On the Closure of Subsets of $\{4, 5, \ldots, 9\}$ which contain $\{4\}$*

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1 Introduction

Let K be a non-empty set of positive integers, and let v be a positive integer. A pairwise balanced design of order v with block sizes from K (that is, a $PBD[v, K]$) is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a v-set and $\mathcal{B}$ is a family of subsets (blocks) of $\mathcal{V}$ which satisfy the properties:

1 If $B \in B$ then $|B| \in K$;

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2 Every pair of distinct elements of $\mathcal{V}$ occurs in precisely one block of $\mathcal{B}$.

Let $B(K) = \{v: \exists \text{ a } PBD[v,k]\}$. Then $B(K)$ is called the PBD-closure of K.

Let $\alpha(K) = \gcd\{k-1: k \in K\}$ and $\beta(K) = \gcd\{k(k-1): k \in K\}$. Then the conditions $v - 1 \equiv 0 \pmod{\alpha(K)}$ and $v(v-1) \equiv 0 \pmod{\beta(k)}$ are easily seen to be necessary for the existence of a $PBD[v,k]$. R.M. Wilson has shown that they are also asymptotically sufficient, that is, there exists a constant $v_0 = v_0(K)$ such that if $v > v_0$ and these conditions are satisfied, then there exists a $PBD[v,k]$. The estimates of the constant $v_0(K)$ as determined by Wilson's theory are extremely large, and in specific instances much stronger results are possible. For example, in [12] the exact closure for all subsets of $\{3,4,5,6,7,8,9,10\}$ which contain 3 are determined. It is our purpose here to investigate the closures of all subsets of $\{4,5,\ldots,9\}$ which contain the element 4. As will be seen in the next section, this extends the work of several authors.

2 Notation and Conventions

For convenience, let $X = \{5,6,7,8,9\}$. For notation and the definitions of group divisible design (GDD), transversal design (TD), balanced incomplete block design (BIBD), and related designs and the notion of essential elements in PBD-closed sets see [4]. Let $X = \{5,6,\ldots,9\}$. Then the sets to be considered are those of the form $\{4\} \cup Y$, where Y belongs to $P(X)$, the power set of X. Thus there are in principle 32 sets to be investigated. Fortunately the size of this investigation is greatly reduced by the work of other investigators. These results are summarized in the next section. To present these results the following notation is introduced. Let $N_{\geq n} = \{x: x \in Z, x \geq n\}$ and $N_{\geq n}(a_1, a_2, \ldots, a_s \bmod m) = \bigcup_{i=1}^{s}\{x: x \in N_{\geq n}, x \equiv a_i \pmod{m}\}$. If K is a non-empty set of positive integers and if $m = \min(K)$, then clearly the integers x satisfying $1 < x < m$ cannot occur in $B(K)$. Therefore we adopt the convention that $B(K)$ is a subset of $N_{\geq m}$. Let K be any subset of positive integers. For convenience, we also define $B_f(K)$ by $B_f(K) = \{v: \exists \text{ a } PBD[v, K \cup \{f^*\}\}$, that is, $B_f(K)$ is the set of orders of PBDs with block sizes in K which contain a flat of size f. Similarly if S is a PBD-closed set, then $S_f = \{v: \exists \text{ a } PBD[v, S \cup \{f^*\}]$. Note that if $v \in S_f$, and if $f \in S$, then $v \in S$.

3 Known results

In this section existing results are surveyed, and in some instances, updated.

Result 3.1 *(Hanani [13])*
$B(\{4\}) = N_{\geq 4}(1, 4 \bmod 12)$.

By removing a point from a $PBD[12t+1, \{4\}]$ or a $PBD[12t+4, \{4\}]$, a $\{4\}$-GDD of type 3^{4t} or a $\{4\}$-GDD of type 3^{4t+1} is obtained. This is the most frequently used ingredient GDD for recursive constructions.□

Result 3.2 *(Wilson [4])*
$B(\{4,5\}) = N_{\geq 4}(0, 1 \bmod 4) \setminus \{8, 9, 12\}$.

Result 3.3 *(Brouwer [5])*
$B(\{4,7\}) = N_{\geq 4}(1 \bmod 3) \setminus \{10, 19\}$.

Result 3.4 *(Bennett [2])*
Let

$$E = \{5, 9, 12, 17, 20, 21, 24, 33, 41, 44, 45\}$$

and $S = \{48, 53, 60, 65, 68, 69, 72, 77, 81, 89, 93, 96, 101, 105, 108, 117, 129, 153, 156, 161, 164, 165, 168, 173, 177\}$. *Then* $B(\{4,8\}) \supseteq N_{\geq 4}(0, 1 \bmod 4) \setminus (E \cup S)$. *Further if* $v \in E$, *then* $v \notin B(\{4,8\})$.

The following is an update of that result.

Lemma 3.1 *Let* $S_1 = \{48, 53, 60, 65, 69, 77, 89, 101, 161, 164, 173\}$, *and let* E *be as in Result 3.4. Then* $B(\{4,8\}) \supseteq N_{\geq 4}(0, 1 \bmod 4) \setminus (E \cup S)$. *Further if* $v \in E$, *then* $v \notin B(\{4,8\})$.

Proof. It is necessary to show that $\{68, 72, 81, 93, 96, 105, 108, 117, 129, 153, 156, 165, 168, 177\} \in B(\{4,8\})$.

It is shown in [8] that $\{68, 72\} \subseteq B(\{4,8\})$.

In the following, an automorphism group and a set of base blocks for each design on the specified number of points is given. (If no group is specified, the design is based on the cyclic group of the given order.)

$v = 81 : \{0, 3, 5, 9, 22, 33, 47, 74\},\ \{0, 15, 60, 61\},\ \{0, 18, 26, 49\}$.
$v = 93 : G = Z_3 \times Z_{31}$.

Let $B_1 = \{(0,0), (0,1), (0,5), (0,25), (1,0), (1,2), (1,10), (1,19)\}$ and $B_2 = \{(0,0), (0,16), (1,29), (2,9)\}$. We obtain two more blocks by multiplying B_2 by $(1,5)$ and $(1,25)$.

$v = 96:$ $\{0,24,48,72\}$ $\{0,1,3,7,18,32,40,45\}$ $\{0,9,28,75\}$ $\{0,10,26,60\}$ $\{0,12,35,55\}$.

$v = 105:$ $\{0,1,8,33,35,38,48,52\}$ $\{0,24,46,55\}$ $\{0,16,36,79\}$ $\{0,6,18,82\}$ $\{0,21,49,60\}$.

$v = 108:$ $\{0,27,54,81\}$ $\{0,1,4,10,21,46,78,90\}$ $\{0,2,15,73\}$ $\{0,5,52,60\}$ $\{0,7,23,82\}$ $\{0,14,38,79\}$.

$v = 117:$ $\{0,1,21,33,44,50,52,113\}$ $\{0,24,27,103\}$ $\{0,15,28,87\}$ $\{0,9,16,55\}$ $\{0,18,40,82\}$ $\{0,10,36,70\}$.

If $v = 129:$ $G = Z_3 \times Z_{43}$.
Let $B_1 = \{(0,1),(0,6),(0,36),(0,3),(0,18),(0,22),(1,0),(2,0)\}$, $B_2 = \{(0,0),(0,1),(1,5),(2,33)\}$, and $B_3 = \{(0,0),(0,9),(1,35),(2,23)\}$. We obtain four more blocks by multiplying B_2 and B_3 by $(1,6)$ and $(1,36)$.

$v = 153:$ $\{0,1,38,41,49,51,63,68\}$ $\{0,18,24,52\}$ $\{0,9,69,130\}$ $\{0,15,73,94\}$ $\{0,33,72,108\}$ $\{0,16,42,98\}$ $\{0,7,54,83\}$ $\{0,4,57,122\}$ $\{0,46,66,110\}$.

$v = 156:$ $\{0,1,25,62,69,71,92,105\}$ $\{0,27,75,148\}$ $\{0,22,60,63\}$ $\{0,19,45,74\}$ $\{0,15,33,99\}$ $\{0,5,103,107\}$ $\{0,14,128,139\}$ $\{0,10,50,150\}$ $\{0,20,97,144\}$ $\{0,39,78,117\}$.

$v = 165:$ $\{0,1,19,47,63,74,95,99\}$ $\{0,13,30,54\}$ $\{0,39,100,114\}$ $\{0,40,60,82\}$ $\{0,12,69,162\}$ $\{0,45,79,160\}$ $\{0,35,112,122\}$ $\{0,38,64,97\}$ $\{0,23,31,159\}$ $\{0,2,9,58\}$.

$v = 168:$ $\{0,1,25,62,69,71,92,105\}$ $\{0,22,27,60\}$ $\{0,75,78,94\}$ $\{0,4,57,83\}$ $\{0,12,51,66\}$ $\{0,45,73,154\}$ $\{0,32,118,128\}$ $\{0,8,55,120\}$ $\{0,11,31,150\}$ $\{0,35,52,162\}$ $\{0,42,84,126\}$.

$v = 177:$ $\{0,1,19,41,57,68,93,101\}$ $\{0,4,30,51\}$ $\{0,39,48,163\}$ $\{0,64,69,160\}$ $\{0,12,54,99\}$ $\{0,43,63,118\}$ $\{0,72,79,145\}$ $\{0,24,34,131\}$ $\{0,35,37,50\}$ $\{0,23,106,171\}$ $\{0,58,61,89\}$.

□

Result 3.5 *(Lenz [15])*

(a) Let $B_1 = \{6, 11, 12, 14, 15, 18, 19, 23, 26, 27, 30, 38, 42, 51, 86, 90\}$, $B_2 = \{62, 66, 74, 78\}$, *and* $B_3 = \{39, 50, 54, 63\}$.

(i) $B(\{4,5,7\}) \supseteq N_{\geq 4} \setminus (B_1 \cup B_2 \cup B_3 \cup \{8,9\})$.

(ii) $B(\{4,5,7,8\}) \supseteq N_{\geq 4} \setminus (B_1 \cup B_2 \cup \{9\})$.

(iii) $B(\{4,5,7,8,9\}) \supseteq N_{\geq 4} \setminus B_1$.

(b) Let $C_1 = \{8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$ *and* $C_2 = \{7, 43, 47\}$. *Then*

(i) $B(\{4,5,6\}) \supseteq N_{\geq 4} \setminus (C_1 \cup C_2)$,

(ii) $B(\{4,5,6,7\}) \supseteq N_{\geq 4} \setminus C_1$. □

It is an immediate consequence of Result 3.2 that $B(\{4,5,8\}) = N_{\geq 4}(0, 1 \bmod 4) \setminus \{9, 12\}$, $B(\{4,5,9\}) = N_{\geq 4}(0, 1 \bmod 4) \setminus \{8, 12\}$, and $B(\{4,5,8,9\}) = N_{\geq 4}(0, 1 \bmod 4) \setminus \{12\}$.

In the other direction, Drake and Larson [7] have proved the following result.

Result 3.6 *[7] Let K be a set of positive integers and let m denote the smallest integer in K. Suppose that there exists a $PBD[v, k]$ which contains blocks B_h and B_k of sizes h and k respectively. Then*

(i) $v \geq (m-1)k + h - m + 1$; *hence*

(ii) $v \geq (m-1)k + 1$, *with equality if and only if there exists a resolvable BIBD*$(k(m-2)+1, m-1, 1)$.

It is known that $23 \notin B(\{4,5,6,7\})$ (See [16]). Therefore the set C_1 of Result 3.5(b) contains only elements which do not lie in $B(\{4,5,6\})$ and $B(\{4,5,6,7\})$.

Let $F = \{10, 11, 12, 14, 15, 18, 19, 23\}$. Then the above observations together with Results 3.5(b) and 3.6 show that $B(\{4,5,6,8\}) = N_{\geq 4} \setminus (\{7,9\} \cup F)$, and that $B(\{4,5,6,9\}) = N_{\geq 4} \setminus (\{7,8\} \cup F)$, $B(\{4,5,6,7,8\}) = N_{\geq 4} \setminus (\{9\} \cup F)$, $B(\{4,5,6,7,9\}) = N_{\geq 4} \setminus (\{8\} \cup F)$, and $B(\{4,5,6,7,8,9\}) = N_{\geq 4} \setminus (F)$.

Lemma 3.2 $\{38, 50, 62, 63, 66, 74, 78, 86, 90\} \subset B(\{4,5,7\})$.

Proof. Greig [8] has shown that $38 \in B(\{4,5,7\})$.

Gronau and Bennett [10] have shown that there exist a $PBD[90,\{4,5,7\}]$ and a $PBD[86,\{4,5,7\}]$. These results were obtained by deleting all but one point on a block in a TD[5, 18] and TD[5, 17] to obtain a {4, 5}-GDD of types $17^4 18^1$ and $16^4 17$ respectively. A further ingredient employed is a $PBD[21,\{4,5,4^*\}]$ (which is also a $PBD[21,\{4,5,5^*\}]$) which can be created by deleting 4 points from the group of a TD[5, 5]. Applying the singular direct product (see [11] , Theorem 2.3). using the {4, 5}-GDD of type $17^4 18^1$ and the above ingredient (viewed as a $PBD[21,\{4,5,4^*\}]$), a $PBD[90,\{4,5,22^*\}]$ is obtained. Similarly using the {4, 5}-GDD of type $16^4 17^1$ with this ingredient (viewed as a $PBD[21,\{4,5,5^*\}]$), a $PBD[86,\{4,5,2$ is obtained. Further there is a $PBD[22,\{4,7\}]$ (which can be obtained by adjoining 7 "infinite" points to a Kirkman Triple System of order 15), so the block of size 22 can be "broken up" to yield a $PBD[86,\{4,5,7\}]$ and a $PBD[90,\{4,5,7\}]$. Further we have $\{50,62,74\} \subseteq B(\{4,5,7\})$ as is shown below.

$v = 50$: Let $V = \{0,1\} \times (GF(5^2)$, generated by $x^2 = x+3)$, under the action $(-, GF(5^2))$.

Then the base blocks are

$$\{(0,3x),(0,3x+2),(0,4x+3),(1,0),(1,1),(1,2x+1),(1,3x+3)\}$$
$$\{(0,0),(1,0),(1,2),(1,4x+2),(1,x+1)\}$$
$$\{(0,0),(0,1),(0,2x),(1,3x)\}.$$

Two additional base blocks are obtained by multiplying the second components of the last block by $3x+3$ and $2x+1$.

$v = 62$: $V = \{0,1\} \times Z_{31}$ under the action $(-, \bmod 31)$. Then the base blocks are

$$\{(0,1),(0,5),(0,25),(1,0),(1,16),(1,18),(1,28)\}$$
$$\{(0,0),(1,0),(1,1),(1,5)(1,25)\}\{(0,0),(1,12),(1,21),(1,29)\}$$
$$\{(0,11),(0,24),(0,27),(1,0)\}\{(0,0),(0,21),(0,22),(1,9)\}$$

Two additional base blocks are obtained by multiplying the second component in the last blocks by 5 and 25 (mod 31).

