GAME1**.

On page 27 of their paper, Moritz and Williams left a few questions unanswered. Before coming to the questions let us make our notations clear most of which have been borrowed from [8]. Let $p = \text{Prob}(\textit{head})$ and $q = 1 - p = \text{Prob}(\textit{tail})$. We will always write a permutation $\sigma \in \mathcal{S}_n$ in one line notation. That is, if $\sigma = [i_1, i_2, \ldots, i_n]$ then we have $\sigma(j) = i_j$ for $1 \leq j \leq n$. For the above $\sigma \in \mathcal{S}_n$, we write $\text{Prob}(P_{i_1}, P_{i_2}, \ldots, P_{i_n})$ to mean $\text{Prob}(\sigma)$, i.e. we have deleted the parentheses. For any permutation $\sigma \in \mathcal{S}_n$, let us define $L_A(\sigma)$ to be the *least (minimum)* number of players who must toss tails in order for σ to be the order in which the players go out. Define for $n = 1, 2, \ldots,$

$$n!_q = (1+q)(1+q+q^2)\ldots(1+q+q^2+\ldots+q^{n-1}).$$

Note that for $q = 1$, $n!_q$ specializes to the order of the group S_n. The statement and proof of Theorem 1 has been taken from [8] for the sake of completeness.

Theorem 1. *Let us consider GAME 1. Then for $\sigma \in S_n$,*

$$Prob(\sigma) = \frac{q^{L_A(\sigma)}}{n!_q}.$$

Proof: One can easily verify the result for $n = 1$. Let the result be true if the number of players is less than or equal to $n-1$. We need to show the validity for n players. Let $\sigma(1) = k$. Then

$$\text{Prob}(\sigma(1)\text{is the first player to go out of the game}) = \frac{q^{k-1}(1-q)}{1-q^n}.$$

For

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & k & k+1 & k+2 & \cdots & n \\ k & \sigma(2) & \sigma(3) & \cdots & \sigma(k) & \sigma(k+1) & \sigma(k+2) & \cdots & \sigma(n) \end{pmatrix},$$

we define

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & \cdots & n-k & \cdots & n-1 \\ \tilde{\sigma}(2) & \tilde{\sigma}(3) & \cdots & \tilde{\sigma}(n-k+1) & \cdots & \tilde{\sigma}(n) \end{pmatrix},$$

where $\tilde{\sigma}(i) = \sigma(i) - k \pmod{n}$ for $2 \le i \le n$.

Claim: $L_A(\sigma) = k - 1 + L_A(\tilde{\sigma})$.

The game starts with player P_1. We want the player $P_k (= P_{\sigma(1)})$ to be the first player to go out with the least number of tails being tossed. This number is $k-1$ as each player beginning with player P_1 and ending with player P_{k-1} must all toss tail and the player P_k comes out of the game after tossing a head. Now we are left with $n-1$ players with player P_{k+1} supposed to start the tossing of the coin. Therefore, in the definition of $\tilde{\sigma}$ we need to make a cyclic shift of k (the game moves cyclically and the definition of $\tilde{\sigma}$ clearly fits into this). Hence $L_A(\sigma) = k - 1 + L_A(\tilde{\sigma})$. Therefore,

$$\begin{aligned} \text{Prob}(\sigma) &= \frac{q^{k-1}(1-q)}{1-q^n} \cdot \text{Prob}(\tilde{\sigma}) \\ &= \frac{q^{L_A(\sigma)}}{n!_q} \end{aligned}$$

using the induction hypothesis and the Claim. Hence the result follows. □

Theorem 1 tells us that for $\sigma_1, \sigma_2 \in S_n$, $\text{Prob}(\sigma_1) = \text{Prob}(\sigma_2)$ if and only if $L_A(\sigma_1) = L_A(\sigma_2)$. Hence, a partition of S_n is induced by taking equal probabilities as an equivalence relation. These set partitions in turn raised the following questions:

(a) Is there some other way to characterize this set partition of S_n?

(b) Is there a more algebraic method for computing $L_A(\sigma)$?

(c) Let us write $b(n,k) = \text{Card}\{\sigma \in S_n : L_A(\sigma) = k\}$. Moritz and Williams in [8] show that $b(n,k)$ equals the coefficient $n!_q$, which also follows from our interpretation of $L_A(\sigma)$ (and known fact about major index which we discuss presently). The authors, Moritz and Williams, also asked whether there is a simple formula for the numbers $b(n,k)$?

