Near Exact and Envy Free Cake Division

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ABSTRACT. A division of a cake $X = X_1 \cup \cdots \cup X_n$ among n players with associated probablity measures $\mu_1, \ldots, \mu_n$ on X is said to be:

(a) exact in the ratios of $\alpha_1 : \alpha_2 : \ldots : \alpha_n$ provided whenever $1 \leq i, j \leq n$, $\frac{\mu_i(X_j)}{\mu_i(X)} = (\alpha_j/(\alpha_1 + \cdots + \alpha_n))$

(b) ϵ-near exact in the ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n$ provided whenever $1 \leq i, j \leq n$, $|\frac{\mu_i(X_j)}{\mu_i(X)} - \frac{\alpha_j}{\alpha_1 + \cdots + \alpha_n}| < \epsilon$.

(c) envy free in ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n$ provided whenever $1 \leq i, j \leq n$, $\frac{\mu_i(X_i)}{\mu_i(X_j)} \geq \frac{\alpha_i}{\alpha_j}$.

A moving knife exact division is described for two players and it is shown there can be no finite exact algorithm for $n \geq 2$ players. A bounded finite ϵ-near exact algorithm is given which is used to produce a finite envy free, ϵ-near exact algorithm.

1 Introduction

The problem of "fairly" dividing a cake has a growing literature since being introduced by Steinhaus in 1948 [27]. "Fairly" has a number of interpretations and the focus of this work is on two interpretations of fair: exact and envy free.

We will assume that the cake X is a Lebesgue measurable compact set in E^n, and players $P_1, \ldots, P_n$ are to divide the cake. Associated with each player is a probability measure μ_i on X and we assume Lebesgue measurable subsets of X are μ_i measurable. Further we assume each measure is absolutely continuous with respect to Lebesgue measure. (This assumption insures that the μ_i value of cake under a moving knife is continuous with respect to the position of the knife.) All measures used are assumed to come from this class, and all pieces considered are assumed to be Lebesgue measurable.

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Definition 1.1: A partition of a cake (or a piece of cake) $X = X_1 \cup \ldots \cup X_n$ is envy free in the ratios $\alpha_1 : \alpha_2 \ldots : \alpha_n, (\alpha_i > 0)$, among $P_1, \ldots, P_n$ provided whenever $1 \leq i,\ j \leq n$, $\frac{\mu_i(X_i)}{\mu_i(X_j)} \geq \frac{\alpha_i}{\alpha_j}$.

In particular, this implies that $\mu_i(X_i) \geq (\alpha_i/(\alpha_1 + \ldots + \alpha_n))\,\mu_i(X)$. Also if $X_1 \cup \ldots \cup X_n$ and $Y_1 \cup \ldots \cup Y_n$ are both envy free in ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n$ on different cakes then so are pieces $X_1 \cup Y_1, \ldots, X_n \cup Y_n$ on $X \cup Y$. In most cases we will assume $\Sigma \alpha_i = 1$ but situations will arise in proofs where that is not the case.

Definition 1.2: A partition of a cake (or piece of cake) $X = X_1 \cup \ldots \cup X_n$ is exact in the ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n, (\alpha_i > 0)$, among $P_1, \ldots, P_n$ provided for all $1 \leq i,\ j \leq n$,

$$\mu_i(X_j) = (\alpha_j/(\alpha_1 + \ldots + \alpha_n))\,\mu_i(X).$$

Definition 1.3: A partition of a cake (or piece of cake) $X = X_1 \cup \ldots \cup X_n$ is ϵ-near exact in ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n, (\alpha_i > 0)$ provided for all $1 \leq i,\ j \leq n$,

$$|(\mu_i(X_j)/\mu_i(X)) - \frac{\alpha_j}{\alpha_1 + \ldots + \alpha_n}| < \epsilon.$$

A finite algorithm for envy free division of a cake among three players each getting a third has been given by Selfridge [32]. A continuous algorithm accomplishing the same task is found in [29]. Brams, Taylor and Zwicker have described an envy free moving knife solution for four players [8]. A finite envy free algorithm for n players with equal shares has been given by Brams and Taylor [9]. Below we describe a different finite envy free algorithm for n players in any given ratios.

Existence theorems for envy free and exact partitions with equal ratios have been given earlier [1, 10, 29, 32]. We will describe a moving knife type algorithm for exact division among two players for any ratio, show that no such finite algorithm exists, and also give a ϵ-near exact algorithm for n players in any ratios. The last result utilizes a theorem of Bergstrom from 1930 [4] and is used to produce the envy free algorithm.

A finite algorithm is understood to have the following properties.