$v = 63$: $V = \{0,1\} \times Z_{28}$ under the action $(-, \text{mod } 28)$ together with the points $\infty_1, \infty_2, \cdots, \infty_7$. Then the base blocks are

$$\{\infty_1, \infty_2, \cdots, \infty_7\}$$
$$\{(0,0),(0,14),(1,0),(1,14)\}^*$$
$$\{(0,0),(0,1),(1,2),(1,5)\}^*$$
$$\{(0,0),(0,3),(1,6),(1,13)\}^*$$
$$\{(0,0),(0,7),(1,15),(1,26)\}^*$$
$$\{(0,0),(0,2),(1,18),(1,24)\}$$
$$\{(0,8),(1,0),(1,1),(1,5),(1,20)\}$$
$$\{(0,19),(1,0),(1,2),(1,18)\}$$
$$\{(0,0),(0,4),(0,9),(0,17)\}$$
$$\{(0,0),(0,6),(0,16),(1,23)\}$$

$(*)$ The first four blocks generate 7 parallel classes, say, $C_1, C_2, \ldots, C_7$. The point ∞_i is added to each block of class C_i, $i = 1, 2, \ldots, 7$.

$v = 66$: $V = \{0,1\} \times Z_{33}$ under the action $(-, \text{mod } 33)$. The base blocks are

$$\{(0,0),(2,0),(5,0),(9,0),(15,0),(0,1),(8,1)\}$$
$$\{(0,0),(0,4),(0,18),(1,2),(1,24)\}$$
$$\{(0,4),(0,15),(1,0),(1,8),(1,23)\}$$
$$\{(1,0),(1,1),(1,3),(1,7)\}$$
$$\{(0,0),(0,6),(1,7),(1,21)\}$$
$$\{(0,0),(0,13),(1,3),(1,27)\}$$
$$\{(0,22),(1,0),(1,5),(1,17)\}$$

$v = 74$: $V = \{0,1\} \times Z_{37}$ under the action $(-, \text{mod } 37)$. Then the base blocks are

$$\{(0,0),(0,1),(0,10),(0,26),(1,2),(1,15),(1,20)\}$$
$$\{(0,0),(1,0),(1,7),(1,33),(1,34)\}, \{(1,0),(1,6),(1,8),(1,23)\}$$
$$\{(0,0),(0,2),(0,5),(1,32)\}, \{(0,0),(0,8),(1,16),(1,25)\}.$$

Two additional base blocks are obtained by multiplying the second components of the last block by 10 and 26 (mod 37). □

$v = 78$: $V = \{0,1\} \times Z_{39}$ under the action $(-, \text{mod } 39)$. Then the base blocks are

$$\{(0,0),(0,2),(0,5),(0,9),(0,15),(1,0),(1,8)\}$$
$$\{(0,0),(0,8),(0,19),(1,15),(1,29)\}$$
$$\{(0,8),(0,26),(1,0),(1,10),(1,27)\}$$
$$\{(1,0),(1,2),(1,5),(1,9)\}$$
$$\{(0,0),(0,12),(1,17),(1,28)\}$$
$$\{(0,0),(0,14),(1,18),(1,36)\}$$
$$\{(0,28),(1,0),(1,1),(1,16)\}$$
$$\{(0,25),(1,0),(1,6),(1,19)\}$$
$$\{(0,0),(0,1),(0,17),(1,26)\}$$

Lemma 3.3 $\{43,47\} \subseteq B(4,5,6)$.

Proof. See [18].

Let $D_{457} = \{6,8,9,10,11,12,14,15,18,19,23,26,27\}$ and let $P_{457} = \{30,39,42,51,54\}$.

Result 3.7 *(i) If $v \geq 4$, then $v \in B(\{4,5,7\})$ with the exception of $v \in D_{457}$ and possible exception of $v \in P_{457}$.*

(ii) If $v \geq 4$, then $v \in B(\{4,5,7,8\})$ with the exception of $v \in D_{457} \setminus \{8\}$ and the possible exception of $v \in \{30,42,51\}$.

(iii) If $v \geq 4$, then $v \in B(\{4,5,7,9\})$ with the exception of $v \in D_{457} \setminus \{9\}$ and the possible exception of $v \in \{30,39,42,50,51,54\}$.

(iv) If $v \geq 4$, then $v \in B(\{4,5,7,8,9\})$ with the exeption of $v \in D_{457} \setminus \{8,9\}$ and possible exception of $v \in \{30,51\}$.

Proof. The result follows from those in [13] together with the results of Lemma 3.2. □

4 Closure of the remaining two element sets

In this section, the closure of $\{4,6\}$ and $\{4,9\}$ is investigated. We use [1] as our source for the TDs and ITDs employed in any constructions requiring such ingredients.

Construction 4.1 *Let S be a PBD-closed set which contains $\{4\}$. Suppose that there exists a TD$[5,m]$, and that t is an integer satisfying $0 \leq t \leq m$. If $3m + f \in S_f$, then $12m + 3t + f \in S_{3t+f}$.*

Proof. By deleting a point from PG(2,3) and AG(2,4), {4}-GDDs of types 3^4 and 3^5 respectively are obtained. The result follows immediately from Wilson's Fundamental Construction (WFC), see ([11], Theorem 2.5) and the singular direct product (SDP) (see [11], Theorem 2.3). □

Construction 4.2 *Let S be a PBD-closed set which contains $\{4,6\}$. Suppose that there exists a $TD[6,m]$. Let a and b be positive integers satisfying $a+b \le m$. If $3m+f \in S_f$, then $15m+5a+6b+f \in S_{5a+6b+f}$.*

Proof. As noted in Construction 4.1, there exists a {4}-GDD of type 3^5. By adjoining a new point to the groups of a TD[4,5] then deleting another point, a {4,6}-GDD of type $3^5 5^1$ is obtained. Also, by adjoining six points to a KTS(15), a {4}-GDD of type $3^5 6^1$ is obtained. Let $G_0, G_1, \dots, G_5$ be the groups of a TD$[6,m]$. Assign a weight of three to every point in $G_1, G_2, \dots, G_5$, then assign a weight of five to a points of G_0, a weight of six to b such points, and a weight of zero to the remaining points. The result follows from WFC and SDP.□

Construction 4.3 *[11] (Singular Indirect Product)*

Let S be a PBD-closed set which contains an integer k. Suppose that $m+f \in S_f$ and that there exists a $TD[k,m+a]$-$TD[k,a]$ for some a satisfying $0 \le a \le f$. Then $km+(k-1)a+f \in S_{(k-1)a+f}$.

For sake of completeness the following construction is also included.

Construction 4.4 *Let S be a PBD-closed set which contains $\{4\}$. If $3s+1 \in S$, then $9s+4 \in S$.*

Proof. Adjoin $3s+1$ points to a KTS$(6s+3)$.□

Lemma 4.1 *Suppose that v is an integer satisfying $v \equiv 1 \pmod 3$, $v \ge 25$. If*

$$v \notin \{34,46,55,67,70,79,82\},$$

then $v \in B(\{4,6\})$.

Proof. If $v \equiv 1,4 \pmod{12}$, $v \in B(\{4\})$. We only need to consider when $v \equiv 7,10 \pmod{12}$. If $v=31$, or 91, then $v \in B(\{6\})$. If $v=43$, use the Extended Brouwer Construction [11] with $q=t=3$ to obtain $43 \in B(\{4,6\})$. If $v=58$, take a $TD[6,9]$ with a flat of order 4. If $v \ge 94$, then $v \in B(\{4,31^*\})$ by [20]. The result follows.□

Lemma 4.2 *Suppose that v is an integer satisfying $v \equiv 0 \pmod 3$, $v \geq 21$. If*

$$v \notin \{24,27,33,39,45,51,57,63,75,87,93,99,123,129,159\},$$

then $v \in B(\{4,6\})$.

Proof. If $v \equiv 0 \pmod 6$, $v \geq 30$, then there exists a $\{4\}$-GDD of type 6^r for all $r \geq 5$. We only need to deal with the case when $v \equiv 3 \pmod 6$. If $v = 21$, take a $TD[4,5]$ and add a point to each group to obtain $21 \in B(\{4,6\})$. If $v = 69$, take a resolvable 3-GDD of type 6^8 and add one point to each parallel class of blocks to obtain a $\{4\}$-GDD of type $6^8 21^1$. This gives $69 \in B(\{4,6\})$. If $v = 81$, take a $TD[4,20]$ and add a point at infinity to obtain $81 \in B(\{4,6\})$. If $v = 105$, apply Construction 4.2 with $m = 5$, $a = 1$, $b = 4$ and $f = 1$. For $v = 111$, the result follows easily from the fact that $111 \in B(\{6\})$. If $v = 117$, take a $TD[4,29]$ and add a point at infinity. If $v = 135$, apply Construction 4.2 with $m = 7$, $a = 6$ and $b = f = 0$. If $v = 141$, take a $\{4\}$-GDD of type 20^7 and add a point to each group. If $v = 147$, apply Construction 4.2 with $m = 7$, $a = 6$, $b = 2$ and $f = 0$. If $v = 153$, apply Construction 4.2 with $m = 7$, $a = 4$, $b = 3$ and $f = 10$ by noting that $31 \in B(\{4,10^*\})$ [20]. Apply Construction 4.2 with $a = 1$, $b \in \{4,5,6,7,8\}$, $m = 9$ and $f = 1$ to obtain $165,171,177,183,189 \in B(\{4,6\})$. If $v = 195$, apply Construction 4.2 with $m = 11$, $a = 1$, $b = 8$ and $f = 13$, the required $IPBD$ is obtaining by adjoining infinite points from $KTS(27)$. Applying Construction 4.2 with $a = 4$, $b \in \{0,1,2,3,4,5,6,7\}$, $m = 11$ and $f = 16$ to obtain $201,207,213,219,225,231,237,243 \in B(\{4,6\})$. Finally, applying Construction 4.2 with $a = 1$, $b \in \{4,5,\ldots,m-1\}$, $f = 1$ and $m \geq 13$, odd, $m \neq 15,23,27$, to obtain $15m+30, 15m+36, \ldots, 21m \in B(\{4,6\})$. This gives $v \in B(\{4,6\})$ for all $v \geq 225$ and $v \equiv 3 \pmod 6$ with the exception of $v = 279$. If $v = 279$, apply Construction 4.2 with $m = 13$, $a = 13$, $b = 0$ and $f = 13$. The existence of the ingredient $IPBD$ is obtained by adjoining infinite points to $KTS(39)$. The result follows.□

We also have some direct constructions.

Lemma 4.3 $\{55,57,63,67,70,82,93,99,123,129,159\} \subset B(\{4,6\})$.

Proof. If $v = 55$, let $G = Z_{55}$. Then the base blocks are

$$\{0,1,3,8,23,44\},$$
$$\{0,4,10,28\},$$
$$\{0,9,25,38\}.$$

If $v = 67$, let $G = \mathbf{Z}_{67}$, $B_1 = \{1, 2, 7, 29, 37, 58\}$ and
$B_2 = \{0, 2, 14, 17\}$.
We can multiply B_2 by 37 and 29 to obtain 2 further blocks. It is easy to check that they form a difference family.

For $v = 57$, we use a non-abelian group. Let $V = \mathbf{Z}_3 \times \mathbf{Z}_{19}$ under the action of addition $(-, \bmod 19)$ and $\tau : (t, u) \to (t + 1, 7u)$. Then the base blocks are

$$\{(0,0),(0,8),(1,0),(1,18),(2,0),(2,12)\}$$
$$\{(0,6),(0,10),(1,4),(1,13),(2,9),(2,15)\}$$
$$\{(0,0),(0,13),(0,18),(1,15)\}$$
$$\{(0,0),(0,12),(0,10),(1,5)\}$$
$$\{(0,1),(0,3),(1,4),(2,10)\}$$
$$\{(0,13),(1,0),(1,2),(2,9)\}$$
$$\{(0,6),(1,15),(2,0),(2,14)\}.$$

For $v = 63$, we use the group used for $v = 57$, together with the six invariant points $\infty_3, \infty_7, \infty_{10}, \infty_{12}, \infty_{13}, \infty_{15}$. Then the base blocks are

$$\{(0,0),(0,3),(1,0),(1,2),(2,0),(2,14)\}$$
$$\{(0,6),(0,15),(1,4),(1,10),(2,9),(2,13)\}$$
$$\{(0,0),(0,1),(0,5),(0,7)\}$$
$$\{(1,0),(1,7),(1,11),(1,16)\}$$
$$\{(2,0),(2,1),(2,11),(2,17)\}$$
$$\{(0,0),(0,8),(1,9),(2,10)\}$$
$$\{(0,13),(1,0),(1,18),(2,6)\}$$
$$\{(0,4),(1,15),(2,0),(2,12)\}$$
$$\{\infty_i\ (0,16_i),(1,17_i),(2,5_i)\}\ i \in \{3,7,10,12,13,15\}$$
$$\{\infty_3,\infty_7,\infty_{10},\infty_{12},\infty_{13},\infty_{15}\}.$$

For $v = 70$, let $V = \mathbf{Z}_{69} \cup \{\infty\}$. Then the base blocks are

$$\{0,1,3,7,15,40\}$$
$$\{0,5,18,27\}$$
$$\{0,10,26,45\}$$
$$\{0,11,28,49\}$$
$$\{\infty,0,23,46\}.$$

If $v = 79$, let $G = \mathbf{Z}_{79}$. For base blocks, take $B_1 = \{1, 2, 23, 31, 46, 55\}$,
$B_2 = \{0, 12, 28, 39\}$ and
$B_3 = \{0, 6, 37, 75\}$.
We multiply B_3 by 23 and 55 to obtain 2 further blocks. It is easy to check that this is the required difference family. □

For $v = 82$, let $V = \mathbf{Z}_{81} \cup \{\infty\}$. Then the base blocks are

$$\{0,1,5,11,23,53\}$$
$$\{0,2,9,64\}$$
$$\{0,3,16,47\}$$
$$\{0,8,32,56\}$$
$$\{\infty,0,27,54\}.$$

For $v = 93$, again we use a non-abelian group. Let $V = \mathbf{Z}_3 \times \mathbf{Z}_{31}$ under the action of addition $(-,31)$ and $\tau(t,u) \rightarrow (t+1,5u)$. Then the base blocks are

$$\{(0,0),(0,1),(1,0),(1,5),(2,0),(2,25)\}$$
$$\{(0,2),(0,4),(1,10),(1,20),(2,7),(2,19)\}$$
$$\{(0,0),(0,3),(1,1),(2,4)\}$$
$$\{(0,0),(0,4),(1,7),(1,21)\}$$
$$\{(0,0),(0,5),(1,15),(1,28)\}$$
$$\{(0,0),(0,10),(1,22),(1,24)\}$$
$$\{(0,0),(0,6),(0,23),(1,25)\}$$
$$\{(0,0),(0,7),(0,18),(1,27)\}.$$

