On The Local Structure Of Well-Covered Graphs Without 4-Cycles

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Introduction

A well-covered graph (introduced by Plummer [9] in 1970) is one in which every maximal independent set of vertices is a maximum. Whereas determining the independence number of an arbitrary graph is NP-complete, for a well-covered graph one can simply apply the greedy algorithm. The problem of characterizing the well-covered graphs has not been so straightforward however. The reader is referred to [1] to [12] for results on this problem and to [10], especially, for an excellent survey of progress. The particular attack that the author has been involved with involves restricting the girth (size of the smallest cycle). In particular, Finbow and Hartnell [4] characterized well-covered graphs of girth 8 or more and then Finbow, Hartnell and Nowakowski [5] extended this characterization to include the well-covered graphs of girth 5 or more. In addition, the same three authors [6] characterized those well-covered graphs in which there were no 4-cycles nor 5-cycles (triangles, however, are allowed).

As background and to help the reader appreciate the motivation for the investigation carried out in this paper, we first need to review the general structure of the well-covered graphs in [5] and [6]. A vertex v in a well-covered graph G is said to be *extendible* if and only if $G - v$ is also well-covered and the independence number of $G - v$ is the same as G. For example, any vertex of a graph which is a 5-cycle (or a 3-cycle or a K_2) is extendible. On the other hand, the leaves of a path on 4 vertices are not extendible whereas the other 2 vertices are. Extendible vertices play an important role as they can be used as attachment points to join well-covered graphs to form larger ones. In particular, two 5-cycles can be joined by an edge or a 5-cycle and a K_2 can be joined by an edge (see Figure 1 where the extendible vertices are colored black).

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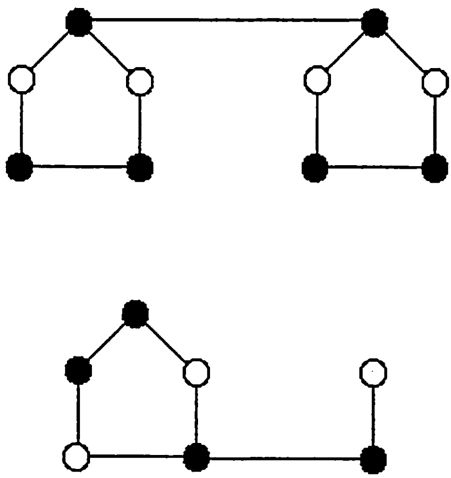

Figure 1

In fact, one can start with any collection of K_2's and 5-cycles and designate one vertex of each K_2 as well as any two non-adjacent vertices of each 5-cycle as attachment points and then form a connected graph by arbitrarily joining attachment points. The graph so formed will be well covered. In [5] it is shown that a well-covered graph of girth 5 or more with an extendible vertex must in fact belong to this family. The rather startling result is that there are only six other well-covered graphs (namely, having no extendible vertices) of girth 5 or more. Hence the K_2 and the C_5 are essentially the two basic building blocks used to form the family. In [6], the authors focus on a collection of K_2's, with exactly one attachment point, and 3-cycles, with either one or two attachment points. Again one can form a well-covered graph that is connected by arbitrarily joining attachment points with new edges.

The authors establish that any well-covered graph with no 4- nor 5-cycle, but with an extendible vertex, must belong to this family. Again, it turns out there are only two other graphs (having no extendible vertices) in the collection.

Gasquoine, Hartnell, Nowakowski and Whitehead [7], in their attempt to generalize the characterization to include all well-covered graphs without 4-cycles, have determined over a dozen new basic building blocks which can be used to build well-covered graphs without 4-cycles (see Figure 2, for some examples).