1. At each step, any player, say P_i, can cut an existing piece A into k Lebesgue measurable pieces $A_1, \ldots, A_k$ of specified sizes $\mu_i(A_j) = a_j$ such that $a_1 + \ldots + a_k = \mu_i(A)$. (This can also be viewed as $k - 1$ steps in which only one cut is made.)

2. This cut is to be made without consulting with any other player as to their opinion of any of the A_i before it is cut.

3. After the cut is made all of the other players evaluate $A_1, \ldots, A_k$. On the basis of these values and all previously known values, the algorithm specifies the next cut.

4. The above procedures are repeated a finite number of times to accomplish fair division.

We note that the definition above excludes any form of moving knife which requires simultaneous and continuous evaluations by all of the players. Also, since no consultation is allowed, our class of measures is such that after the cut (1) above is made, it may be the case that for $\ell \neq i$ and any $b_j \geq 0$ satisfying $b_1 + \ldots + b_k = \mu_\ell(A)$ that $\mu_\ell(A_j) = b_j$.

Two possibilities arise. Some algorithms will have an absolute bound on the number of cuts required to accomplish fair division independent of the evaluations given in 3. Other algorithms will not have such an absolute bound; although the algorithm will always terminate in a finite number of steps, the number of steps required depends on the numbers given in 3. This paper contains algorithms of both types. In Section 2, continuous variations of moving knife are utilized. In Sections 3 and 4 divisions are given with finite algorithms, some bounded and some unbounded.

2 Cutting Portions On Which There Is Agreement

Although there are existence proofs for a partition of the cake on which every player would agree that all n pieces are equal, producing algorithms for such a division is another matter. The next two results address this issue.

Theorem 2.1. *Given $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and measures μ_1 and μ_2 which are absolutely continuous with respect to Lebesgue measure, there is a continuous algorithm which produces a partition $X = X_1 \cup X_2$ which is exact in the ratio $\alpha : \beta$.*

An outline of the proof is given. Player P_1 is asked to move two parallel knives continuously across the cake so that an αth of the cake always lies between the knives. At some point P_2 will agree.

With the cake X given in E^n we will cut by a pair of moving hyperplanes perpendicular to the first axis. Thus it suffices to assume the μ_i measures, $i = 1, 2$, are distributed on [0, 1] of the real line with distribution functions f_1 and f_2 respectively. Extend each function periodically to all of R^+ so that for any $a \geq 0$, $\int_a^{a+1} f_i = 1$.

Form all pairs (x_1, x_2) so that $\int_{x_1}^{x_2} f_1 = \alpha$. Since the measure is zero on sets with Lebesgue measure zero, these pairs form a set containing a continuous path in the plane on which x_2 is monotonically non-decreasing. For example, Figure 1(a) shows a distribution function on $[0, 1]$ which has been periodically extended to R^+, and 1(b) shows the corresponding path in the plane for $\alpha = 1/3$.

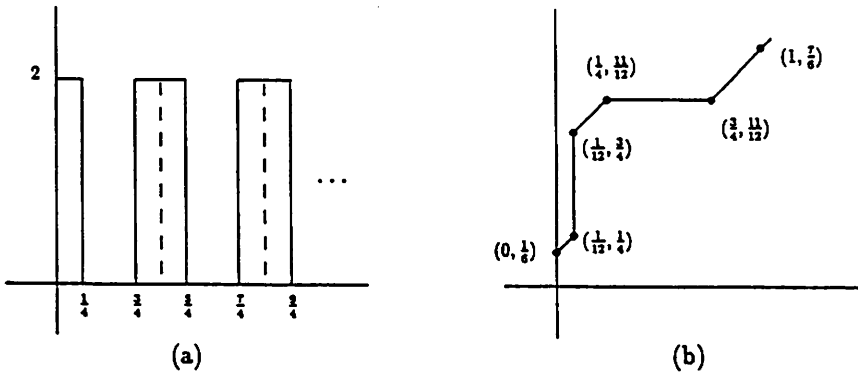

Figure 1

For any (x_1, x_2) on the path with $\int_{x_1}^{x_2} f_2 = \alpha$ the required agreement is found. Assuming w.l.o.g. that for all such pairs $\int_{x_1}^{x_2} f_2 < \alpha = \frac{m}{n}$ (where α is rational) contradicts the fact that $\int_0^m f_2 = m$. The cases for α irrational can be established using a standard compactness argument on intervals associated with a sequence of rational numbers which converges to α.

We remark in passing that one can also show by similar methods that if parallel knives move across the cake with a fixed distance between the knives, at some point the two players will agree on the amount of cake between the knives.