For $v = 99$, we use the group used for $v = 93$, with six invariant points $\infty_2, \infty_{10}, \infty_{12}, \infty_{14}, \infty_{19}, \infty_{24}$. The base blocks are

$$\{(0,0),(0,1),(1,0),(1,5),(2,0),(2,25)\}$$
$$\{(0,2),(0,4),(1,10),(1,20),(2,7),(2,19)\}$$
$$\{(0,0),(0,3),(1,1),(2,4)\}$$
$$\{(0,0),(0,4),(0,9),(0,15)\}$$
$$\{(0,0),(0,7),(1,3),(1,22)\}$$
$$\{(0,0),(0.8),(1,25),(1,28)\}$$
$$\{(0,0),(0,14),(1,21),(1,23)\}$$
$$\{\infty_x,(0,8x),(1,9x),(2,14x)\}\ x \in \{2,10,12,14,19,24\}$$
$$\{\infty_2,\infty_{10},\infty_{12},\infty_{14},\infty_{19},\infty_{24}\}.$$

For $v = 123$, we use $(Z_3 \times Z_{39}) \cup \{\infty_i,\ i \in \{1,3,4,7,11,12\}$ where $Z_3 \times Z_{39}$ is under the action of $(-, 39)$ and $(t,u) \to (t+1, 16u)$. The base blocks are

$$\begin{array}{l}
\{(0,0),(0,1),(1,0),(1,16),(2,0),(2,22)\} \\
\{(0,3),(0,7),(1,9),(1,34),(2,27),(2,37)\} \\
\{(0,1),(0,8),(1,6),(2,18)\} \\
\{(0,1),(0,11),(1,18),(2,12)\} \\
\{(0,0),(0,6),(0,26),(1,34)\} \\
\{(0,0),(0,9),(0,11),(1,22)\} \\
\{(0,0),(0,18),(0,23),(1,4)\} \\
\{(0,0),(0,3),(0,15),(1,29)\} \\
\{(0,0),(0,22),(0,14),(0,32)\} \\
\{\infty_i,(0,8i),(1,11i),(2,20i)\},\ i \in \{1,3,4,7,11,12\} \\
\{\infty_i :\ i \in \{1,3,4,7,11,12\}.
\end{array}$$

For $v = 129$, we use a non-abelian group. Let $V = Z_3 \times Z_{43}$ under the action of addition $(-, \bmod 43)$ and $\tau : (t,u) \to (t+1, 6u)$. Then the base blocks are

$$\begin{array}{l}
\{(0,0),(0,1),(1,0),(1,6),(2,0),(2,36)\} \\
\{(0,2),(0,4),(1,12),(1,24),(2,15),(2,29)\} \\
\{(0,0),(0,3),(1,1),(2,5)\} \\
\{(0,0),(0,4),(1,7),(1,16)\} \\
\{(0,0),(0,5),(1,2),(1,9)\} \\
\{(0,0),(0,8),(1,34),(1,37)\} \\
\{(0,0),(0,7),(0,32),(1,25)\} \\
\{(0,0),(0,9),(0,19),(1,33)\} \\
\{(0,0),(0,12),(1,35),(1,39)\} \\
\{(0,0),(0,13),(1,30),(1,32)\} \\
\{(0,0),(0,17),(1,28),(1,38)\}.
\end{array}$$

For $v = 159$, we use the same group as for $v = 129$, together with the set of 30 invariant points $\{\infty_i : i \in S\}$ where

$$S = \{2,3,4,7,9,11,12,14,16,17,18,19,21,23,24,25,26,27,28,29,30,32, 33,34,35,36,37,38,39,40\}$$

The base blocks are

$$\{(0,0),(0,1),(1,0),(1,6),(2,0),(2,36)\}$$
$$\{(0,2),(0,4),(1,12),(1,24),(2,29),(2,15)\}$$
$$\{(0,1),(0,4),(1,2),(2,6)\}$$
$$\{(0,0),(0,4),(0,9),(0,31)\}$$
$$\{(0,0),(0,6),(0,19),(0,29)\}$$
$$\{(0,0),(0,7),(0,15),(0,32)\}$$
$$\{\infty_i,(0,26i),(1,27i),(2,33i)\}, i \in S,$$

together with a PBD(30, $\{4,6\}$) defined on the set of 30 invariant points. □

Let D_{46}={7, 9, 10, 12, 15, 18, 19, 22, 24, 27} and P_{46}= {33, 34, 39, 45, 46, 51, 55, 75, 87 }.

The following proof requires the use of packing numbers (see [19]).

Lemma 4.4 *If $v \in D_{46}$, then $v \notin B(\{4,6\})$.*

Proof. If $v \not\equiv 1,4 \pmod{12}$, then $v \notin B(\{4\})$. So if $v \in B(\{4,6\})$, then any such PBD must have a block of size six. Applying Result 3.6, we have $v \geq 19$.

If $v = 19$, then there must be a point lying on at least two blocks of size six. Apply Result 3.6 to obtain $v \geq 21$, a contradiction.

If $v = 22$, every point must be on $0 \pmod 3$ lines of size six. In particular, there must be a point on at least three lines. If the point is on six six-lines, then there is a contradiction since the structure has more than 22 points. Assume that there is a point which lies on exactly three six-lines. By removing the point, a $\{4,6\}$-GDD of type $3^2 5^3$ is obtained. Now, a point on a group of size three must also lie on a six-line, but it is impossible to have a line of size six because there are only five groups.

If $v = 24$, and every point is on exactly one six-line, then this is equivalent to a $TD[4,6]$ which is known not to exist (See [4]). Again, it is trivial to see that every point must be on $1 \pmod 3$ lines of size six. It is impossible to have a point on seven or more six-lines because the structure must have at least 36 points. Hence, there must be a point lying on exactly three six-lines. By removing the point, we obtain a $\{4,6\}$-GDD of type $5^4 3^1$. By a simple parity argument, a point on the group of size three must be on at least one block of size six. However, there are only five groups, so this is impossible. Hence, we have $24 \notin B(\{4,6\})$.

If $v = 27$, every point must be on $1 \pmod 3$ blocks of size six. Suppose there is a point on at least seven blocks of size six, this gives a contradiction since the structure would have at least 31 points. Hence, every point is on at most four blocks of size six, it must have at most 18 blocks of size six. By counting pairs, it is easy to see that the number of block of size six must be odd. By removing a point on four blocks of size six, we obtain a $\{4,6\}$-GDD of type $5^4 3^2$, this means that we must have at least three more blocks of size six. Hence, the number of blocks of size six is at least seven. Let b_i be the number of point on i blocks of size six. Trivially, $b_{3k+2} = b_{3k} = 0$ for all k and $b_{3k+7} = 0$ for all $k \geq 0$. Let b be the number of blocks of size 6. A simple counting argument reveals that $2b - 9 = b_4$. If $b = 7$, there are five points lying on four blocks of size six, dually, this means that we can pack seven points into 5 blocks of size 4 without repeating a pair, and this is impossible. If there exists a $PBD[27,\{4,6\}]$ containing b blocks of size six, we must able to pack b points blocks of size four in $2b - 9$ points for some $9 \leq b \leq 17$. However, this violates the pair packing number (see [19]). Hence, there does not exist a $PBD[27,\{4,6\}]$.□

Theorem 4.1 *In view of the above, if $v \geq 4$, $v \equiv 0,1 \pmod 3$ and $v \notin D_{46} \cup P_{46}$, then $v \in B(\{4,6\})$. Moreover, if $v \in D_{46}$, then $v \notin B(\{4,6\})$.*

We now consider the closure of $\{4,9\}$.

It is easy to see that the necessary condition for $v \in B(\{4,9\})$ is $v \equiv 0,1,4,9 \pmod{12}$. In the case $v \equiv 1,4 \pmod{12}$, we have $v \in B(\{4\}) \subseteq B(\{4,9\})$. Hence, we only need to deal with the case when $v \equiv 0,9 \pmod{12}$.

Lemma 4.5 $36t, 36t+9 \in B(\{4,9\})$ *for all $t \geq 0$.*

Proof. The result follows from the existence of $\{4\}$-GDD of type 9^{4t} and 9^{4t+1} (See [14]).□

Lemma 4.6 $24t+9 \in B(\{4,9\})$.

Proof. This follows by taking a $\{4\}$-GDD of type 8^{3t+1} (See [14]) and add a point at infinity.□

From the above lemmas, we only need to consider v if $v \equiv$ 12, 21, 24, 48, 60, 69 (mod 72).

Lemma 4.7 $120, 132, 141, 165, 168, 213 \in B(\{4,9\})$.

Proof. If $v = 120, 136$, apply Construction 4.3 with $k = 4$, $m = 29, 32$, $a = 0$ and $f = 4$ respectively. If $v = 141$, apply Construction 4.1 with

$m = 9$, $t = 8$ and $f = 9$. If $v = 165$, apply Construction 4.1 with $m = 11$, $t = 11$ and $f = 0$. If $v = 168$, apply Construction 4.3 with $k = 4$, $m = 41$, $a = 0$ and $f = 4$. If $v = 213$, apply Construction 4.1 with $m = 15$, $t = 11$ and $f = 0$. □

Lemma 4.8 $\{228, 237, 240, 264, 276, 285\} \subset B(\{4, 9\})$.

Proof. If $v = 228$ or $v = 237$, apply Construction 4.1 with $m = 16$, $t = 9$ or $t = 12$ and $f = 9$. If $v = 240$, take a $TD[9, 9]$ and remove four points from a block to obtain a $\{5, 8, 9\}$-GDD of type $8^4 9^5$. Assign weight three to every point and use ingredient $\{4\}$-GDDs of type 3^r for $r = 5, 8, 9$ to get a $\{4\}$-GDD of type $24^4 27^5$. Adjoin a flat of size nine to obtain $240 \in B(\{4, 9\})$. If $v = 264$, apply Construction 4.1 with $m = 19$, $t = 12$ and $f = 0$. If $v = 276$, apply Construction 4.3 with $k = 4$, $a = 0$, $m = 68$ and $f = 4$. If $v = 285$, apply Construction 4.1 with $m = 19$, $t = 19$ and $f = 0$. □

Lemma 4.9 $\{300, 309, 312, 336, 348, 357\} \subset B(\{4, 9\})$.

Proof. If $v = 300$, take an $RTD[9, 27]$ and adjoin four points to each of the 14 parallel classes of blocks, and one point to the groups for a total addition of 57 points. Replace each block of size 13 by a $PBD[13, \{4\}]$ in such a fashion that the four adjoined points form a block, then delete that block. Fill in the hole by a $PBD[57, \{4, 9\}]$. If $v = 309$, take a $\{4\}$-GDD of type 4^7 and assign weight 11 to each point to obtain a $\{4\}$-GDD of type 44^7. The result follows by adding an infinite point to each group. If $v = 312$, apply Construction 4.3 with $m = 77$, $k = 4$, $a = 0$ and $f = 4$. If $v = 336$, take a $RTD[9, 13]$ and remove eight points in one block. Use another parallel class to define a $\{8, 9, 13\}$-GDD of type $8^8 9^6$. Assign weight three to each point and take a flat of size nine to obtain $336 \in B(\{4, 9\})$. If $v = 348$, take a $RTD[9, 9]$ and add 4 infinite points each to the 26 parallel class of block and a point to each group in analogy with the case $v = 300$. Fill in the hole with a $PBD[105, \{4, 9\}]$. If $v = 357$, apply Construction 4.1 with $m = 27$, $t = 11$ and $f = 0$. □

Lemma 4.10 $\{372, 381, 384, 408, 420, 429\} \subset B(\{4, 9\})$.

Proof. If $v = 372$, by adding an exterior point in the $\{0, 8\}$-arc of order 120 embedded in the $PG(2, 16)$ to obtain a $\{8, 9\}$-GDD of type $8^{14} 9^1$. Assigning weight three to each point and take a flat of order nine to obtain $372 \in B(\{4, 9\})$. If $v = 384$, apply Construction 4.3 with $k = 4$, $m = 93$, $a = 1$ and $f = 13$. The required $IPBD$ is obtained by taking a $\{4\}$-IGDD of type $(8,1)^{13}$ [21] and adding a point together with filling in the hole with a block of size 13. If $v = 408$ or $v = 420$, apply Construction 4.3 with $k = 4$, $m = 101$ or $m = 104$, $a = 0$ and $f = 4$. If $v = 429$, apply Construction 4.1 with $m = 32$, $t = 12$ and $f = 9$. □

Lemma 4.11 *If* $r \equiv 0 \pmod 4$, $r \geq 40$ *and* $3(r-4)+9, 3(r-3)+9, 3r+9, 3(r+1)+9 \in B(\{4, 9^*\})$, *then* $12r+33, 12r+36, 12r+45, 12r+48, 12r+57, 12r+60, 12r+69, 12r+72 \in B(\{4,9\})$.

Proof. Apply Construction 4.1 with $m = r$, $t = 8$ and $f = 9$ to obtain $12r + 33 \in B(\{4,9\})$. Take $m = r$, $t = 9$ and $f = 9$ to obtain $12r + 36 \in B(\{4,9\})$. Take $m = r$, $t = 12$ and $f = 8$ to obtain $12r + 45 \in B(\{4,9\})$. Take $m = r + 1$, $t = 9$ and $f = 9$ to obtain $12r + 48 \in B(\{4,9\})$. Take $m = r$, $t = 12$ and $f = 9$ to obtain $12r + 57 \in B(\{4,9\})$. Take $m = r - 4$, $t = 33$ and $f = 9$ to obtain $12r + 60 \in B(\{4,9\})$. Take $m = r + 1$, $t = 16$ and $f = 9$ to obtain $12r + 69 \in B(\{4,9\})$. Take $m = r - 3$, $t = 33$ and $f = 9$ to obtain $12r + 72 \in B(\{4,9\})$. □

The next two lemmas are a variant of Lemma 4.11.

Lemma 4.12 *If* $r \equiv 0 \pmod 4$, $r \geq 56$ *and* $3(r-4)+9, 3r+9, 3(r+4)+9 \in B(\{4,9\})$, *then* $12r+33, 12r+36, 12r+45, 12r+48, 12r+57, 12r+60, 12r+69, 12r+72 \in B(\{4,9\})$.

Proof. In view of the proof of Lemma 4.11, we only need to concern with the case $12t+48, 12t+69$ and $12t+72$. Apply Construction 4.1 with $m = r-8$, $t = 45$ and $f = 9$ to obtain $12r+48 \in B(\{4,9\})$. Take $m = r-4$, $t = 36$ and $f = 9$ to obtain $12r + 69 \in B(\{4,9\})$. Take $m = r - 4$, $t = 37$ and $f = 9$ to obtain $12t + 72 \in B(\{4,9\})$. □

Lemma 4.13 *If* $v \geq 432$ *and* $v \equiv 0, 9 \pmod{12}$, *then* $v \in B(\{4,9\})$.