To the best of my knowledge, there is no simple answer to (c), except the following recursions:

1. For n and k both positive, one has $b(n,k) = \sum b(n-1,j)$, where the sum is over those integers j that lie between $\max\{0, k-n+1\}$ and k; with initial values $b(n,0) = 1$ for all $n \geq 0$, and $b(0,k) = 0$ for $k \geq 1$.

2. $b(n,k) = b(n-1,k) + b(n,k-1)$ if $k < n$.

Before going to the answers, let us note the following definition.

A Coxeter system is a pair (W, S) where W is the Coxeter group and S is the set of generators for the group W. The elements of the set S have relations of the form $s_i^2 = 1(\textit{identity})$, and $(s_i s_j)^{m_{ij}} = 1$ for $s_i \neq s_j \in S$ with m_{ij} a positive integer. To get the Coxeter group W using S, it suffices to have relations only of the above form. Each element $w \neq 1$ in W can be written in the form $w = s_1 s_2 \ldots s_q$ for some s_i (not necessarily distinct) in S. If q is the smallest integer for which the above expression is possible, then q is called the *length* of w, written $\ell(w)$.

For questions (a) and (b), we define a set $\text{Des}(\sigma)$ (known in the literature as the descent set of $\sigma \in S_n$, first defined by MacMahon, see e.g. page 16, [5]). For any $\sigma \in S_n$, define $\textbf{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$ and $\textbf{inv}(\sigma) = \textbf{Card}\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$. The inversion number of any permutation $\sigma \in S_n$ is a combinatorial object which we have denoted by 'inv'. In the language of Coxeter groups this number is known as 'length function'. For this section, we will write $\ell_A(\sigma)$ in place of $\ell(\sigma)$ to emphasise that the Permutation group S_n is a Coxeter group of type A_{n-1}. For example, if $\sigma = [68125743]$, then $\text{Des}(\sigma) = \{2,6,7\}$ and $\text{inv}(\sigma) = 16$. MacMahon defined the *index* (now a days commonly known as major index or in short "maj") of a permutation $\sigma \in S_n$ to be equal to $\sum_{i \in \text{Des}(\sigma)} i$. Using the above definition he showed that the number of permutations having a given major index k, is the same as the number of permutations having (k inversions) length k. One can also use this fact to get our interpretation in Lemma 2 below. We now give the answer to the questions (a) and (b). Check that $L_A(\sigma) = (8-2)+(8-6)+(8-7) = 6+2+1 = (5+1)+2+1 = 9$.

Lemma 2. *In GAME 1, for* $\sigma \in S_n$,

$$\begin{aligned} L_A(\sigma) &= \sum_{i \in Des\,(\sigma)} (n-i) \\ &= n\, Card\{Des\,(\sigma)\} - maj(\sigma). \end{aligned}$$

For $t \geq 1$, let us understand a term 'tth cycle of the game', before coming to the proof of Lemma 2. Let the game start with n players, $P_1, P_2, \ldots, P_n$ with P_{i+1} tossing the coin after P_i for $1 \leq i \leq n-1$. Then the *first cycle of the game* ends as soon as the player P_n tosses the coin, i.e. the first cycle consists of exactly n tosses with each player tossing the coin (beginning with player P_1 and ending with player P_n) exactly once. Suppose that the players $P_{i_1}, P_{i_2}, \cdots, P_{i_l}$, $1 \leq i_1 < i_2 < \cdots, i_l \leq n$ are in the game after the completion of the first cycle. Since player P_n has already tossed the coin, the next player to toss the coin according to the rule of GAME 1 is P_{i_1}. Hence the second cycle of GAME 1 begins with the tossing of the coin by player P_{i_1} and ends with the tossing of the coin by the player P_{i_l}, i.e. for the 2nd cycle of the game to end each of the l players $P_{i_1}, P_{i_2}, \cdots, P_{i_l}$ has tossed the coin exactly twice from the initial stage of the GAME 1. In general, suppose that after $(t-1)$st *cycle*, the players still playing the game are $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ with $1 \leq i_1 < i_2 \cdots < i_k \leq n$. Then the tth *cycle* consists of k tosses, tossed successively by $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$. That is, for the tth *cycle* to be over all the k players starting with P_{i_1} and ending with P_{i_k} must have tossed exactly t times after the game began.