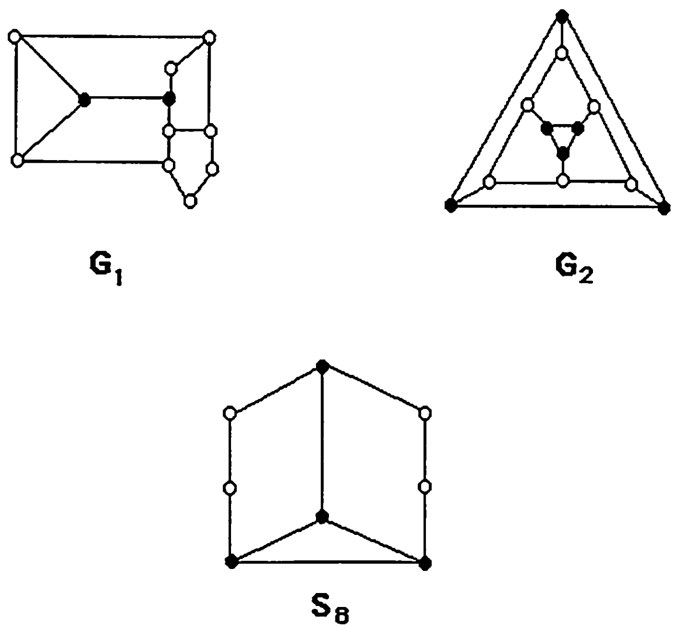

Figure 2

For instance, the well-covered graph in Figure 3 is formed by joining various basic building blocks with edges between extendible vertices.

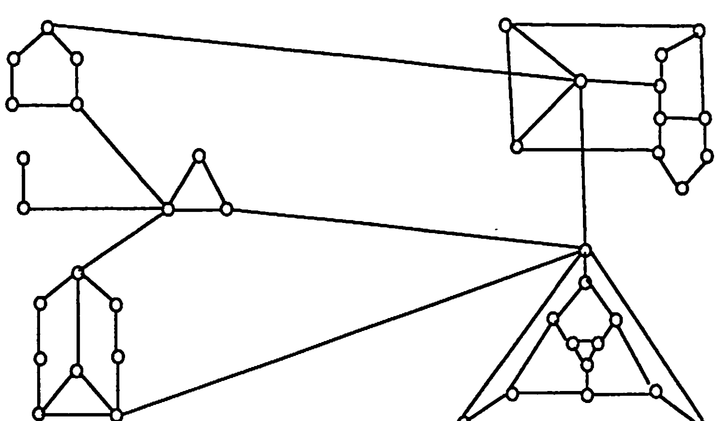

Figure 3

The hope is that, similar to the well-covered graphs of girth 5 or more as well as those without 4-cycles and 5-cycles, one can show there are a reasonably small number of basic building blocks (each has at least one extendible vertex) and then a limited number of exceptional graphs (those with no extendible vertices). The purpose of this paper is to gain insight into the structure of the basic building blocks (other than the already known ones of K_2, the 3-cycle and the 5-cycle). Our main result is that an extendible vertex v in a well-covered graph without 4-cycles is either part of one of the already known special K_2's, 3-cycles or 5-cycles or is a vertex of a special induced subgraph which we shall call S_8 (see Figure 2).

For instance, the reader can readily verify that an induced S_8 is present in G_1 and G_2 shown in Figure 2.

Before we proceed, a few definitions are required. A vertex will be called a *stem* if it has a leaf as a neighbor. A 3-cycle will be called *basic* if at least one of its vertices is of degree two. A 5-cycle will be called *basic* if it contains no adjacent vertices of degree three or more.

If G is well-covered and I is an independent set of vertices in G and $N[I]$ represents the set consisting of all vertices in I as well as any neighboring vertex, then $G - N[I]$ must also be well-covered (see [5], for example). It is also shown in [5] that if v is an extendible vertex in a well-covered graph G, then it is not possible to find an independent set I in G such that $G - N[I] \cong \{v\}$. That is, it is not possible to isolate v. It also follows that if v is extendible in G, that v is still extendible in the component of $G - N[J]$, for any independent set J, containing v. If G has at least 3 vertices, a leaf in G is not extendible (since it can be isolated). If v is an extendible vertex on a basic 5-cycle, then both neighbors of v on the 5-cycle must be of degree two. Similarly, if v is extendible and on a basic 3-cycle, then at least one of its neighbors on the 3-cycle must be of degree two.

It will be very useful to note that certain subgraphs are impossible in a well-covered graph. In particular, we observe that the induced subgraphs illustrated in Figure 4, where x and y are not adjacent to any other vertices in G, but s and others may be, are impossible in a well-covered graph. This follows by extending $\{s\}$ to a maximal independent set, say I, of G and then by considering $J = I \cup \{x, y\} - \{s\}$. J is an independent set of larger size than I which cannot occur in a well-covered graph.