Proof. From the proof of Lemma 4.11, if we take $r = 32$, we establish the result for $12(32)+48 = 432$ and $12(32)+69 = 453$. If $v = 12(32)+60 = 444$, this is obtained by applying Construction 4.3 with $m = 109$, $k = 4$, $a = 5$ and $f = 13$, the ingredient $IPBD$ is simply obtained by taking a $TD[9,13]$. If $v = 12(32) + 72 = 456$, apply Construction 4.3 with $m = 113$, $k = 4$, $a = 0$ and $f = 4$. If we take $r = 36$, we obtain $12(36) + 33, 12(36) + 36, 12(36) + 45, 12(36) + 48, 12(36) + 57, 12(36) + 69 \in B(\{4,9\})$. For $v = 12(36) + 60 = 492$, apply Construction 4.1 with $m = 35$, $t = 24$ and $f = 0$. For $v = 12(36) + 72 = 504$, this is simply obtained by taking a $\{4\}$-GDD of type 9^{56}. For any $r \equiv 0 \pmod 4$ with $40 \leq r \leq 140$ and $r \neq 48$, it is easy to verify that r satisfies the conditions of at least one of the Lemma 4.11 or Lemma 4.12. Hence, we obtain the conclusion for $v \geq 432$ and $v \leq 1752$ with the possible exceptions in the interval from 609 to 648. When $r = 48$, from the proof of Lemma 4.11, we only need to deal with $12(48)+48 = 624$ and $12(48) + 69 = 645$. If $v = 624$, apply Construction 4.1 with $m = 41$, $t = 41$ and $f = 9$. If $v = 645$, apply Construction 4.1 with $m = 43$, $t = 43$ and $f = 0$. Therefore, we have obtained for any $v \in [432, 1752]$ and $v \equiv 0, 9 \pmod{12}$, we have $v \in B(\{4, 9\})$. The result follows by induction and Lemma 4.11. □

Let $D_{49}= \{12, 21, 24, 48\}$ and
$P_{49} = \{60, 69, 84, 93, 96, 192\}$.

Lemma 4.14 $\{156, 204\} \subset B(4, 9)$.

Proof. For $v = 156$, note that there are 4-GDDs of type 6^5 and $6^4 9^1$ (see[11]). Take a $\{5\}$-GDD of type 4^6 and give all but one of the points weight 6 and give the remaining point weight 9. This produces a $\{4\}$-GDD of type $24^5 27^1$. Adjoin 9 new points to this GDD, making use of the fact that $\{33, 36\} \subset B(4, 9, 9^*)$. The result is a PBD$[156, \{4, 9\}]$. For $v = 204$, it is know that there exists a $\{4\}$-GDD of type 8^7 which contains nine parallel classes (Greig, [8]). Adjoin 9 points to this to obtain a $\{4, 5\}$-GDD of type $8^7 9^1$. Give all points of this GDD weight 3 to produce a 4-GDD of type $24^7 27^1$. Adjoin a further 9 points to this GDD using the fact that $\{33, 36\} \subset B(4, 9, 9^*)$ to produce a PBD$[204, \{4, 9\}]$. □

Lemma 4.15 $12, 21, 24, 48 \notin B(\{4, 9\})$.

Proof. Trivially, $12, 21, 24, 48 \notin B(\{4\})$. So if $12, 21, 24, 48 \in B(\{4, 9\})$, a corresponding PBD must have at least one block of size nine. Applying Result 3.6, we have $v \geq 28$. This proves $12, 21, 24 \notin B(\{4, 9\})$. If $v = 48$, suppose to the contrary, a $PBD[48, \{4, 9\}]$ exists. Let x be a point in the design and r_i be the number of blocks of size i that point x is on. Evidently, $47 = 3r_4 + 8r_9$. This gives $r_9 \equiv 1 \pmod 3$. Hence, every point in on at least a block of size nine. Let b be the number of blocks of size nine. Trivially, $b \geq 6$ holds. Let a_i be the number of points in the design so that it is on i blocks of size nine. We have shown that $a_{3k} = a_{3k+2} = 0$ for all k, a positive integer. It is easy to see that $a_i = 0$ for all $i \geq 7$ since otherwise, there must be at least 48 points in this design. So, we must have $48 = a_1 + a_4$. Also, we know that $9b = a_1 + 4a_4$. This gives $a_4 = 3b - 16$. Consider only the blocks of size 9. In the dual, it forms a packing design with b points and

$3b - 16$ blocks of size 4 with replication number at most 9. The pair packing number for b points is at most $\lfloor \frac{b}{4} \lfloor \frac{b-1}{3} \rfloor \rfloor$. This gives a contradiction. □

Theorem 4.2 *If $v \equiv 0,1,4,9 \pmod{12}$, $v \geq 4$ and $v \notin D_{49} \cup P_{49}$, then $v \in B(\{4,9\})$. Moreover, if $v \in D_{49}$, then $v \notin B(\{4,9\})$.*

Proof. This is a summary of the foregoing. □

5 Closure of three element sets containing 4 and 6 but not 5

In this section, the closures of $\{4,6,7\}$, $\{4,6,8\}$ and $\{4,6,9\}$ are investigated.

We now deal with the closure of $\{4,6,7\}$.

It is easy to check that the necessary conditions are $v \equiv 0,1 \pmod 3$.

Since the necessary conditions for the closure of the set $\{4,6\}$ are also $v \equiv 0,1 \pmod 3$, we eliminate numbers that are in the list of possible exceptions for $B(\{4,6\})$. First of all, we note that if $v \equiv 1 \pmod 3$ and $v \neq 10, 19$, then $v \in B(\{4,7\})$. So, we only need to deal with multiples of 3.

Lemma 5.1 $\{39,51,75\} \subset B(4,6,7)$.

Proof. If $v = 39$, take a $TD[5,8]$ embedded in the $PG(2,8)$ and line-flip a five-line [11].

If $v = 51$, note that $D = \{0,1,3,9,27,81,61,49,56,77\}$ is a difference set for $PG(2,9)$. Let $X = \{1, 2, 3, 6, 8, 9, 11, 13, 14, 16, 18, 19, 20, 22, 24, 26, 27, 29, 31, 33, 34, 35, 37, 39, 40, 42, 48, 52, 53, 54, 55, 57, 58, 60, 61, 65, 66, 67, 68, 71, 72, 73, 74, 78, 79, 80, 81, 83, 85, 87, 89\}$. It is possible to verify that X is an $\{1,4,6,7\}$-arc of order 51 in $PG(2,9)$. Hence, we have $51 \in B(\{4,6,7\})$.

For $v = 75$, note that $21 \in B(4,6,4^*)$. Apply Construction 4.3 with $k = 4$, $m = 17$, and $a = 1$ to get $75 \in B(\{4,6,7\}$. □

Let $D_{467} = \{9,10,12,15,18,19,24,27\}$ and $P_{467} = \{33,45,75,87\}$.

Lemma 5.2 *For any $v \in D_{467}$, $v \notin B(\{4,6,7\})$.*

Proof. Evidently, $D_{467} \subseteq D_{46}$ and it is proved in Theorem 4.1 that if $v \in D_{46}$, then $v \notin B(\{4,6\})$. If $v \in D_{467} \subseteq D_{46}$ and $v \in B(\{4,6,7\})$, then a $PBD[v,\{4,6,7\}]$ must have a block of size seven. Applying Result 3.6, we have $v \geq 22$.

If $v = 24$ and a design exists, by a simple argument, every point is on 1 (mod 3) blocks of size six. If every point is on one block of size six, then it cannot have a block of size seven. Also, if there is a point on at least seven blocks of size six, then this must have at least 43 points. Hence, there must be a point on exactly four blocks of size six. Removing this point yields a $\{4,6,7\}$-GDD of type $5^4 3^1$. Since such a design cannot have a block of size seven, this implies $24 \in B(\{4,6\})$, a contradiction.

If $v = 27$ and a design exists, by a similar argument, there must be a point lying on exactly four blocks of size six. Removing this point yields a $\{4,6,7\}$-GDD of type $5^4 3^2$, but such a design cannot have a block of size seven. This implies $27 \in B(\{4,6\})$, a contradiction. □

This yields the following theorem.

Theorem 5.1 *If* $v \geq 4$, $v \equiv 0,1 \pmod 3$ *and* $v \notin D_{467} \cup P_{467}$, *then* $v \in B(\{4,6,7\})$. *Moreover, if* $v \in D_{467}$, *then* $v \notin B(\{4,6,7\})$.

Proof. Clearly $B(\{4,6\}) \subseteq B(\{4,6,7\})$, so $B(\{4,6,7\}) \subseteq N_{n\geq 0}(0,1 \bmod 3)\backslash\{7,9,10,12,18,19,22,24,27,33,34,39,45,46,51,55,75,87\}$. But $\{22,34,46,55\} \subset B(\{4,7\})$, and by Lemma 5.1, we have $\{39,51\} \subseteq B(\{4,6,7\})$, and trivially 7 is also in this set. The result follows. □

We now deal with the closure of $\{4,6,8\}$.

First of all, we note that by the results for $B(\{4,6\})$ and $B(\{4,8\})$, a simple argument shows that we only have to get a closure result for $v \equiv 2,11 \pmod{12}$. For the other congruence classes, all we need to do is to eliminate some exceptions from the list of $B(\{4,6\})$ and $B(\{4,8\})$ using the extra flexibility gained by permitting all three block sizes.

We start off with some direct constructions from finite projective planes.

The following construction is due to M. Greig [9].

Lemma 5.3 $\{38,44,45,46\} \subset B(\{4,6,8\})$.

Proof. For $v = 38$, note that in [3], it is established that there is a $\{0,1,5,7,9\}$-arc of order 47 in the projective plane of order 8. By deleting a line of size nine from this arc, we obtain $38 \in B(\{4,6,8\})$. For $v = 44$, note that the Denniston arcs in [11] are nested. In a $PG(2,8)$, remove 28 points of the $\{0,4\}$-arc to obtain a $\{5,9\}$-arc of order 45. The result follows by performing a line-flip of a 5-line. □

For $v = 45$, note that $D = \{0,1,3,9,27,81,61,49,56,77\}$ is a difference set for the desarguesian projective plane of order nine. Let X ={1, 2, 3, 6, 7, 8, 9, 10, 11, 16, 18, 20, 21, 22, 24, 27, 29, 30, 31, 33, 34, 37,40, 48, 53, 54, 55, 60, 61, 63, 64, 66, 68, 71, 72, 73, 74, 79, 81, 82, 85, 87, 88, 89, 90}. It is easily verified that X is a $\{1,4,6,8\}$-arc of order 45 in this plane.

For $v = 46$, note that it is shown in [3] that there is a $\{5,7,9\}$-arc of order 55 in $PG(2,8)$. By deleting an exterior line, we obtain a PBD[46, $\{4,6,8\}$]. □

Next, we deal with the cases where v is relatively small. In dealing with the next cases, we often use the fact that if $v \equiv 7,10 \pmod{12}$, and $v \neq 10,19$ then $v \in B(\{4,7^*\})$.

Lemma 5.4 $\{74,83,86,98,119,122,146,155,182,215,266\} \subset B(\{4,6,8\}]$.

$v = 74$: $V = \{0,1\} \times Z_{37}$. under the action $(-,37)$. Then the base blocks are

$$\{(0,0),(0,5),(0,13),(0,19),(1,0),(1,3),(1,4),(1,30)\}$$
$$\{(0,9),(0,12),(0,16),(1,18),(1,24),(1,32)\}$$
$$\{(1,0),(1,2),(1,15),(1,20)\}\ \{(0,0),(1,1),(1,10),(1,26)\}$$
$$\{(0,0),(0,2),(0,28),(1,33)\}$$

Multiply the 2nd components in the last block by 10 and 26 mod 37, to obtain two more additional base blocks.

Proof. $v = 83$: $V = Z_{75} \cup \{\infty_0, \infty_1, \ldots, \infty_7\}$.
Then the base blocks are

$$\{\infty_i, 5j+i, 6+5j+i, 27+5j+i, 13+5j+i, 29+5j+i\},$$
$$i = 0,1,\ldots,4;\ j = 0,1,\ldots,14.$$
$$\{\infty_{5+i}, 3j+i, 55+3j+i, 38+3j+i\}, i = 0,1,2; j = 0,\ldots,24$$
$$\{\infty_0,\ldots,\infty_7\}$$
$$\{0,3,12,36\},\ \{0,15,34,26\},\ \{0,18,28,50\},\ \{0,30,31,35\}.$$

$v = 86$: $V = Z_{85} \cup \{\infty\}$. Then the base blocks are

$$\{0,2,22,26,66,69,74,75\}, \{0,7,30,57\}, \{0,15,46,71\},$$
$$\{\infty,0,17,34,51,68\}.$$

$v = 98$: $V\{0,1\} \times Z_{49}$, under the action $(-,49)$. Then the base blocks are

$$\{(0,y),(0,18y),(0,30y),(1,0)\} \text{ for } y = 1,2,3,7$$
$$\{(0,0),(0,4),(0,22),(0,23),(1,0),(1,24),(1,34),(1,40)\}$$
$$\{(0,24),(0,34),(0,40),(1,7),(1,14),(1,28)\}$$
$$\{(1,0),(1,3),(1,5),(1,41)\}\ \{(1,0),(1,1),(1,18),(1,30)\}$$
$$\{(0,0),(0,41),(1,7),(1,33)\}$$

Multiply the 2nd components in the last block by 18 and 30 mod 49, to obtain two additional base blocks.

For $v = 119$, we use a non-abelian group. Let $V = (Z_3 \times Z_{37}) \cup \{\infty_i, i \in \{14, 28, 23, 24, 28, 30, 31, 33\}$ where the action $Z_3 \times Z_{37}$ is given by $(-, 37)$ and $T(t, u) \rightarrow (t + 1, 10u)$. Then the base blocks are

$$\begin{array}{l}
\{(0,0),(0,1),(1,0),(1,10),(2,0),(2,26)\} \\
\{(0,0),(0,2),(1,3),(2,4)\} \\
\{(0,0),(0,3),(1,5),(2,1)\} \\
\{(0,0),(0,4),(0,9),(1,15)\} \\
\{(0,0),(0,6),(0,18),(1,22)\} \\
\{(0,0),(0,7),(0,20),(1,22)\} \\
\{(0,0),(0,8),(0,23),(1,21)\} \\
\{(0,0),(0,10),(0,21),(1,29)\} \\
\{\infty_i : i \in \{14,18,23,24,28,30,31,33\}\}.
\end{array}$$

$v = 122$: $V = \{0, 1\} \times Z_{61}$, under the action $(-, 61)$. Then the base blocks are

$$\begin{array}{l}
\{(0,y),(0,13y),(0,47y),(1,0) \text{ for } y = 1,18,31 \\
\{(0,0),(1,y),(1,13y),(1,47y)\} \text{ for } y = 28,36 \\
\{(0,0),(0,3),(0,19),(0,39),(1,0),(1,32),(1,40),(1,50)\} \\
\{(0,16),(0,20),(0,25),(1,43),(1,10),(1,8)\} \\
\{(1,0),(1,1),(1,13),(1,47)\} \\
\{(1,0),(1,3),(1,19),(1,39)\} \\
\{(0,0),(0,13),(0,53),(1,17)\} \ \{(0,0),(0,24),(1,33),(1,39)\}
\end{array}$$

Multiply the 2nd components in the last 2 blocks by 13 and 47 mod 61, to obtain four additional base blocks.