Observation: A cycle of the game starts with the player having the least number tag and ends with the tossing of the coin by the player having the highest number tag, i.e. each player present in a given cycle has tossed the coin exactly once in that particular cycle.

Consider $\sigma = [68125743]$. As mentioned earlier $\text{Des}(\sigma) = \{2, 6, 7\}$. Since we want to compute $L_A(\sigma)$ which by definition is the minimum number of tails required for the players to go out according to the permutation σ, we must have the following:

1. In the first cycle of the game, only the players P_6 and P_8 toss head and all others toss tails.

2. The players in the second cycle are P_1, P_2, P_3, P_4, P_5 and P_7 ($8-2=6$ players in number) who all had tossed tails in the first cycle.

3. In the second cycle of the game, the players P_1, P_2, P_5 and P_7 must all toss head. The players P_3 and P_4 ($8 - 6 = 2$ players in number) have to toss tail for the game to move according to the permutation σ (when one is interested in getting the minimum number of tails).

4. In the third cycle of the game, player P_3 ($8 - 7 = 1$ player) tosses a tail and player P_4 tosses a head and goes out of the game.

5. Lastly player P_3 tosses a head in the fourth cycle and the game ends.

Proof of Lemma 2: Let $\mathrm{D}es\ (\sigma) = \{i_1, i_2, \dots, i_m\}$. Consider the case of minimum number of tails. In the first cycle, obviously $\sigma(1), \dots, \sigma(i_1)$ toss heads, but $\sigma(i_1 + 1)$ does not, since $\sigma(i_1 + 1) < \sigma(i_1)$. Similarly, in the $(t+1)$st cycle, $\sigma(i_t + 1), \dots, \sigma(i_{t+1})$ toss heads. Thus, in the $(t+1)$st cycle there are $n - i_t$ players playing the game, and all these must have tossed tail in the previous, tth cycle, in order to remain in the game. Thus, the total number of tails in the minimal case is $\sum_t (n - i_t)$, as required. □

We now modify GAME1 so that we need to find the length (inversion number) of the permutation σ for computing Prob(σ) in the modified game. We note that in GAME1, the movement of the game was cyclic, i.e. the player P_{i_1+1} tosses the coin after the player P_{i_1} had tossed a head and gone out of the game, i.e. one can think of the players sitting in a circle and playing the game. But in the modified game, we replace the above rule by another rule. The modified game for S_n is as follows:

GAME 2: The players $P_1, P_2, \dots, P_n$ sit in a row. As soon as the player P_{i_1} tosses a head, he goes out of the game and doesn't toss again. But now the game proceeds a fresh with the first player in the row starting to toss the coin, i.e. everytime a player goes out, the game starts with the first player seated in the row.

Note that to compute Prob(σ) for $\sigma \in S_n$ in GAME2, it is enough to compute the *least (minimum)* number of players who must toss tail in order for σ to be the order in which the players go out. Suppose $\sigma = [P_{i_1} P_{i_2} \dots P_{i_n}]$ then we need minimum $i_1 - 1$ tails before the player P_{i_1} goes out of the game. But that is same thing as saying that the number of inversions is $i_1 - 1$. In general, if P_{i_k} is the kth player to go out then we just need to check how many players P_i are after P_{i_k} with $i < i_k$ to get the least (minimum) number of tails required for P_{i_k} to be the kth player to go out. Hence, we have the following result.

Theorem 3. *In GAME2, for $\sigma \in S_n$, Prob(σ) $= \frac{q^{\ell_A(\sigma)}}{n!_q}$ where $\ell_A(\sigma)$ is defined on page 4.*

Remark: 1. Note that, in general $L_A(\sigma) \neq L_A(\sigma^{-1})$. For example, for $n = 4$ and $\sigma = [2341]$ we get $L_A(\sigma) = L_A(P_2 P_3 P_4 P_1) = 1 \neq 3 =$

$L_A(P_4P_1P_2P_3) = L_A([4123]) = L_A(\sigma^{-1})$, whereas it is well-known that $\ell_A(\sigma) = \ell_A(\sigma)^{-1}$.