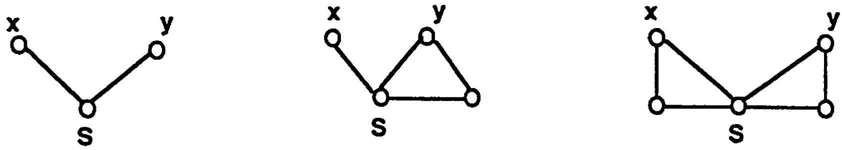

Figure 4

To simplify the explanation we shall refer to these subgraphs as a double-leaf (namely, x and y at s), a leaf-triangle (namely, x and y at s) and a double-triangle (again x and y at s) in the work that follows.

We will use $N_2(v)$ to refer to the set of vertices at distance two from v.

The Main Result

Theorem. *Let G be a well-covered graph without 4-cycles. Let v be a vertex in G satisfying the following conditions:*

(i) v is extendible.

(ii) v is not a stem nor on either a basic 3-cycle or a basic 5-cycle.

Then v must be a vertex on an induced S_8.

Proof: Let G and v be as described in the hypothesis. We first note that v is of degree three or more. This is easy to verify since a leaf is not extendible and if v were of degree two then the lack of 4-cycles implies v is on a basic 3- or 5-cycle or is a stem. If G has a vertex w such that v still satisfies (i) and (ii) in $G - N[w]$, then consider the component of $G - N[w]$ containing v instead. That is, we consider the minimal graph G containing v where G and v have the required property. This implies that for any w not belonging to $N[v]$, that $G - N[w]$ no longer satisfies the hypothesis. Certainly $G - N[w]$ is still well-covered and without 4-cycles and (i) still holds. Thus it must be the case that now v either has a leaf as a neighbor (in this case we say w is of type [2] in G), or is on a basic 3-cycle in $G - N[w]$ (in this event we say w is of type [3] in G) or v is on a basic 5-cycle in $G - N[w]$ (we say w is of type [5] in G). Hence we may assume that all vertices other than v and its neighbors are of type [2], [3] or [5]. Since G contains no 4-cycles, no neighbor of v is adjacent to more than one neighbor of any vertex w. Thus if w is of type [2], the resulting leaf in $G - N[w]$ must be of degree 2 in G itself. If w is of type [3], then in $G - N[w]$ the vertex v is on a basic 3-cycle implying that a neighbor of v is now of degree 2. Hence, in G, that neighbor must be of degree 3. Similarly, if w is of type [5], then in $G - N[w]$ the vertex v is on a basic 5-cycle meaning that two neighbors of v are now of degree two. Thus, in G, those neighbors must be of degree two or three.

Hence, in G, v must have at least one neighbor of degree two or three. For instance, if we consider (see Figure 2) a vertex v that is on both the 7-cycle and 3-cycle in S_8, and any other vertex, say w, not adjacent to v, we observe that in $S_8 - N[w]$, the vertex v is now either a stem or on a basic 3- or 5-cycle.

Case (1):

Assume v has at least two neighbors of degree 2, say x_1 and x_2, with neighbors y_1 and y_2 respectively. If y_1 and y_2 had a common neighbor, say z, then $G-N[z]$ would contain a double-leaf (namely x_1 and x_2 at v) which is forbidden. Similarly if y_1 and y_2 had independent neighbors, say z_1 and z_2 respectively, then $G - N[\{z_1, z_2\}]$ would contain the same double-leaf. Hence any neighbor ($\neq x_1$) of y_1 must be adjacent to any neighbor ($\neq x_2$) of y_2. Since G has no 4-cycles, this forces both y_1 and y_2 to be of degree two.

If y_1 and y_2 were adjacent, then v would be already on a basic 5-cycle, namely, $vx_1y_1y_2x_2$ which is not the case. Furthermore, if either y_1 or y_2, y_1 say, were of type [2], then y_1 must be adjacent to y_3, say, where y_3 and v share a degree two neighbor, x_3 say. But again this is impossible as $vx_1y_1y_3x_3$ would be a basic 5-cycle.

Let the neighbor ($\neq x_1$) of y_1 be y_m and neighbor ($\neq x_2$) of y_2 be y_n where y_m and y_n are adjacent (by preceding paragraph). Since y_1 and y_2 are of type [3] or [5], v and y_m have a common neighbor x_m of degree 3 and v and y_n have a common neighbor x_n of degree 3.