$v = 146$: $V = Z_{145} \cup \{\infty\}$. Then the base blocks are

$$\begin{array}{l}
\{0,1,5,17,49,72,80,94\}, \{0,25,60,117\}, \{0,11,20,30\}, \\
\{0,21,27,64\}, \{0,13,47,54\}, \{0,26,62,95\}, \{0,2,40,86\}, \\
\{0,3,18,42\}, \{\infty\ 0,29,58,87,116\}.
\end{array}$$

$v = 155$: $V = (Z_3 \times Z_{49}) \cup \{\infty_i : i \in \{6, 20, 24, 39, 43, 44, 45, 46\}$ where the action on $Z_3 \times Z_{49}$ is $(-, 49)$ and $T(t, u) \rightarrow (t + 1, 30u)$. Then the base blocks are

$$\{(0,0),(0,1),(1,0),(1,30),(2,0),(2,0),(2,18)\}$$
$$\{(0,5),(0,21),(1,3),(1,42),(2,41),(2,35)\}$$
$$\{(0,7),(0,35),(1,19),(1,21),(2,28),(2,42)\}$$
$$\{(0,8),(0,19),(1,44),(1,31),(2,46),(2,48)\}$$
$$\{(0,9),(0,40),(1,25),(1,24),(2,15),(2,34)\}$$
$$\{(0,18),(0,42),(1,1),(1,35),(2,30),(2,21)\}$$
$$\{(0,24),(0,43),(1,34),(1,16),(2,40),(2,39)\}$$
$$\{(0,0),(0,2),(1,3),(2,4)\}$$
$$\{(0,0),(0,3),(1,5),(2,1)\}$$
$$\{(0,0),(0,4),(0,9),(1,13)\}$$
$$\{(0,0),(0,6),(0,13),(0,23)\}$$
$$\{(0,0),(0,8),(0,20),(0,35)\}$$
$$\{\infty_i,\ (0,22i),(1,23i),(2,4i)\} i \in \{6,20,39,43,44,45,46\}$$
$$\{\infty_i :\ i \in \{6,20,24,39,43,44,45,46\}\}.$$

$v = 182$: $V = \{0,1\} \times \boldsymbol{Z}_{91}$ under the action $(-,91)$. Then the base blocks are

$$\{(0,y),(0,16y),(0,74y),(1,0)\} \text{ for } y = 1,7,22,45$$
$$\{(0,0),(1,y),(1,16y),(1,74y)\} \text{ for } y = 3,9,25,33$$
$$\{(0,0),(0,3),(0,40),(0,48),(1,0),(1,34),(1,59),(1,89)\}$$
$$\{(0,34),(0,59),(0,89),(1,50),(1,60),(1,72)\}$$
$$\{(0,50),(0,60),(0,72),(1,13),(1,26),(1,52)\}$$
$$\{(1,4),(1,8),(1,23),(1,37),(1,46),(1,64)\}$$
$$\{(0,0),(0,44),(0,67),(0,71)\}$$
$$\{(0,0),(0,1),(0,7),(1,65)\}$$
$$\{(0,0),(0,13),(1,20),(1,27)\}$$
$$\{(0,0),(0,19),(1,79),(1,80)\}$$

Multiply the second component in each of the last three blocks by 16 and 74 (mod 91) to obtain an additional six base blocks.

$v = 215$: $V = \boldsymbol{Z}_5 \times \boldsymbol{Z}_{43}$. Then the base blocks are

$$\{(0,0),(1,0),(2,9),(2,11),(2,23),(3,1),(3,6),(3,36)\}$$
$$\{(0,0),(1,10),(1,16),(1,17),(2,4),(2,15),(2,24),(3,0)\}$$
$$\{(0,5),(0,8),(0,30),(2,3),(2,18),(2,22)\}$$
$$\{(0,0),(0,10),(1,1),(3,13)\},\ \{(0,0),(1,2),(2,20),(3,24)\}$$

Multiply the last 2 blocks of size 4 by $(1,y)$ for $y = 6$ and 36, to obtain four additional base blocks.

$v = 266$: $V = \{0,1\} \times Z_{133}$ under the action $(-, \bmod 133)$. Then the base blocks are

$\{(0,y),(0,11y),(0,121y),(1,0)\}$ for $y = 1,7,14,17,28,29,30,45,47$
$\{(0,0),(1,y),(1,11y),(1,121y)\}$ for $y = 1,2,16,23,26,29,34,40$
$\{(0,0),(0,36),(0,100),(0,130),(1,0),(1,60),(1,78),(1,128)\}$
$\{(0,18),(0,50),(0,65),(1,27),(1,31),(1,75)\}$
$\{(1,0),(1,y),(1,11y),(1,121y)\}$ for $y = 7,30$
$\{(0,0),(0,1),(0,19),(1,66)\}, \{(0,0),(0,72),(0,113),(1,7)\}$
$\{(0,0),(0,43),(0,128),(1,62)\}, \{(0,0),(0,27),(1,48),(1,85)\}$
$\{(0,0),(1,39),(1,90),(1,125)\}, \{(0,0),(1,73),(1,113),(1,130)\}$

Multiply the second components by the last six blocks by 11 and 121 (mod 133) to obtain an additional twelve base blocks. □

Lemma 5.5 $\{107,134,143,158,164,167,173,179,191,194,203,218\} \subset B(\{4,6,8\}$

Proof. The results follows from Construction 4.2, using the following table.

v	m	a	b	f
107	5	5	0	7
134	7	1	6	0
143	7	4	3	0
158	8	5	2	1
164	9	2	2	7
167	9	5	0	7
173	9	5	1	7
179	9	5	2	7
191	9	5	4	7
194	11	5	0	4
203	11	2	4	4
218	12	7	3	0

This establishes the lemma. □

Lemma 5.6 *If* $v \equiv 11 \pmod{12}$ *and* $v \geq 227$, *then* $v \in B(\{4,6,8\})$.

Proof. Apply Construction 4.2 with $m = 4t+1$ where $t \geq 3$, $f = 7$ and choose $5a + 6b$ so that $a + b \leq 13$ and $5a + 6b + 7 \in \{32,44,56,68,80\}$. □

We now only deal with the case when $v \equiv 2 \pmod{12}$.

To aid in this, we introduce the following construction.

Construction 5.1 *Suppose that there exists a TD*$[8, m]$ *and that* x, y *and* z *are nonnegative integers satisfying* $x + y + z \leq m$. *Let* $w = x + 18y + 6z$. *If* $\{6m, w\} \subset B(\{4, 6, 8\})$, *then* $42m + w \in B(\{4, 6, 8\})$.

Proof. For the RTDs, RGDDs, and GDDs required below, see [11]. From an RTD(6, 7) together with a new point adjoined to the groups, a $\{6, 8\}$-GDD of type $6^7 1^1$ is obtained. From a 3-RGDD of type 6^7, a 4-GDD of type $6^7 18^1$ is obtained by adjoining 18 new points one to each resolution class, together with a new group consisting of the 18 new points. Also there exist 4-GDDs of type 6^7 and 6^8. Let G be a group of TD$[8, m]$. Assign a weight of 6 to every point in the remaining groups of this design and assign a weight of 1 to x points, a weight of 18 to y points and a weight of 6 to z points of G_1, and assign a weight of 0 to the remaining $m - (x + y + z)$ points of G_1. Then apply Wilson's Fundamental Construction to obtain a $\{4, 6, 8\}$-GDD of type $(6m)^7 w^1$. This can be viewed as a PBD$[42m + w, \{4, 6, 8\}]$. □

Lemma 5.7 $\{242, 254, 278, 290, 302, 314, 326, 338, 350, 362, 374, 386, 398, 410\} \subset B(\{4, 6, 8\})$.

Proof. For $v = 242$, start with a TD[7, 17] and give all points weight two. (There exists a 4-GDD of type 2^n, see [11]), to obtain a 4-GDD of type 34^7. Since $38 \in B(\{4, 6, 8, 4^*\})$, we can adjoin four new points to the group of this GDD to get $242 \in B(\{4, 6, 8\})$.

For $v = 290$, note that there exists an RTD[6, 8]. First view this as a $\{6, 8\}$-GDD of type 6^8. In the last group of this TD, give one point weight zero, and all remaining points weight six. It is noted in [11] that there exists a 4-GDD of type 6^n for all $n \geq 5$, so the necessary ingredients exists to produce a 4-GDD of type $36^7 30^1$ from the above weighting. Since 38 and 44 are in $B(\{4, 6, 8, 8^*\})$, we can adjoint eight new points to obtain a PBD$[290, \{4, 6, 8\}]$.

For $v = 302$, note that there exists a $\{6, 8\}$-GDD of type $7^7 2^1$ (see [11]). Adjoin a point to each group to obtain a PBD$[52, \{6, 8, 3^*\}]$. Now apply Construction 4.3 with $k = 6, f = 3$ and $a = 1$ to obtain a PBD$[302, \{6, 8\}]$. For $N \in \{326, 350, 386, 410\}$, we apply Construction 5.1 in accordance with the following table.

v	m	x	y	z	
326	7	32	2	1	2
350	7	56	2	3	0
386	9	8	2	0	1
410	9	32	2	1	2

For the remaining values we use Construction 4.2 in accordance with the following table.

v	m	a	b	f
254	15	5	0	4
278	16	5	2	1
314	19	2	3	1
338	20	5	2	1
362	17	2	15	7
374	23	5	0	4
398	19	2	17	1

This establish the lemma. □

Lemma 5.8 *If* $v \equiv 2 \pmod{12}$, $v \geq 422$ *then* $v \in B(\{4,6,8\})$.

Proof. It is easy to see that for any $v \equiv 2 \pmod 4$, $v \geq 26$ then $3v+1 \in B(\{4,6,8\})$. For $v \equiv 2 \pmod 4$, $v \geq 26$ and $v \neq 40$, apply Construction 4.2 with $f = 1$ and a and b chosen so that $5a+6b+1 \in \{32,44,56,68,80\}$. This takes care all $v \geq 422$, $v \neq 662,674,686,698,710$. For the remaining values, we use construction 4.2 in accordance with the following table.

n	m	a	b	f
662	37	1	17	0
674	43	3	2	1
686	38	4	16	9
698	44	4	3	0
710	38	4	20	0

This establishes the lemma. □

Let D_{468}={5, 7, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22, 23, 24, 26, 27} and P_{468}={33, 34, 35, 39, 41, 47, 50, 51, 53, 59, 62, 65, 71, 75, 77, 87, 89, 95, 101, 110, 131, 161, 170}.

Lemma 5.9 *If $v \in D_{468}$, then $v \notin B(\{4,6,8\})$.*

Proof. If a $PBD[v,\{4,6,8\}]$ which contains a block of size 8 exists, then by Result 3.6, $v \geq 25$. Hence, if $v \leq 24$, $v \in B(\{4,6,8\})$ implies $v \in B(\{4,6\})$ and by the results for $B(\{4,6\})$, this eliminates $v \in \{5,7,9,10,11,12,14,15,17,18,19,20,23,24\}$. If $v = 26$, it is easy to see a $PBD[26,\{4,6,8\}]$ must have a block of size six and a block of size eight. Applying Result 3.6, we have $v \geq 27$, a contradiction. If $v = 27$ and a $PBD[27,\{4,6,8\}]$ exists, a simple counting argument reveals that either it has zero blocks of size eight or at least three blocks of size eight. In the first case, this implies $27 \in B(\{4,6\})$, a contradiction. In the second case, apply Result 3.6 to obtain $v \geq 29$, a contradiction. □

Theorem 5.2 *If $v \geq 4$ and $v \notin D_{468} \cup P_{468}$, then $v \in B(\{4,6,8\})$. Moreover, if $v \in D_{468}$, then $v \notin B(\{4,6,8\})$.*

Proof. For $v \not\equiv 2$ or $11 \pmod{12}$, the result follows from the results for $B(\{4,6\})$ and $B(\{4,8\})$ with the exceptions of $v \in \{44,45,46,164,173\}$, which have been treated in this section.

For $v \equiv 11 \pmod{12}$, the result follows from Lemma 5.6, with the exception of $v \in \{83,107,119,143,155,167,179,191,203,215\}$, all of which have been treated in this section.

For $v \equiv 2 \pmod{12}$, the results follows from Lemma 5.8 with the exception of $v \in \{38,86,134,146,158,182,206,218,230,254,278,302,314,326,338,350,362,374,386,398,410\}$, all of which, with the exception of $v \in \{206,230\}$ have been treated in this section.

For $v = 206$, apply Construction 4.1 with $m = 14$, $t = 10$ and $f = 8$.

For $v = 230$, apply Construction 4.1 with $m = 16$, $t = 10$ and $f = 8$. □

We now consider the closure of $\{4,6,9\}$.

It is very easy to check that the necessary conditions for $v \in B(\{4,6,9\})$ are $v \equiv 0$ or $1 \pmod 3$. As in the situation of the closure of $\{4,6,7\}$, we only eliminate possible exceptions from the closure of $\{4,6\}$ and $\{4,9\}$.

Lemma 5.10 $\{39,46,51\} \subset B(4,6,9)$.

Proof. For $v = 39$, there exists a 4-GDD of type $6^5 9^1$ (see [11]). For $v = 46$, apply Greig's $q-2$ construction with $q-9$ and $x = 5$ (see [11]). For $v = 51$, there exists a 4-GDD of type $9^5 6^1$ (see [11]). □

Let $D_{469} = \{7,10,12,15,18,19,22,24,27\}$ and $P_{469} = \{34,75,87\}$.

Lemma 5.11 *If $v \in D_{469}$, then $v \notin B(\{4,6,9\})$.*

Proof. Note that $D_{469} \subseteq D_{46}$, by Theorem 4.1, we can assume if $v \in D_{469}$ and a $PBD[v, \{4,6,9\}]$ exists, then it must have a block of size nine. Apply Result 3.6, we have $v \geq 28$, a contradiction. □

Theorem 5.3 *If $v \equiv 0,1 \pmod 3$, $v \geq 4$ and $v \notin D_{469} \cup P_{469}$, then $v \in B(\{4,6,9\})$. Moreover, if $v \in D_{469}$, then $v \notin B(\{4,6,9\})$.*

Proof. This is a summary of the foregoing. □

6 Closure of three element sets containing 4 and 7 but not 5 or 6

In this section, we deal with the closure of the set $\{4,7,8\}$ and $\{4,7,9\}$.