2. From Lemma 2, for any $\sigma \in \mathcal{S}_n$ we have $L_A(\sigma) = \sum_{i \in \mathrm{Des}\,(\sigma)}(n-i)$. Now we get the obvious bijection, say ψ, between "$L_A(\cdot)$" and "maj" index, given by $\psi(\sigma) = [n+1-\sigma(n), n+1-\sigma(n-1), \ldots, n+1-\sigma(1)]$. It is easily verified that

$$L_A(\psi(\sigma)) = \sum_{j \in \mathrm{Des}\,(\psi(\sigma))} (n-j) = \sum_{i \in \mathrm{Des}\,(\sigma)} i = maj(\sigma)$$

and vice-versa:

$$\begin{aligned} i \in \mathrm{Des}\,(\psi(\sigma)) &\iff \psi(\sigma)(i) > \psi(\sigma)(i+1) \\ &\iff n+1-\sigma(n-i+1) > n+1-\sigma(n-i) \\ &\iff \sigma(n-i) > \sigma(n-i+1) \\ &\iff n-i \in \mathrm{Des}(\sigma). \end{aligned}$$

3. Using GAME1 and GAME2, we get another bijection (basically composition of two bijections) between the statistics major index and inversion number via the bijection ψ mentioned in Remark 2.

Let $\tau : \mathcal{S}_n \longrightarrow \mathcal{S}_n$ be a bijection such that for $\sigma \in \mathcal{S}_n$, $L_A(\sigma) = \ell_A(\tau(\sigma))$. We now give the algorithm for the map τ : Let the n players be numbered $P_1, P_2, \ldots, P_n$. Suppose $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$. Note that for GAME1, the n players are seated in cyclic order whereas in GAME2 the n players are seated in a row.

(*i*) Define $\tau(\sigma)(1) = \sigma(1)$; and assume that we have already defined $\tau(\sigma)(i)$ for $1 \le i \le k$. Thus there are exactly $n-k$ players left in both games.

(*ii*) Suppose according to GAME1 the players $P_{i_1}, P_{i_2}, \ldots, P_{i_{n-k}}$ are still in the game and the player P_{i_1} has to begin coin-tossing. We also suppose that in GAME2 the players still in the game are $P_{j_1}, P_{j_2}, \ldots, P_{j_{n-k}}$ with P_{j_1} the first person seated in the row and is to begin the tossing of the coin.

Suppose the player P_{i_r} is the next player to go out according to σ (i.e. $\sigma(k+1) = i_r$) in GAME1. Then the minimum number of tails required for the player P_{i_r} to go out is $r-1$. We define $\tau(\sigma)(k+1) = j_r$.

(*iii*) If $k = n$, then we have got the permutation $\tau(\sigma)$ and the algorithm clearly indicates that $L_A(\sigma) = \ell_A(\tau(\sigma))$.

For example let $\sigma = [385972146]$. Then $\mathrm{Des}(\sigma) = \{2,4,5,6\}$ and $L_A(\sigma) = (9-6)+(9-5)+(9-4)+(9-2) = 19$. The algorithm gives: $\tau(\sigma)(1) = 3$, $\tau(\sigma)(2) = 6$, $\tau(\sigma)(3) = 7$, $\tau(\sigma)(4) = 4$, $\tau(\sigma)(5) = 9$, $\tau(\sigma)(6) = 2$, $\tau(\sigma)(7) = 8$, $\tau(\sigma)(8) = 1$, and $\tau(\sigma)(9) = 5$. That is, $\tau(\sigma) = [367492815]$ and check that $\ell_A(\tau(\sigma)) = 19$.

4. The two bijections between major index and inversion number, namely, the one given by Foata (cf., [3]) using the map, say F and the other due to us are different. For example, let $n = 5$ and $\sigma = [53124]$. Then $\psi(\sigma) = [24531]$ and $\tau(\sigma) = [53214]$ whereas $F(\psi(\sigma)) = F([24531]) = [42531]$. Note that

$$\underbrace{\ell_A([42531]) = \text{maj}([24531])}_{F} = \underbrace{L_A([53124]) = \ell_A([53214])}_{\tau} = 7.$$

Section 2

In this section, we propose to modify GAME2 of Section 1 in such a way that the answer in Theorem 3 which involved the length function for the Symmetric group (viewed as a Coxeter group) now gets replaced by the corresponding object for the Coxeter group of type B_n $(n \geq 2)$.