First assume $x_m \neq x_n$. If the remaining neighbor ($\neq v, \neq y_m$) of x_m, say a, as well as the remaining neighbor ($\neq v, \neq y_n$) of x_n, say b, are both adjacent to v, then $G-N[\{y_1, y_2\}]$ would contain a double-triangle (namely, x_n and x_m at v) which is impossible since G is well-covered.

On the other hand if a, say, is not adjacent to v, then y_1 cannot be of type [3]. But y_1 of type [5] implies that $G - N[y_1]$ has a basic 5-cycle vx_m a y_s x_s where x_s is adjacent to v. This implies x_s is of degree 2 in G since the only neighbors of y_1 are x_1 and y_m (and y_m adjacent to x_s would yield a 4-cycle). But then, noting that y_n and a are not adjacent since there are no 4-cycles, we have that $G - N[\{y_n, a\}]$ contains a double-leaf (namely, x_2, x_s at v).

Hence we must have $x_m = x_n$. However, $vx_1y_1y_my_ny_2x_2$ and x_m form an S_8 (since x_1 and x_2 are of degree 2 in G, it is, in fact, an induced S_8) and we are done.

Case (2):

Next we consider the case in which v has exactly one neighbor, say x, of degree 2. Let y be the other neighbor of x.

Case 2(a): Let y be of type [5] and $vx_1y_1y_2x_2$ be the basic 5-cycle in $G - N[y]$. Let the shared neighbor of y and x_1 be y_1' and the common neighbor of y and x_2 be y_2'. Note that both x_1 and x_2 are of degree 3 in G. Furthermore, either y_1 or y_2, say y_1, is of degree 2 in $G - N[y]$ since $vx_1y_1y_2x_2$ is a basic 5-cycle. Thus, in G, y_1 is of degree 2 or 3.

If it is of degree 2, then y_1 must be of type [5] and the basic 5-cycle in $G - N[y_1]$ is $vx_2y_2'yx$. This implies that y_2' must be of degree 2 (since y

not) in $G-N[y_1]$ and hence of degree 2 in G as well (if y_2' and y_2 adjacent, then $vx_1y_1y_2y_2'yx$ and x_2 form an induced S_8). But this is impossible as $G-N[\{x_1, y_2\}]$ would contain a double-leaf (namely, x and y_2' at y). Hence y_1 is of degree 3 in G.

Let the neighbor, other than x_1 and y_2, of y_1 be w. Then w is also adjacent to y. If $w = y_1'$, then $vx_2y_2y_1y_1'yx$ and x_1 form an induced S_8.

If y_1 is of type [5], we first note that the basic 5-cycle in $G-N[y_1]$ cannot be vx_3y_3yx as then x_3y_3yw would form a 4-cycle since w must be adjacent to x_3. Now let the basic 5-cycle in $G-N[y_1]$ be $vx_2y_2'y_3x_3$ where $x_3 \neq x$ and w is adjacent to x_3 (and y) in G. But this is impossible since $G-N[\{y_1, y_2'\}]$, has a double-leaf (namely, x and x_3 at v). Thus $x_3 = x$ and the basic 5-cycle in $G-N[y_1]$ must be $vxyy_2'x_2$. Since one of y and y_2' must be of degree 2 in $G-N[y_1]$ (basic 5-cycle) it must be y_2'. In G, y_2' is either of degree 2 or of degree 3 in which case it is adjacent to w (recall that y_2 and y_2' adjacent forces an S_8). In either case, $G-N[\{x_1, y_2\}]$ contains a double-leaf or leaf-triangle (namely, x and y_2' at y) which is impossible. Thus y_1 cannot be of type [5].

If y_1 is of type [3], then y_1 is adjacent to y_3 which shares a neighbor, say x_3, with v. In G, x_3 is of degree 3. But this is impossible since $G-N[\{y, y_1\}]$ contains a leaf-triangle (namely, x_2 and x_3 at v). This completes case 2(a).

Case 2(b): Let y be of type [3] and vx_1x_1' be the basic 3-cycle in $G-N[y]$ where y and x_1 share a neighbor, say y_1, and x_1 is of degree 3 in G.

First observe that y and y_1 cannot both be of degree 3 or more. If they were, then they either have a common neighbor, say w, or independent neighbors, say w_1 and w_2 (since G has no 4-cycles). But then $G-N[w]$ or $G-N[\{w_1, w_2\}]$ would contain a leaf-triangle (namely, x and x_1 at v) which is impossible.