Construction 6.1 *Let S be a PBD-closed set which contains $\{7,8\}$. Suppose that there exists a $TD[8,m]$. Let a be a non-negative integer satisfying $a \leq m$. If $m \in S$, then $7m + a \in S_a$.*

Proof. Truncate one group to a points. □

Construction 6.2 *Suppose there exists a $TD[9,m]$, and u, v, x, and y are non-negative integers satisfying $x + y \leq m$ and $u + v \leq m$. Suppose further that $\{3m+a,\ 3u+7v+a\} \epsilon S_a$. Then $21m+3u+7v+3x+7y+a \in S_{3x+7y+a}$.*

Proof. In [2], it is shown that a resolvable $PBD[36, \{4,8\}]$ exists. It is easy to check that every point of the design lies on exactly two of the 8-lines. By deleting one point of the design, one obtains a $\{4,8\}$-GDD of type $3^7 7^2$. As noted in [11], there exists $\{4\}$-GDDs of type 3^8 and 3^9. A $\{4,7\}$-GDD of type 3^7 can be obtained by removing a point not on the block of size 7 in a $PBD[22, \{4, 7^*\}]$. Take a $TD[4,7]$ and add a point to each group, then delete another point to obtain a $\{4,7,8\}$-GDD of type $3^7 7^1$. Also, take a $TD[4,8]$ and delete a point to obtain a $\{4,8\}$-GDD of type $3^8 7^1$. In the $TD[9,m]$, assign weight three to every point in the first seven groups. Then one can assign weights of $0, 3$ or 7 to points in the last two groups. The result follows by Wilson's Fundamental Construction and filling in holes. □

We first consider the closure of $\{4,7,8\}$.

We first deal with small cases.

Lemma 6.1 $\{50,53,60,63,95,98,99,107,119,135,155,170,171,179,182,183,191\} \subseteq B(\{4,7,8\})$.

Proof. For v= 50, 53, 63, 95, 98, 99, 119, 179, 182, 183 and 191, apply Construction 6.1 with $(m,a) = (7,1)$, $(7,4)$, $(8,7)$, $(13,4)$ $(13,7)$, $(13,8)$, $(16,7)$, $(25,4)$, $(25,7)$, $(25,8)$ and $(25,16)$ respectively.

If $v = 60$, remove four points from $AG(2,8)$.

For $v = 107$ and $v = 135$, apply Construction 4.3 taking $k = 4$, $m = 25$, $f = 4$ and $a = 1$, noting that $29 \in B(\{4,8,4^*\})$ and $k = 4$, $m = 32$, $f = 4$ and $a = 1$, using $36 \in B(\{4,8,4^*\})$, respectively.

If $v = 155$, take a $\{4,7\}$-GDD of type 4^7 and 7^7 together with a $PBD[22,\{4,7\}]$ to obtain a $\{4,7\}$-GDD of type 22^7. The result follows by adding an infinite point.

For $v = 170$, note that there exist $\{4\}$-GDDs of type 5^5 and $3^5 6^1$. By deleting a point from a TD(6,8), a $\{5,6\}$-GDD of type $8^5 7^1$ can be obtained, and this can be inflated by the above ingredients, by assigning weight 3 to all points in the groups of size 8 and weight 6 to the remaining points, to obtain a $\{4\}$-GDD of type $24^5 42^1$. Since 32 and 50 belong to $B(\{4,7,8,8^*\})$, we can "fill in holes" to obtain $170 \in B(\{4,7,8\})$. □

Lemma 6.2 $\{86,122,134,146,147,158,194\} \subseteq B(\{4,7,8\})$.

$v = 86$: $V = \{0,1\} \times Z_{43}$ under the action $(-, \bmod 43)$. The base blocks are

$$\{(0,0),(0,1),(0,6),(0,36),(1,0),(1,9),(1,11),(1,23)\}$$
$$\{(0,0),(0,2),(0,12),(0,29),(1,26),(1,27),(1,33)\}$$
$$\{(0,0),(0,4),(0,15),(0,24)\}$$
$$\{(0,0),(1,2),(1,12),(1,29)\},\{(1,0),(1,5),(1,8),(1,30)\}$$
$$\{(0,0),(0,3),(1,35),(1,39)\}$$

Multiply the second components in the last block by 6 and 36 (mod 43) to obtain two additional base blocks.

$v = 122$: $V = \{0,1\} \times Z_{61}$ under the action $(-, \bmod 61)$

$$\{(0,y),(0,13y),(0,47y),(1,0)\} \text{ for } y = 1,6$$
$$\{(0,0),(1,y),(1,13y),(1,47y)\} \text{ for } y = 8,12$$
$$\{(0,0),(0\,2),(0,26),(0,33),(1,0),(1,11),(1,21),(1,29)\}$$
$$\{(0,0),(0,9),(0,56),(0,57),(1,1),(1,13),(1,47)\}$$
$$\{(1,0),(1,y),(1,13y),(1,47y)\} \text{ for } y = 16,31$$
$$\{(0,0),(0,19),(0,44),(1,16)\},\{(0,0),(0,18),(1,7),(1,54)\}$$

Multiply the second components in the last two blocks by 13 and 47 (mod 61) to obtain four additional base blocks.

Proof. $v = 134$: Let $V = \{0,1\} \times \boldsymbol{Z}_{67}$ under the action $(-, 67)$. Then the base blocks are

$$\{(0,0),(0,1),(0,29),(0,37),(1,0),(1,4),(1,14),(1,49)\}$$
$$\{(0,9),(0,60),(0,65),(1,3),(1,20),(1,44)\}$$
$$\{(0,10),(0,22),(0,35),(1,8),(1,28),(1,31)\}$$
$$\{(1,25),(1,27),(1,46),(1,54),(1,55),(1,61)\}$$
$$\{(0,0),(0,6),(0,21),(0,40)\}$$
$$\{(0,0),(0,2),(0,20),(1,45)\}$$
$$\{(0,0),(0,24),(1,19),(1,31)\}$$
$$\{(0,0),(0,22),(1,23),(1,39)\}.$$

Multiply the 2nd components in the last 3 blocks by 29 and 37 (mod 67) to obtain six additional base blocks.

$v = 146$: $V = \{0,1\} \times \boldsymbol{Z}_{73}$ under the action $(-, \text{mod } 73)$. Then the base blocks are

$$\{(0,y),(0,8y),(0,64y),(1,0)\} \text{ for } y = 1,2,5,6$$
$$\{(0,0),(1,y),(1,8y),(1,64y)\} \text{ for } y = 1,4,7,12$$
$$\{(0,0),(0,4),(0,32),(0,37),(1,0),(1,43),(1,51),(1,52)\}$$
$$\{(0,0),(0,43),(0,51),(0,52),(1,5),(1,28),(1,40)\}$$
$$\{(1,0),(1,y),(1,8y),(1,64y)\} \text{ for } y = 13,14$$
$$\{(0,0),(0,15),(0,70),(1,31)\}, \{(0,0),(0,19),(1,22),(1,63)\}$$

Multiply the second components in the last two blocks by 8 and 64 (mod 73) to obtain an additional four base blocks.

$v = 147$: Let $V = \boldsymbol{Z}_3 \times \boldsymbol{Z}_{49}$. Then the base blocks are

$B_1 = \{(0,0),(0,7),(0,14),(0,21),(0,28),(0,35),(0,42)\}$

$B_2 = \{(0,0),(1,0),(1,1),(1,18),(1,30),(2,5),(2,3),(2,31)\}$

$B_3 = \{(0,0),(0,3),(1,29),(2,43)\}$

$B_4 = \{(0,0),(0,24),(1,17),(2,30)\}$.

Multiply B_3 and B_4 by $(1,30)$ and $(1,18)$ to obtain 4 additional base blocks.

$v = 158$: $V = \{0,1\} \times \boldsymbol{Z}_{79}$ under the action $(-, \text{mod } 79)$. The base blocks are

$$\{(0,y),(0,23y),(0,55y),(1,0)\} \text{ for } y = 1,3,4,11,25$$
$$\{(0,0),(1,y),(1,23y),(1,55y)\} \text{ for } y = 3,9,11,12,18$$
$$\{(0,0),(0,2),(0,31),(0,46),(1,0),(1,22),(1,25),(1,32)\}$$
$$\{(0,0),(0,12),(0,28),(0,39),(1,41),(1,43),(1,74)\}$$
$$\{(1,0),(1,y),(1,23y),(1,55y)\} \text{ for } y = 15,34$$
$$\{(0,0),(0,23),(0,65),(1,37)\}, \{(0,0),(0,26),(0,34),(1,64)\}$$

Multiply the second components in the last two blocks by 23 and 55 (mod 79) to obtain an additional four base blocks.

$v = 171$: $V = Z_{171}$. The base blocks are

$$\{0,1,5,7,35,49,57,74\}, \{0,12,75,84,94,145,160\}$$
$$\{0,3,46,138\}, \{0,18,47,106\}$$

Multiply the last two blocks of size 4 by 7 and 49 (mod 171) to obtain four additional base blocks.

$v = 194$: $V = \{0,1\} \times Z_{97}$ under the action $(-, \text{mod } 97)$. The base blocks are

$$\{(0,y),(0,35y),(0,61y),(1,0)\} \text{ for } y = 1,2,4,5,7$$
$$\{(0,0),(1,y),(1,35y),(1,61y)\} \text{ for } y = 1,4,6,11,12$$
$$\{(0,0),(0,6),(0,16),(0,75),(1,0),(1,30),(1,80),(1,84)\}$$
$$\{(0,0),(0,49),(0,66),(0,79),(1,7),(1,39),(1,51)\}$$
$$\{(1,0),(1,y),(1,35y),(1,61y)\} \text{ for } y = 23,33$$
$$\{(0,0),(0,20),(0,55),(1,37)\}, \{(0,0),(0,11),(0,83),(1,42)\}$$
$$\{(0,0),(1,3),(1,65),(1,73)\}, \{(0,0),(0,43),(1,23),(1,71)\}$$

Multiply the second component of the last four blocks by 35 and 61 (mod 97) to obtain an additional eight base blocks. □

The following observation is very important.

Let A_n be the set of integers $\{0,3,6,\ldots,3n\}$. Then it is easy to see that every integer in the set {0, 3, 6, 12, 15, 21, 24, 27, 28, 30, 31, 33, 35, 36, 56, 59, 62} can be written in the form $3a+7b$ where $0 \le a,b$ and $a+b \le 8$. Every integer satisfying $v \equiv 0 \pmod 3$ and $v \le 57$ can be written in the form $3a+7b$ where $0 \le a,b$ and $a+b \le 11$. Every integer $v \equiv 0 \pmod 3$ and $v \le 75$ can be written in the form $3a+7b$ where $0 \le a,b$ and $a+b \le 13$. Every integer $v \equiv 0 \pmod 3$ and $v \le 117$ can be written in the form $3a+7b$ where $0 \le a,b$ and $a+b \le 19$. Finally, every integer $v \equiv 0 \pmod 3$ and $v \le 159$ can be written in the form $3a+7b$ where $0 \le a,b$ and $a+b \le 25$.

We now consider the case when $v \equiv 0 \pmod 3$.

Lemma 6.3 *If* $v \equiv 0 \pmod 3$ *and* $v \ge 204$, *then* $v \in B(\{4,7,8\})$.

Proof. We apply Construction 6.2. Take $m = 8$, $7u+3v = 35$ and $3x+7y \in \{0,3,6,12,15,21,24,27,30,33,36\}$ with a flat of size one to obtain $v \in B(\{4,7,8\})$ for $204 \le v \le 240$ and $v \ne 213, 222$. Note that $213 \in B(\{4,8\})$ and $222 \in B(\{4,7,8\})$ by taking $7u+3v = 31$ and $3x+7y = 18$ with a flat of size 4. Take $m = 9$, $7u+3v = 35, 56, 59$ and $3x+7y \in \{0,3,6,12,15,21,24,27,30,33,36\}$ with a flat of size one to obtain $v \in$

$B(\{4,7,8\})$ for $225 \le v \le 285$ with the exception of $v = 234$ and $v = 243$. When $v = 243$, take $7u + 3v = 31$ or and $3x + 7y = 18$ with a flat of size four. Further $234 \in B(\{4,8\})$. Take $m = 11$, $7u + 3v = 35, 56, 59, 62$ and $3x + 7y \in A_{19}$ with a flat of size one to obtain $v \in B(\{4,7,8\})$ for $288 \le v \le 329$. Take $m = 13$, $7u+3v = 35, 56, 59, 62$ and $3x+7y \in A_{25}$ with a flat of size one to obtain $v \in B(\{4,7,8\})$ for $330 \le v \le 411$. If $m = 16$, we can similarly obtain $v \in B(\{4,7,8\})$ for $393 \le v \le 474$. If $m \ge 19$ and there exists a $TD[9,m]$, $7u + 3v = 35, 56, 59, 62, 71, 83, 98$ and $7x + 3y \in A_{39}$ to obtain $v \in B(\{4,7,8\})$ for $21m+35+22 \le v \le 21m+98+117$. Since there is a $TD[9,m]$ for one of every six consecutive integers (See [1]), and we have produced a full interval of size 158. Hence, we can apply the construction recursively to obtain the result. □

We now consider the case when $v \equiv 2 \pmod 3$.

Lemma 6.4 *If* $v \ge 197$ *and* $v \equiv 2 \pmod 3$, *then* $v \in B(\{4,7,8\})$.

Proof. We apply Construction 6.2. If we take $7v + 3u = 28, 31$, $7x + 3y \in A_{f_m}$ for appropriate m, we obtain $v \in B(\{4,7,8\})$ for $21m + 28 + 1 \le v \le 21m + 31 + 3f_m + 1$. Apply with $(m, f_m) = (8,12), (9,12), (11,19), (13,25), (19,39), (25,53)$ and $(k, 53)$ for all k such that $k \ge 25$ and there exists a $TD[9,k]$. It is a simple matter to check that it gives the desired result. □

Lemma 6.5 *If* $v \in \{71, 198\}$, *then* $v \in B(\{4,7,8\})$.

Proof. For $v = 71$, delete 5 points from the groups of a TD[8, 9] to obtain a $\{7,8\}$-GDD of type $9^7 4^1$. Now consider PG(3, 2) as a PBD$[13, \{4, 4^*\}]$ and use this in a singular direct product (see [[11], Theorem 2.3]) to fill in the holes.

