Recognizing Clique-Helly Graphs

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ABSTRACT. A family of subsets satisfies the Helly property when every subfamily of it, formed by pairwise intersecting subsets has a common element. A graph is clique-Helly when the family of subsets of vertices which induces the maximal cliques of the graph satisfies the Helly property. We describe a characterization of clique-Helly graphs, leading to a polynomial time algorithm for recognizing them.

1 Introduction

The purpose of this note is to describe a characterization of clique-Helly graphs. It leads to a polynomial time algorithm for recognizing graphs of this class. The problem of whether or not clique-Helly graphs can be recognized in polynomial time has been mentioned as open by Brandstädt [2] and Prisner [6].

Clique-Helly graphs have been introduced in the context of clique graphs. Hamelink [5] showed that if G is clique-Helly then it is the clique graph of some graph. Escalante [4] proved that the second iterated clique graph of a clique-Helly graph G is an induced subgraph of G. More recently, cliqueHelly graphs have been employed in the definition or characterization of other classes of graphs. Those include disk-Helly graphs (Bandelt

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and Prisner [1]), dually chordal graphs (Bransdtädt, Dragan, Chepoi and Voloshin [2]), hereditary clique-Helly graphs (Prisner [6]), clique irreducible graphs (Wallis and Zhang [7]).

G denotes an undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Write $n = |V(G)|$ and $m = |E(G)|$. G' is a *subgraph* of G when it is a graph satisfying $V(G') \subset V(G)$ and $E(G') \subset E(G)$. If $V(G) = V(G')$ then G' is a *spanning subgraph* of G. For $v, w \in V(G')$, when $(v, w) \in E(G)$ implies $(v, w) \in E(G')$ then G' is an *induced subgraph* of G. A *clique* is a complete subgraph of G. A *maximul clique* is one not properly contained in any other. Let S be a family of subsets S_i of some set. The *intersection graph* of S is a graph whose vertices correspond to the subsets S_i, two vertices being adjacent if the corresponding subsets of S intersect. S satisfies the *Helly property* when every subfamily of S consisting of pairwise intersecting subsets has a nonempty intersection. The *clique graph* of G is the intersection graph of the vertex sets of its maximal cliques. G is *clique-Helly* when the vertex sets of its maximal cliques satisfy the Helly property. Figure 1 depicts the smallest graph which is not clique-Helly.

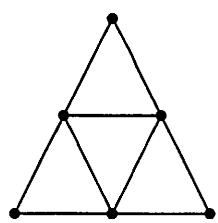

Figure 1

2 Characterization

Let G be a graph and T a triangle of it. The *extended triangle* of G, relative to T, is the subgraph of G induced by the vertices which form a triangle with at least one edge of T.

Let C be a subset of the maximal cliques of G. The *clique subgraph* induced by C in G is the subgraph of G formed exactly by the vertices and edges belonging to some clique of C.

The following observations are clear.

A clique subgraph of G is not necessarily an induced subgraph of it.

Every maximal clique of C is a maximal clique of the clique subgraph induced by C in G. The converse is not true.

The lemma below relates extended triangles and clique subgraphs.

Lemma 1. *Let G be a graph, T a triangle of it and C the subset of maximal cliques of G containing at least one of the edges of T. Then the clique subgraph induced by C in G is a spanning subgraph of the extended triangle relative to T.*

Proof: Let H be the extended triangle relative to T and H' the clique subgraph induced by C in G. We show that $V(H) = V(H')$. Let $v \in V(H)$. Then v forms a triangle with some edge e of T. That is, v belongs to some maximal clique containing e. Hence $v \in V(H')$. Conversely, let $v \in V(H')$. Then v belongs to some maximal clique containing an edge e of T. That is, v forms a triangle with e, and therefore $v \in V(H)$. Consequently, $V(H) = V(H')$. In addition, H and H' are both subgracphs of G, but H is induced. Consequently, $E(H') \subset E(H)$. □

Let H be a subgraph of G. A vertex $v \in V(H)$ is *universal in* H whenever v is adjacent to every other vertex of H.

The following theorem characterizes clique-Helly graphs.

Theorem 1. *G is a clique-Helly graph if and only if every of its extended triangles contains a universal vertex.*

Proof ($\Longrightarrow$): Suppose the contrary. That is, let G be a clique-Helly graph such that it contains an extended triangle H, relative to some triangle T, having no universal vertex. Let C be the subset of maximal cliques of G containing at least one of the edges of T. Clearly, any two edges of T have a common vertex. Therefore any two maximal cliques of C have also a common vertex. However, H has no universal vertex. According to Lemma 1, the clique subgraph H' induced by C is a spanning subgraph of H. Hence neither H' contains a universal vertex. Consequently, C must be a set of pairwise intersecting maximal cliques having no common vertex. Hence G is not clique-Helly, a contradiction.

Proof ($\Longleftarrow$): Suppose the theorem is not true. Then there is some graph G such that all its extended triangles contain universal vertices, but G is not clique-Helly. Consequently, G has a family C of pairwise intersecting maximal cliques with no common vertex. Let $C' = \{C_1, \ldots, C_k\}$ be a minimal subfamily of C not satisfying the Helly property. That is, C' does not satisfy the Helly property, but every subfamily of it does. Since every two maximal cliques of G satisfy the Helly property it follows that C' exists and $|C'| \geq 3$. By the minimality of C' it follows that the cliques of $C' - C_i$ contain a common vertex, denoted by v_i, $i = 1, \ldots k$. Since C' does not satisfy the Helly property, $v_i \notin V(C_i)$. Because $k \geq 3$, every two distinct vertices v_i, v_j belong to a same clique and therefore must be adjacent. Hence the vertices v_1, v_2, v_3 form a triangle, denoted by T. Let H' be the clique subgraph induced by C' . Since C' does not satisfy the Helly property it follows that H' does not contain a universal vertex. Denote by H'' the clique subgraph induced by the maximal cliques of G, having at least one edge of T. By construction, every maximal clique $C_i \in C'$ contains all vertices v_j, $i \neq j$. That is, every maximal clique of C' has at least one edge of T. Hence H' is a subgraph of H''. Let H be the extended triangle of G relative to T. By lemma 1, H'' is a spanning subgraph of H.

By hypothesis, H contains a universal vertex z. Then z is adjacent in G to all vertices of H', in particular. So z must belong to H', otherwise the maximal cliques which form H' would not be maximal. But in this case, z would be a universal vertex in H', a contradiction. Therefore G must be a clique-Helly graph. □

3 Algorithm

The recognition algorithm for clique-Helly graphs follows directly from Theorem 1. Given a connected graph G, for each of its triangles T, find the extended triangle H relative to T, and verify if H has a universal vertex. G is clique-Helly if and only if a universal vertex exists for each extended triangle H. Finding all triangles of G requires $O(nm)$ steps. The extended triangle H can be constructed in $O(n)$ time, for each triangle T. To verify if H has a universal vertex requires $O(m)$ time, for each H. Hence a straightforward implementation of this algorithm has time complexity $O((n+t)m)$, where t is the number of triangles of G.

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