On the other hand, if y_1 is of degree 2, then $G-N[x_1']$ contains a double-leaf (namely, x and y_1 at y) which is not possible.

Hence we conclude that y_1 is of degree 3 or more, and y is of degree 2.

Now, let the neighbors of x_1' (other than v and x_1) be Y_1'. Let the set of vertices (other than x_1' and Y_1') which have a neighbor in Y_1' be W. Note that y_1 does not belong to W since there are no 4-cycles.

First assume that y_1 is not adjacent to any vertex in W. Select a maximal independent set, say I, of vertices in W. If every vertex in Y_1' is adjacent to some vertex in I, then $G-N[I \cup \{y_1\}]$ contains a double-leaf (namely, x and x_1' at v) which is impossible. But if some vertex, say y_1', in Y_1' is not adjacent to any vertex in I, then $G-N[I \cup \{y\}]$ contains either a leaf-triangle or double-triangle (in either case x_1 and y_1' at x_1') which is impossible. Hence y_1 must be adjacent to a vertex in W.

In the case that y_1 is adjacent to a vertex, say w, in W, where w is adjacent to y_1', say, in Y_1' we have an S_8. In particular, $vx_1'y_1'wy_1yx$ and x_1

form an induced S_8.

This completes case 2(b) and hence case 2 since y cannot be of type [2] as x is the only degree two neighbor of v.

Case (3):

Finally, assume v has no degree two neighbors. Hence no vertex is of type [2].

Case 3(a): First consider the case where y is a vertex belonging to $N_2[v]$, with x the common neighbor of v and y, where y is of type [5]. Let $vx_1y_1y_2x_2$ be the basic 5-cycle containing v in $G - N[y]$.

3(a)(i): First assume that neither x_1 nor x_2 is adjacent to x.

Since no neighbor of v is of degree two in G, we note that y and x_1 share a neighbor y_1' and that y and x_2 share a neighbor y_2'. Also x_1 and x_2 are of degree 3 in G. Furthermore, either y_1 or y_2, say y_1, must be of degree 2 in $G - N[y]$ since $vx_1y_1y_2x_2$ is a basic 5-cycle. Thus, in G, y_1 is of degree 2 or 3. But y_1 cannot be of degree 2, else y_1 could not be of type [3] nor of type [5] (keeping in mind that v has no degree 2 neighbors). Hence y_1 is of degree 3 in G. Let the neighbor, other than x_1 and y_2, of y_1 be w. Then w must also be adjacent to y. If $w = y_1'$, then $vx_2y_2y_1y_1'yx$ and x_1 form an S_8 (induced, since x_2 and x are not adjacent).

3(a)(i1): If y_1 is of type [5], then the basic 5-cycle in $G - N[y_1]$ must be $vx_2y_2'y_3x_3$ where w is adjacent to x_3 (and y) in G and x_3 is of degree 3 in G.

Assume $x_3 \neq x$. Since one of y_2' and y_3 must be of degree 2 in $G - N[y_1]$, y_3 must be. But y_3 could not be of degree 2 in G as then it could not be of type [3] nor of type [5] in G (again recalling that there are no degree 2 neighbors of v). Thus y_3 is of degree 3 in G. Therefore, y_3 must be adjacent to a neighbor of y_1 but it could not be adjacent to y_2 (4-cycle) and thus y_3 and w must be adjacent. But then $vx_1y_1wy_3y_2'x_2$ and x_3 form an induced S_8.

If $x_3 = x$, then the basic 5-cycle in $G - N[y_1]$ is $vx_2y_2'yx$ where $x = x_3$ is of degree 3 in G. Since one of y and y_2' must be of degree 2 in $G - N[y_1]$, y_2' must be. Again y_2' could not be of degree 2 in G else y_2' not of type [3] nor of type [5]. Hence, in G, y_2' is adjacent to a neighbor of y_1. But y_2' and w adjacent is impossible (4-cycle) so y_2 and y_2' must be adjacent. But then $vx_1y_1y_2y_2'yx$ and x_2 form an induced S_8.