For $v = 198$, we need as ingredients $\{4,7,8\}$-GDDs of types $4^6 1^1, 4^7 1^1, 4^7$ and 4^8. The latter two are readily obtained from transversal designs. A $\{4\}$-GDD of type $4^6 1^1$ by adjoining a new point to the groups of a resolvable $\{4\}$-GDD of type 3^8 (such a design exists, see [11]). A $\{4\}$-GDD of type $4^7 1^1$ is obtainable by adjoining a new point to the group of an RTD[4, 7]. Now take a TD(8, 7). In the first six groups, give all points weight 4. In the second last group, give two points weight one and five points weight zero. In the last group give two points weight 4 and the remaining points weight zero. By using the above input designs, this produces a $\{4,7,8\}$-GDD of type $28^6 22^1 8^1$. Since $\{8, 28, 22\} \subset B(\{4,7,8\})$, the result follows. □

Let $D_{478} = \{5,6,9,10,11,12,14,15,17,18,19,20,21,23,24,26,27,30,$ $33,35,38,39,41,42,44\}$ and $P_{478} = \{45,47,48,51,54,59,62,65,66,69,74,$ $75,77,78,83,87,89,90,101,102,110,111,114,123,126,131,138,143,150,$ $159,161,162,164,167,173,174,186,195\}$.

Lemma 6.6 *If $v \in D_{478}$, then $v \notin B(\{4,7,8\})$.*

Proof. If $v = 10$ or 19, it is known that $v \notin B(\{4,7\})$ by Result 3.3. So if there exists a $PBD[v,\{4,7,8\}]$ then it must have a block of size eight. Also, if $v \not\equiv 1 \pmod 3$, then a $PBD[v,\{4,7,8\}]$ must contain a block of size eight. Apply Result 3.6 to obtain $v \geq 25$. If $v = 26$ or $v = 27$, by counting pairs, we can assert that if a $PBD[v,\{4,7,8\}]$ exists then it must have a block of size seven. Apply Result 3.6 to get $v \geq 28$, a contradiction. If $v = 30$ or $v = 33$ and a $PBD[v,\{4,7,8\}]$ exists, then every point must be on $2 \pmod 3$ blocks of size eight. Hence, there must be a point on at least five blocks of size eight, which is impossible. If $v = 35$ and a $PBD[35,\{4,7,8\}]$ exists, then every point must be on $1 \pmod 3$ blocks of size eight. Hence, there must be a point which lies on exactly four blocks of size eight. Removing this point yields a $\{4,7,8\}$-GDD of type either $3^2 7^4$ or $6^1 7^4$. In either case, there are at most six groups, so all blocks must be of size four. Calculating the number of blocks of size 4 through a point on a group of size either three or six yields a fractional result, which is impossible. The case of $v = 38$ is treated similarly. If a PBD$[39,\{4,7,8\}]$ exists, then every point must be on $2 \pmod 3$ blocks of size eight. As above, there must be a point on five blocks of size eight. By removing a point, we obtain a $\{4,7,8\}$-GDD of type $3^1 7^5$. But since the number of groups is six, all remaining blocks must be of size 4, which is again impossible. If a PBD$[41,\{4,7,8\}]$ exists, every point must be on $1 \pmod 3$ blocks of size eight. In particular, every point must be on either one or four blocks of size eight. Let b_i be the number of points on i blocks of size eight and b be the number of blocks of size eight in the PBD. We must have $b_4 = \frac{8b-41}{3}$. Dually, this yields a packing of b points in b_4 blocks of size 4. This is impossible for $b \in \{7,10,13,16,20\}$ (See [19]). Since every point is on at most four blocks of size eight, it is easily seen that $b \leq 20$. Hence, $41 \notin B(\{4,7,8\})$. If $v = 42$ and a $PBD[42,\{4,7,8\}]$ exists, as in the case when $v = 39$, there must be a point on five blocks of size eight. By removing this point, we obtain a $\{4,7,8\}$-GDD of type $7^5 3^2$ or $7^5 6^1$. In either case, there must be a block of size eight in the design. This is impossible since there are at most seven groups. If $v = 44$, we proceed as in the case when $v = 41$ and we are required to pack $b_4 = \frac{8b-44}{3}$ blocks of size four on b points. This is impossible for $b \in \{7, 10, 13, 16, 20\}$ but $b \le 22$ by a simple counting argument. We conclude that $44 \notin B(\{4, 7, 8\})$. □

Theorem 6.1 *If $v \ge 4$ and $v \notin D_{478} \cup P_{478}$, then $v \in B(\{4, 7, 8\})$. Moreover, if $v \in D_{478}$, then $v \notin B(\{4, 7, 8\})$.*

Proof. For $v \equiv 1 \pmod 3$, the lemma follows from the result for $B(\{4, 7\})$.

For $v \equiv 0 \pmod 3$, if $v \ge 204$, the result follows from Lemma 6.3. Apart from those cases handled by the result for $B(\{4, 8\})$, we have $v \in \{60, 63, 99, 147, 183\}$ to consider. These values were treated in this section.

For $v \equiv 2 \pmod 3$, if $v \ge 197$, the result follows from Lemma 6.4. Apart from those cases handled by the result for $B(\{4, 8\})$, we have $v \in \{50, 53, 71, 95, 98, 119, 134, 155, 179, 182, 191\}$ to consider. All of these values were treated in this section. □

We now consider the closure of $\{4, 7, 9\}$.

It is easy to see that the necessary conditions are $v \equiv 0, 1 \pmod 3$. In view of the result of $\{4, 7\}$ and $\{4, 9\}$, we only need to establish the closure result for $v \equiv 3, 6 \pmod{12}$.

We need constructions.

Construction 6.3 *Let S be a PBD-closed set containing $\{4, 9\}$. Suppose there exists a $TD[10, m]$. Let a and b be integers satisfying $0 \le a, b \le m$. If $3a + 1, 8b + 1, 3m + 1 \in S$, then $24m + 3a + 8b + 1 \in S$.*

Proof. As noted in Construction 4.1, there exist a $\{4\}$-GDD of type 3^8 and 3^9. By adjoining a new point to the groups of a $TD[4, 8]$, then deleting another point, a $\{4, 9\}$-GDD of type $3^8 8^1$ is obtained. Also, adjoining eight points to a $KTS(27)$, a $\{4\}$-GDD of type $3^9 8^1$ is obtained. In a $TD[10, m]$, assign weight three to every point in the first eight groups, weight zero or eight to each point in the ninth group and weight zero or eight to every point in the last group. By Wilson's Fundamental Construction, we obtain $24m + 3a + 8b + 1 \in S$. □

Construction 6.4 *Let S be a PBD-closed set which contains $\{4\}$. Suppose that there exists a $TD[6, m]$. Let a be an integer satisfying $0 \le a \le m$. If $3m + f \in S_f$, then $15m + 6a + f \in S_{6a+f}$.*

Proof. As noted in Construction 4.1, there is a $\{4\}$-GDD of type 3^5. By adding six infinite points to $KTS(15)$, we obtain a $\{4\}$-GDD of type $3^5 6^1$. Take a $TD[6, m]$, assign weight three to every point in the first five groups and weight zero or six to each point in the last group. The result follows by Wilson's Fundamental Construction and Singular Direct Product. □

Construction 6.5 *Let n be an integer satisfying $n \geq 4$, $n \neq 6$. Then $9n \in B(\{4,7,9\})$.*

Proof. In view of the result for $B(\{4\})$ and $B(\{4,7^*\})$, there exists a $\{4,7\}$-GDD of type 3^n for all $n \geq 4$, $n \neq 6$.

By giving all points weight three, we obtain a $\{4,7\}$-GDD of type 9^n, so $9^n \in B(\{4,7,9\})$ for all $n \geq 4$, $n \neq 6$. □

Corollary. If $n \geq 4$, then $9^n \in B(\{4,7,9\})$.

Proof. We need only consider $n = 6$. But there exists a TD[4,9], which disposes of this case. □

Clearly if $v \equiv 0 \pmod 3$, then a $PBD[v,\{4,7,9\}]$ must contain a block of size nine. Since we have 4 and $7 \in B(\{4,7,9\})$, for every $m \neq 3,6$, we have $3m+1 \in S$. We use the notation $[a,b]_3$ to represent the set of integers from a to b which are multiples of 3.

We first deal with small values of v.

Lemma 6.7 $\{123,192,219,222\} \subseteq B(\{4,7,9\})$.

Proof. If $v = 123$, apply Construction 4.3 with $m = 29$, $k = 4$, $a = 1$ and $f = 4$. If $v = 192$, apply Construction 6.3 with $m = 9$, $a = 8$ and $f = 9$. If $v = 219$, apply Construction 6.3 with $k = 4$, $m = 53$, $f = 4$, and $a = 1$, using the fact that $57 \in B(\{4,9,4^*\})$. If $v = 222$, apply Construction 4.3 with $k = 4$, $m = 50$, $a = 5$ and $f = 7$. □

Lemma 6.8 *If $v \equiv 0 \pmod 3$ and $v \geq 225$, then $v \in B(\{4,7,9\})$.*

Proof. If $v \equiv 0$ mod 9, use Construction 6.5. We use a specialization of Construction 6.3. Suppose that there exists a TD$[10,m]$ and r is an integer satisfying $0 \leq a \leq m$ and $a \neq 3$ or 6. Also let t be an integer such that $3t+1 \leq m$. Now r satisfies the conditions for lying in $B(\{4,7\})$, and since $8(3t+1)+1 = 24t+9$, then $8(3t+1)+1 \in B(\{4,9\})$ by Lemma 4.6. Further, since either $m = 0$ or 1, or $m \geq 9$, then $3m+1 \in B(\{4,7\})$, so $24m+3a+8(3t+1)+1 = 24m+3a+8t+9 \in B(\{4,7,9\})$. For $m = 9$, the possible values of a are $\{0,1,2,4,5,7,8,9\}$ and the admissible values of t are $\{0,1,2\}$, so the admissible values of $3a+24t+9$ are $[9,84]_3 \backslash \{18,27,42,51,66,75\}$ so $[225,300]_3 \backslash \{234,243,258,267,282,291\} \subseteq B(\{4,7,9\})$. Similarly, taking $m = 11$, we obtain $[273,378]_3 \backslash \{282,291,315,339,363\} \subseteq B(\{4,7,9\})$. For $m = 13$, we obtain $[321,456]_3 \backslash \{330,339,363,387,411,435\} \subseteq B(\{4,7,9\})$.

For $m = 16$, we obtain $[393, 537]_3 \backslash \{402, 411\} \subseteq B(4, 7, 9)$. For $m \geq 17$, it is easily verified that if there exists a TD$[10, m]$, then $[24m + 30, 24m + 3m + 8(m-2) + 1]_3 \subseteq B(\{4, 7, 9\})$. Since it is known that a TD$[10, m]$ exists for some m in any set of six consecutive integers the least of which is 9, a straightforward induction establishes that if $v \geq 414$ and $v \equiv 0 \pmod 3$ then $v \in B(\{4, 7, 9\})$. To complete this lemma, we need to consider the cases of $v \in \{258, 267, 282, 291, 339, 363, 411\}$. If $v = 258$, we remove points from a $TD[10, 11]$ to obtain a $\{7, 8, 9, 10\}$-GDD of type $11^7 3^3$ and assign weight three to each point to obtain $258 \in B(\{4, 7, 9\})$. If $v = 267$, take a $TD[11, 11]$ and remove points to obtain a $\{7, 8, 9, 10, 11\}$-GDD of type $11^7 3^4$ and assign weight three to each point to obtain $267 \in B(\{4, 7, 9\})$. If $v = 282$, we can remove points from $TD[10, 11]$ to obtain a $\{7, 8, 9, 10\}$-GDD of type $11^8 3^2$ and assign weight three to each point to obtain $282 \in B(\{4, 7, 9\})$. If $v = 291$, we can similarly obtain a $\{7, 8, 9, 10, 11\}$-GDD of type $11^8 3^3$ and gives weight three to each point. If $v = 339$, we can obtain a $\{10, 11\}$-GDD of type $11^{10} 9^1$ and gives weight three to each point. If $v = 363$, take a $TD[11, 11]$ and gives weight three to every point to obtain $363 \in B(\{4, 7, 9\})$. If $v = 411$, apply Construction 4.3 with $k = 4$, $m = 92$, $a = 7$ and $f = 13$ to get $411 \in B(\{4, 7, 9\})$. □

Let $D_{479} = \{6, 10, 12, 15, 18, 19, 21, 24, 27, 30, 39, 42, 48\}$ and $P_{479} = \{51, 54, 60, 66, 69, 75, 78, 84, 87, 93, 96, 102, 111, 114, 138, 147, 150, 159, 174, 183, 186, 195, 210\}$.

Lemma 6.9 *If $v \in D_{479}$, then $v \notin B(\{4, 7, 9\})$.*

Proof. Recall that $10, 19 \notin B(\{4, 7\})$. Also if $v \in B(\{4, 7\})$, then $v \equiv 1 \pmod 3$. Therefore we can conclude that if $v \in D_{479}$ and a PBD $[v, \{4, 7, 9\}]$ exists then it must have a block of size nine. Applying Result 3.6, we obtain $v \geq 28$. If $v = 30$, every point must be on $1 \pmod 3$ blocks of size nine. Also there must be a point on at least four blocks of size nine but this is impossible since there would be too many points in this structure. If $v = 39$, again every point must be on $1 \pmod 3$ blocks of size nine. It is easy to see that every point is on either one or four blocks of size nine, otherwise there would be too many points. Also, there must be a point which is on four blocks of size nine. By removing the point, we obtain a $\{4, 7, 9\}$-GDD of type $8^4 3^2$ or $8^4 6^4$. In either case, a point on the short group must be on a block of size nine by a simple counting argument. However, this is impossible since there are at most eight groups. If $v = 42$, we can apply a similar argument as in the case when $v = 39$ to obtain a contradiction. If $v = 48$, the argument is analogous that that of proving $48 \notin B(\{4,9\})$ and is thus omitted. □

Theorem 6.2 *If $v \geq 4$, $v \equiv 0,1 \pmod 3$ and $v \notin D_{479} \cup P_{479}$, then $v \in B(\{4,7,9\})$. Moreover, if $v \in D_{479}$, then $v \notin B(\{4,7,9\})$.*

Proof. For $v \equiv 1 \pmod 3$, the lemma follows for the result for $B(\{4,7\})$.

For $v \equiv 0 \pmod 3$, if $v \geq 225$, the result follows from Lemma 6.7. Apart from those cases which follows from the result for $B(\{4,9\})$ and Construction 6.5, we have $v \in \{123, 186, 192, 202, 222\}$. But these cases were treated in this section. □

7 Closure of $\{4,8,9\}$

In this section, we consider the closure of $\{4,8,9\}$. It is very easy to show that the necessary conditions for $v \in B(\{4,8,9\})$ are $v \equiv 0,1 \pmod 4$. In view of the results on $B(\{4,8\})$ and $B(\{4,9\})$, not much need to be done.