There is an interesting interpretation for the length function of any finite Coxeter group W in terms of standard combinatorial object known as **root systems** (cf., Humphreys [4]). Since we need this interpretation for our result we briefly recall it.

Given a Coxeter system (W, S), let R be the set of all conjugates of S in W; sometimes these are known as 'reflections'. Taking an abstract second copy $-R$ of the set R, on the disjoint union $\Phi = R \cup (-R)$ it is possible to define an action of the group W with certain interesting geometric properties. Instead of this formal approach (cf., Bourbaki [1]), we can describe the W action on Φ in terms of the set Π of 'positive roots', by which we mean the copy of R inside Φ. By $-\Pi$ we mean the copy of $-R$. In this case the length function $\ell(w)$ for $w \in W$ is also given by, $\ell(w) = \text{Card}\big(\Pi \cap w^{-1}(-\Pi)\big)$, which is equal to the number of roots in Π being sent to $-\Pi$ by w.

By the Coxeter group of type $\mathcal{B}_n$ is meant the group generated by all 'reflections' corresponding to the "type–B_n root systems" which is given by

$$\Phi := \{\pm e_i,\ 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq n\},$$
$$\Pi := \{e_i,\ 1 \leq i \leq n\} \cup \{e_i \pm e_j | 1 \leq i < j \leq n \text{ and } \text{Card}(\Pi) = n^2.$$

For the Coxeter group $\mathcal{B}_n, n \geq 2$ the game is defined as follows. For each of the n players $P_i, 1 \leq i \leq n$, we have a dummy player as well. The players are again made to sit in a row so that the sitting arrangement looks like $P_1, P_2, \dots, P_n, \overline{P_n}, \overline{P_{n-1}}, \dots, \overline{P_1}$; where $\overline{P_i}$ represents the dummy player corresponding to player P_i. If a player or its dummy tosses a head, both of them leave the game and do not toss again. Here we have $2n$ players playing the game and the game ends as soon as n players have tossed a head each.

Any element $w \in \mathcal{B}_n$ is an one-to-one map from $\{1, 2, \dots, n\}$ into $\{-n, -(n-1), \dots, -1, 1, 2, \dots, n\}$ such that $\{|w(i)|\}$ equals the set $\{1, 2, \dots, n\}$.

Note that $\mathcal{S}_n$ is a subgroup of $\mathcal{B}_n$ (to be more precise, $\mathcal{B}_n$ is the semi-direct product of $\mathcal{S}_n$ (which permutes e_i) and $(\mathbb{Z}/2\,\mathbb{Z})^n$ (acting by sign changes over e_i) the latter normal in $\mathcal{B}_n$). For $w \in \mathcal{B}_n$, we have the correspondence between the numbers and the players as follows: if $w(i) \in \{1, 2, \ldots, n\}$ then $w(P_i)$ is not a dummy player, but if $w(i) \in \{-1, -2, \ldots, -n\}$ then $w(P_i)$ is a dummy player and we put a bar above the player to indicate this. For $w \in \mathcal{B}_n$, the last Theorem shows that $\ell_B(w)$ (we are writing $\ell_B(w)$ in place of $\ell(w)$ because we are considering the Coxeter group of type B_n) gives the least (minimum number of players including the dummy players who must toss tails in order for w to be the order in which the players go out.

Theorem 4. *Consider the game defined in section 2. Here for* $w \in \mathcal{B}_n$, $Prob(w) = \frac{q^{\ell_B(w)}}{R_n(q)}$ *where*

$$\begin{aligned} R_n(q) &= \prod_{i=1}^{n}(1 + q + q^2 + \cdots + q^{2i-1}) \\ &= \prod_{i=1}^{n}(1 + q)(1 + q^2 + q^4 + \cdots + q^{2i-2}) \\ &= (1 + q)^n \prod_{i=1}^{n}(1 + q^2 + q^4 + \cdots + q^{2i-2}) \\ &= (1 + q)^n n!_{q^2}. \end{aligned}$$

Note that the order of the group $\mathcal{B}_n$ is equal to $R_n(1) = 2^n\, n!$.