3(a)(i2): If y_1 is of type [3], then w and v must share a common neighbor, say x_3 (of degree 3 in G), with x_3 and v having a common neighbor, say x_3'. But this is impossible as $G - N[\{y, y_1\}]$ has a leaf-triangle (namely, x_2 and x_3 at v).

This completes part 3(a)(i).

Case 3(a)(ii): Next consider the case where one of x_1 or x_2, say x_1, is adjacent to x. Again x_1 and x_2 must be of degree 3 in G. Let the common neighbor of x_2 and y be y_2'. But then $xx_1y_1y_2x_2y_2'y$ and v form an S_8.

If y_2 and y_2' are not adjacent, then the S_8 is induced and we are done.

In the event that y_2 and y_2' are adjacent, we note that y_1 is of degree 3 or more else $G - N[x]$ contains a leaf-triangle (namely, x_2 and y_1 at y_2). Let Y_1 be those vertices (other than y_2 and x_1) adjacent to y_1.

If y_2 is of degree 4 or more, let z be a neighbor other than x_2, y_2' or y_1 If z belongs to Y_1, then $G - N[\{z, y\}]$ contains a double-leaf (namely, x_1 and x_2 at v) which is not possible.

If z is not in Y_1, it cannot be adjacent to any vertex in Y_1 nor to y (no 4-cycles). But then, selecting any w in Y_1, $G - N[\{w, y, z\}]$ contains the same double-leaf (x_1 and x_2 at v). Thus y_2 is of degree 3 in G.

If x is of degree 4 or more, then y_2' must be of type [3] (Case 3(a)(i) rules out type [5]) where y_2' is adjacent to y_3, say, which shares a neighbor, x_3 say, with v. Furthermore x_3 is of degree 3 in G where x_3' is the common neighbor with v. But then $vx_3y_3y_2'y_2y_1x_1$ and x_2 form an induced S_8 (x_1 and x_3 are not adjacent). Thus we are done or x is of degree 3. Let Y be those vertices (other than x and y_2') adjacent to y and recall that Y_1 are those vertices (other than x_1 and y_2) adjacent to y_1. Form any maximal independent set, say I, of $G - N[\{x_2, y_1, y\}]$ that includes at least one vertex that was in $N[v]$ in G (unless v of degree 3 in G). Now consider $G_1 = G - N[I]$ which must be well-covered. Observe that v is not in G_1 (unless v of degree 3 in G). But $G_1 - N[\{y_2\}]$ also must be well-covered which implies there is still at least one neighbor of y, say w, in Y present in $G_1 - N[\{y_2\}]$ else double-leaf at x (namely, x_1 and y) (if v of degree 3 in G, then leaf-triangle at x). Say w is not adjacent to y_1 in G_1.

But then $G_1 - N[\{x_2, y_1\}]$ contains either a double-leaf or a leaf-triangle (namely, x and w at y) which is impossible. Hence w must be adjacent to both y and y_1. But then $vxywy_1y_2x_2$ and x_1 form an induced S_8.

Thus if any $y \in N_2[v]$ is of type [5], G must contain an induced S_8.

Case 3(b): Next consider the case in which some vertex in $N_2(v)$ is of type [3]. This implies that there is a neighbor x_1 of v where x_1 is of degree 3 and v and x_1 have a common neighbor, say x_1'. Let the other neighbor of x_1 be y_1 (y_1 is not adjacent to v). Let another neighbor ($\neq v, \neq x_1$) of x_1' be y_1'. Now consider y_1.

If y_1 is of type [3], then y_1 must be adjacent to y_2, say, which in turn is adjacent to a degree 3 neighbor, say x_2, of v. Let the common neighbor of v and x_2 be x_2'. If y_1 and y_2 had a common neighbor, say z, then $G - N[z]$ would contain a double-triangle (namely, x_1 and x_2 at v) which is impossible. Similarly, if y_1 and y_2 each had neighbors, say z_1 and z_2 respectively, then, since z_1 and z_2 could not be adjacent (no 4-cycles), $G - N[\{z_1, z_2\}]$ would also contain the same double-triangle. Thus at least one of y_1 and y_2, say y_2, must be of degree 2. This too is impossible as $G - N[\{y_1', x_2'\}]$ contains a double-leaf (x_1 and y_2 at y_1). Hence y_1 cannot be of type [3]. But by Case (3a), y_1 of type [5] implies an S_8.

This completes Case 3 and hence the theorem.

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