Lemma 7.1 $\{65, 89\} \subset B(\{4,8,9\})$.

Proof. For $v = 65$, simply take a TD[8,8] and add a point to each group. For $v = 89$, take a TD[10,9], and assign a weight of one to each point of the first nine groups, and a weight of zero to all but two of the points of the remaining group, and assign these points a weight of four. Note that the required ingredient GDDs exist. There is trivially a 9-GDD of type 1^9 and a 4-GDD of type $1^9 4^1$ is easily obtained from PG(3,2). □

Let $D_{489} = \{5, 12, 17, 20, 21, 24\}$ and
$P_{489} = \{41^*, 44^*, 48, 53, 60, 69, 77, 96, 101, 156, 161, 164, 173\}$.

*Note: M. Greig [9] has shown that neither 41 nor 44 lies in $B(4,8,9)$.

Lemma 7.2 *If $v \in D_{489}$, then $v \notin B(\{4,8,9\})$.*

Proof. Since the members of D_{489} do not lie in $B(\{4,8\})$, then if $v \in D_{489}$ and a PBD$[v, \{4,8,9\}]$ exists, then it contains a block of size 9, which contradicts Result 3.6. □

Theorem 7.1 *If $v \geq 4$, $v \equiv 0,1 \pmod 4$ and $v \notin D_{489} \cup P_{489}$, then $v \in B(\{4,8,9\})$. Moreover, if $v \in D_{489}$, then $v \notin B(\{4,8,9\})$.*

Proof. This is a summary of the foregoing.

8 Closure of sets containing more than three elements and excluding 5

In this section, we consider the closure of sets containing more than three elements.

First of all, we consider the closure for $\{4,6,7,8\}$. We obtain the closure based on the closure of $\{4,7,8\}$, $\{4,6,7\}$ and $\{4,6,8\}$.

The following observation is very useful.

Lemma 8.1 *Suppose there exists a $TD[8,m]$. If a and b are integers such that $0 \leq a,b \leq m$ and $a,b,m \in B(\{4,6,7,8\})$, then we have $6m+a+b \in B(\{4,6,7,8\})$.*

Proof. Trivial. □

Lemma 8.2 $\{47,59,87,89,101,110,161,170\} \subseteq B(\{4,6,7,8\})$.

Proof. We use Lemma 8.1 in connection with the following table

v	m	a	b	v	m	a	b
47	7	4	1	101	16	4	1
59	8	7	4	110	16	7	7
87	13	8	1	161	25	7	4
89	13	7	4				

This completes the lemma. □

Let D_{4678}={5, 9, 10, 11, 12 , 14, 15, 17, 18, 19, 20, 23, 24, 26, 27} and P_{4678}={33, 35, 41, 65, 77, 131}.

Lemma 8.3 *If $v \in D_{4678}$, then $v \notin B(\{4,6,7,8\})$.*

Proof. It is shown in Theorem 5.1, 5.2 and 6.1 that if $v \in D_{4678}$, then $v \notin B(\{4,6,7\})$, $v \notin B(\{4,6,8\})$ and $v \notin B(\{4,7,8\})$. Hence, if there exists a $PBD[v,\{4,6,7,8\}]$, then it must have a block of size seven and a block of size eight. Apply Result 3.6, we have $v \geq 28$. □

Theorem 8.1 *If $v \geq 4$ and $v \notin D_{4678} \cup P_{4678}$, then $v \in B(\{4,6,7,8\})$. Moreover, if $v \in D_{4678}$, then $v \notin D_{4678}$.*

Proof. This is a summary of the foregoing. □

We now deal with the closure of the set $\{4, 6, 7, 9\}$. We obtain most of the results by taking the results for $\{4, 6, 7\}$, $\{4, 6, 9\}$ and $\{4, 7, 9\}$.

It is straightforward to check that the necessary conditions for $v \in B(\{4, 6, 7, 9\})$ are $v \equiv 0, 1 \pmod 3$.

Lemma 8.4 $\{87\} \subset B(\{4, 6, 7, 9\}$.

Proof. Take a $TD[7, 13]$ and delete four points in one group to obtain $87 \in B(\{4, 6, 7, 9\})$. □

Let $D_{4679} = \{10, 12, 15, 18, 19, 24, 27\}$.

Lemma 8.5 *If* $v \in D_{4679}$, *then* $v \notin B(\{4, 6, 7, 9\})$.

Proof. It is proved in Theorem 5.1, 5.3 and 6.2 that if $v \in D_{4679}$, then $v \notin B(\{4, 6, 7\})$, $v \notin B(\{4, 6, 9\})$ and $v \notin B(\{4, 7, 9\})$. Hence, if a $PBD[v, \{4, 6, 7, 9\}]$ exists, then it must have a block of size seven and a block of size nine. Apply Result 3.6 to obtain $v \geq 31$. □

Theorem 8.2 *If* $v \geq 4$, $v \equiv 0, 1 \pmod 3$ *and* $v \notin D_{4679}$, *then* $v \in B(\{4, 6, 7, 9\})$. *Moreover, if* $v \in D_{4679}$, *then* $v \notin B(\{4, 6, 7, 9\})$.

Proof. This is a summary of the foregoing. □

We consider the closure of $\{4, 6, 8, 9\}$. In this case, we heavily rely on the results for the $\{4, 6, 8\}$, $\{4, 6, 9\}$ and $\{4, 8, 9\}$.

Lemma 8.6 $\{74, 110, 161, 290\} \subset B(\{4, 6, 8, 9\})$.

Proof. For $v = 74$, M. Greig [9] has pointed out that all non-desarguesian planes of order 9 contain a projective plane of order 2. By deleting the points of such a plane and an external line from a non-desarguesian plane, a PBD$[74, \{6, 8, 9\}]$ is obtained. Therefore $74 \in B(\{6, 8, 9\}) \subset B(\{4, 6, 8, 9\})$.

If $v = 110$, take a $TD[9, 13]$ and remove seven points in one group to obtain $110 \in B(\{4, 6, 8, 9\})$.

If $v = 161$, first note that $44 \in B(\{4, 6, 8, 6^*\})$. Applying Construction 4.3 with $k = 4, m = 38, a = 1$, and $f = 6$ yields a PBD$[161, \{4, 6, 8, 9^*\}]$.

If $v = 290$, apply Construction 4.2 with $m = 15$, $a = 8$, $b = 4$ and $f = 1$. □

Let $D_{4689} = \{5, 7, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22, 23, 24, 26, 27\}$ and $P_{4689} = \{34, 35, 41, 47, 50, 53, 55, 59, 62, 71, 75, 77, 87, 95, 101, 131, 170\}$.

Lemma 8.7 *If* $v \in D_{4689}$*, then* $v \notin B(\{4,6,8,9\})$,

Proof. It is proved in Theorems 5.2, 5.3 and 7.1 that if $v \in D_{4689}$, then $v \notin B(\{4,6,8\})$, $v \notin B(\{4,6,9\})$ and $v \notin B(\{4,8,9\})$. Hence, if a $PBD[v, \{4,6,8,9\}]$ exists, then it must have a block of size eight and a block of size nine. Apply Result 3.6 to obtain $v \geq 35$. □

Theorem 8.3 *If* $v \geq 4$ *and* $v \notin D_{4689} \cup P_{4689}$*, then* $v \in B(\{4,6,8,9\})$. *Moreover if* $v \in D_{4689}$*, then* $v \notin B(\{4,6,8,9\})$.

Proof. This is a summary of the foregoing. □

We consider the closure of $\{4,7,8,9\}$.

The following result is useful below.

Theorem 8.4 *[18] For any* $v \geq 7$, $v \in B(\{7,8,9\})$ *except possibly when* $v \in E_{789}$*, where* $E_{789} = E_{789} = [10,48] \cup [51,55] \cup [59,62] \cup [93,111] \cup [116,118] \cup \{132\} \cup [138,168] \cup [170,216] \cup [219,223] \cup [228,230] \cup [242,279] \cup [283,286] \cup [298,307] \cup [311,342]$.

Lemma 8.8 *Suppose there exists a* $TD[9,m]$*. If* a *and* b *are integers such that* $0 \leq a,\ b \leq m$ *and* $a,b,m \in B(\{4,7,8,9\})$*, then we have* $7m+a+b \in B(\{4,7,8,9\})$.

Proof. Trivial. □

Lemma 8.9 $\{101,102,111,138,186,194,195\} \subseteq B(\{4,7,8,9\})$.

Proof. Apply Lemma 8.8 in accordance with the following table.

v	m	a	b	v	m	a	b
101	13	9	1	138	16	13	13
102	13	7	4	186	25	7	4
111	13	13	7	195	25	13	7

This leaves $v = 194$. For this value, use a (4, $\{0,1,2\}$, 1,1,1,16)-thwart (see [17]). □

Let $D_{4789} = \{5,6,10,11,12,14,15,17,18,19,20,21,23,24,26,27,30\}$ and P_{4789}={35, 38, 39, 41, 42, 44, 47, 48, 51, 54, 59, 62, 110, 143, 150, 159, 161, 164, 167, 173, 174 }.

Lemma 8.10 *If* $v \in D_{4789}$*, then* $v \notin B(\{4,7,8,9\})$.

Proof. It is shown in Theorems 6.1, 6.2 and 7.1 that if $v \in D_{4789}$, then $v \notin B(\{4,7,8\})$, $v \notin B(\{4,7,9\})$ and $v \notin B(\{4,8,9\})$. Hence, if there exists a $PBD[v, \{4,7,8,9\}]$, then it must have a block of size eight and a block of size nine. Apply Result 3.6 to obtain $v \geq 35$. □

Theorem 8.5 *If $v \geq 4$ and $v \notin D_{4789} \cup P_{4789}$, then $v \in B(\{4,7,8,9\})$. Moreover, if $v \in D_{4789}$, then $v \notin B(\{4,7,8,9\})$.*

Proof. This is a summary of the foregoing.

Finally, we consider the closure of $\{4,6,7,8,9\}$.

We can just simply combine the results of $\{4,6,7,8\}$ and $\{4,7,8,9\}$.

Let $D_{46789} = \{5,10,11,12,14,15,17,18,19,20,23,24,26,27\}$ and $E_{46789} = \{35,41\}$.

Lemma 8.11 *If $v \in D_{46789}$, then $v \notin B(\{4,6,7,8,9\})$.*

Proof. It is shown in Theorems 8.1, 8.2, 8.3 and 8.5 that if $v \in D_{46789}$, then $v \notin B(\{4,6,7,8\})$, $v \notin B(\{4,6,7,9\})$, $v \notin B(\{4,6,8,9\})$ and $v \notin B(\{4,7,8,9\})$. Hence, if there exists a $PBD[v, \{4,6,7,8,9\}]$, then it must contain a block of size eight and a block of size nine. Apply Result 3.6 to obtain $v \geq 35$. □

Theorem 8.6 *If $v \geq 4$ and $v \notin D_{46789} \cup P_{46789}$, then $v \in B(\{4,6,7,8,9\})$. Moreover, if $v \in D_{46789}$, then $v \notin B(\{4,6,7,8,9\})$.*

Proof. This is a summary of the foregoing.

9 Conclusion

The results of the foregoing are summarized in the following table.

Subset	Necessary Conditions	Possible exceptions Bold face indicates known exceptions
4	1,4 mod 12	-
4,5	0,1 mod 4	8,9,12
4,6	0,1 mod 3	7,9,10,12,15,18,19,22,24,27,33,34,39,45,46,51,75,87
4,7	1 mod 3	10,19
4,8	0,1 mod 4	5,9,12,17,20,21,24,33,41,44,45,48,53,60,65,69,77,89, 101,161,164,173
4,9	0,1,4,9 mod 12	12,21,24,48,60,69,84,93,96,192
4,5,6	*N*	7,8,9,10,11,12,14,15,18,19,23
4,5,7	*N*	6,8,9,10,11,12,14,15,18,19,23,26,27,30,39,42,51,54
4,5,8	0,1 mod 4	9,12
4,5,9	0,1 mod 4	8,12
4,6,7	0,1 mod 3	9,10,12,15,18,19,24,27,33,45,87
4,6,8	*N*	5,7,9,10,11,12,14,15,17,18,19,20,22,23,24,26,27,33, 34,35,39,41,47,50,51,53,59,62,65,71,75,77,87,89,95, 101,110,131,161,170
4,6,9	0,1 mod 3	7,10,12,15,18,19,22,24,27,34,75,87
4,7,8	*N*	5,6,9,10,11,12,14,15,17,18,19,20,21,23,24,26,27,30, 33,35,38,39,41,42,44,45,47,48,51,54,59,62,65,66,69,74, 75,77,78,83,87,89,90,101,102,110,111,114,123,126,131,138, 143,150,159,161,162,164,167,170,173,174,186,195
4,7,9	0,1 mod 3	6,10,12,15,18,19,21,24,27,30,39,42,48,51,54,60,66, 69,75,78,84,87,93,96,102,111,114,138,147,150,159,174, 183,186,195,210
4,8,9	0,1 mod 4	5,12,17,20,21,24,41,44,48,53,60,69,77,101,161.164,173
4,5,6,7	*N*	8,9,10,11,12,14,15,18,19,23
4,5,6,8	*N*	7,9,10,11,12,14,15,18,19,23
4,5,6,9	*N*	7,8,10,11,12,14,15,18,19,23
4,5,7,8	*N*	6,9,10,11,12,14,15,18,19,23,26,27,30,42,51
4,5,7,9	*N*	6,8,10,11,12,14,15,18,19,23,26,27,30,51,54
4,5,8,9,	0,1 mod 4	12

Subset	Necessary Conditions	Possible exceptions Bold face indicates known exceptions
4,6,7,8	*N*	5,9,10,11,12,14,15,17,18,19,20,23,24,26,27,33,35,41,65, 77,131
4,6,7,9	0,1 mod 3	10,12,15,18,19,24,27
4,6,8,9	*N*	5,7,10,11,12,14,15,17,18,19,20,22,23,24,26,27,34,35,41, 47,50,53,55,59,62,71,75,77,87,95,101,131,161,170
4,7,8,9	*N*	5,6,10,11,12,14,15,17,18,19,20,21,23,24,26,27,30,35,38, 39,41,42,44,47,48,51,54,59,62,110,143,150,159,161,164,167, 173,174
4,5,6,7,8	*N*	9,10,11,12,14,15,18,19,23
4,5,6,7,9	*N*	8,10,11,12,14,15,18,19,23
4,5,6,8,9	*N*	7,10,11,12,14,15,18,19,23
4,5,7,8,9	*N*	6,10,11,12,14,15,18,19,23,26,27,30,51
4,6,7,8,9	*N*	5,10,11,12,14,15,17,18,19,20,23,24,26,27,35,41
4,5,6,7,8,9	*N*	10,11,12,14,15,18,19,23

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