Proof: We shall prove the result by induction. The result is clearly true for $n = 2$. Let the result be true if the number of players playing the game is less than or equal to $2n - 2$. Now let us prove it for $2n$ players. We know that the positive roots for the Coxeter group of type B_n are

$$\Pi := \{e_i,\ 1 \le i \le n\} \cup \{e_i \pm e_j | 1 \le i < j \le n\}.$$

We consider two cases:

Case (i): Let $w \in \mathcal{B}_n$ with $w(1) = k$. Then the minimum number of tails required for the player P_k to be the first to go out of the game is $k - 1$.

We will show that the contribution due to $w(1) = k$ to $\ell_B(w)$ is $k - 1$. Since we are considering $w(1)$, we only take roots of the form $e_1 \pm e_j \in \Pi, 2 \le j \le n$. The action of w sends $e_1 \pm e_j$ to $e_k \pm e_{w(j)}$ if $w(j)$ is a positive integer, and to $e_k \mp e_{|w(j)|}$ if $w(j)$ is a negative integer.

Subcase (a): Suppose that $w(j)$ is a positive integer. Then $e_k + e_{w(j)}$ doesn't belong to $-\Pi$ whereas $e_k - e_{w(j)} \in -\Pi$ if and only if $w(j) < k$.

Subcase (b): If $w(j)$ is a negative integer then $e_k + e_{|w(j)|}$ doesn't belong to $-\Pi$ whereas $e_k - e_{|w(j)|} \in -\Pi$ if and only if $|w(j)| < k$.

In both the subcases we see that the condition for an element of Π to be sent to $-\Pi$ is $|w(j)| < k$. And the cardinality of $\{j : |w(j)| < k\}$ is exactly $k-1$. Hence case (i) is done.

Case (ii): Let $w \in \mathcal{B}_n$ with $w(1) = -k$. Then the minimum number of tails required for the player $\overline{P_k}$ to be the first to go out of the game is $n+n-k = 2n-k$.

We will show that the contribution due to $w(1) = -k$ to $\ell_B(w)$ is also $2n-k$. Since we are considering $w(1)$, we only take roots of the form $e_1 \pm e_j \in \Pi, 2 \leq j \leq n$ and e_1. The action of w sends $e_1 \pm e_j$ to $-e_k \pm e_{w(j)}$ if $w(j)$ is a positive integer, and to $-e_k \mp e_{|w(j)|}$ if $w(j)$ is a negative integer and e_1 is sent to $-e_k \in -\Pi$.

Subcase (a): Suppose that $w(j)$ is a positive integer. Then $-e_k - e_{w(j)}$ always belongs to $-\Pi$ whereas $-e_k + e_{w(j)} \in -\Pi$ if and only if $w(j) > k$.

Subcase (b): If $w(j)$ is a negative integer then $-e_k - e_{|w(j)|}$ always belongs to $-\Pi$ whereas $-e_k + e_{|w(j)|} \in -\Pi$ if and only if $|w(j)| > k$.

Considering both the subcases, we get the contribution due to $w(1) = -k$ to be equal to $1+n-1+n-k = 2n-k$ (1 because of $-e_k$, $n-1$ due to $-e_k - e_{|w(j)|}$ and $n-k$ due to the cardinality of $\{j : |w(j)| > k\}$). So we have the proof for case (ii) as well. Therefore by induction we get the required result as far as the numerator is concerned. It is an easy exercise to show that the denominator is $R_n(q)$. □

Remark 1. Let W be any Coxeter group. Then for $w \in W$, we know that $\ell(w) = \ell(w^{-1})$. Therefore, for the game on $\mathcal{B}_n$ of Section 2, we get $\ell_B(w) = \ell_B(w^{-1})$.

2. For the group $\mathcal{B}_n$ we have observed that $\text{Card}(\Pi) = n^2$ and therefore, for $w \in \mathcal{B}_n$, $0 \leq \ell_B(w) \leq n^2$.

3. One can also generalize the game of Moritz and Williams for the Coxeter group $\mathcal{B}_n$ (or of type B), (i.e. the players are made to sit in a circle in the following order: $P_1, P_2, \ldots, P_n, \overline{P_1}, \overline{P_2}, \ldots, \overline{P_n}$) and get results which are similar to what we have in Section 2. This gives rise to computing another statistics. Now using a slight variation of the map 'τ' of Remark 3 in Section 1, we can get a bijection between the two statistics which arises due to the two games in section 2.

Open problem: Is it possible to define a coin-tossing game for each Coxeter group W such that the problem of finding the required probability boils down to computing the length of the element $w \in W$.

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