We next show that no finite algorithm from the class described above can produce the division accomplished in Theorem 2.1. We will think of the algorithm as a tree and a path in the tree as a sequence of adjacent vertices. At each vertex appears the pieces of X cut up to that point along with the values placed by all players on those pieces. At a next vertex the same sets appear except one previous piece is cut in two. Values of this new collection of pieces are given there in agreement with conditions 1-3 above. At the top vertex appears the uncut set X on which all players place value one. A branch will terminate when the pieces can be partitioned into n subsets and assigned to the n players with their assessments providing their prescribed share.

In the proof which follows, given an algorithm we show how to move from one non-termintating vertex to a next prescribed by the algorithm where the procedure also fails to terminate. Thus an infinite path can be traced and the algorithm is not finite.

We show later in Theorem 4.3, using near exact divisions of Theorem 4.2, that if a piece A is ever cut so that $\mu_1(A) = 0$ while $\mu_2(A) > 0$, using that piece a *finite* exact division can be accomplished between the two players.

Theorem 2.2. *There is no finite algorithm which accomplishes exact fair division for $n \geq 2$ players in the ratios $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\alpha_i > 0$.*

Proof: It suffices to consider the simplest case where $n = 2$ and $\alpha_1 = \alpha_2 = 1/2$. At step k (after $k-1$ cuts) there are k pieces $A_{1k}, A_{2k}, \ldots, A_{kk}$. Let $a_{ij} = \mu_1(A_{ij})$ and $b_{ij} = \mu_2(A_{ij})$, so $a_{11} = b_{11} = 1$ since A_{11} is the whole cake. By renumbering when necessary we may assume that at step k the piece A_{k-1k-1} is cut into the new pieces A_{k-1k} and A_{kk}. In particular this means that $a_{ik} = a_{ik-1}$ and $b_{ik} = b_{ik-1}$ for $1 \leq i \leq k-2$. A branch will terminate at step k if and only if for some $S \subseteq \{1, \ldots, k\}$

$$\sum_{i \in S} a_{ik} = 1/2 = \sum_{i \in S} b_{ik}. \tag{2.1}$$

The algorithm cannot terminate at $k = 1$. Assume we are at a vertex at which we have $k-1$ pieces and values $a_{1k-1}, \ldots, a_{k-1k-1}$ and $b_{1k-1}, \ldots, b_{k-1k-1}$ such that the algorithm does not terminate. We may suppose P_2 cuts A_{k-1k-1}. The algorithm can specify the values b_{k-1k} and b_{kk}, and the branches depend on the values a_{k-1k} and a_{kk}. We may assume that b_{k-1k} and b_{kk} are neither zero. If P_2 is told to cut a piece of measure 0, P_1 may agree and nothing has changed. We will see that a_{k-1k} and a_{kk} are always chosen non-zero along our path. It follows that all a_{ij} and b_{ij} are non-zero along the path.

Suppose along a path the process first stops at a vertex where there is a set S for which (2.1) holds. Then we may assume $k-1 \in S$ but $k \notin S$ since if S contains neither (or equivalently both by considering its complement) then

$$\sum_{i \in S} a_{ik-1} = \sum_{i \in S} a_{ik} = \frac{1}{2} = \sum_{i \in S} b_{ik} = \sum_{i \in S} b_{ik-1}$$

which contradicts the fact that the algorithm did not terminate at step $k-1$.

We now describe how to follow the algorithm from one nonterminating vertex to a next that also fails to terminate. If $T \subseteq \{1, \ldots, k-2\}$ the sum $\sum_{i \in T} a_{ik-1}$ may or may not equal $1/2$. Let $\eta = \min |1/2 - \sum_{i \in T} a_{ik-1}|$ where the minimum is over all such T for which the sum is not $1/2$. So $\eta > 0$. (For $k = 2$ let $\eta = 1/2$.)

Regardless how the algorithm prescribes b_{k-1k} and b_{kk} there is (in accordance with the comment following 1-4) a next vertex where $a_{k-1k} = \frac{1}{2}\min(a_{k-1k-1}, \eta)$. Then for any S as described above with $k-1 \in S$ and $k \notin S$,

$$|\frac{1}{2} - \sum_{i \in S} a_{ik}| = |\frac{1}{2} - \left(\sum_{i \in S - \{k-1\}} a_{ik-1} + a_{k-1k} \right)|$$

is $a_{k-1k} > 0$ if $\sum a_{ik-1} = \frac{1}{2}$ and at least $\frac{\eta}{2}$ otherwise.

Hence, there is no set S satisfying (2.1) at this next vertex. Continuing in this way we produce an infinite path in the tree.

3 Cutting Portions On Which There Is Near Agreement

In this section we give an ϵ-near exact finite algorithm for dividing a cake among n players and arbitrary ratios. Furthermore, once ϵ is set, the algorithm has an absolute bound on the numbers of steps required. The process utilizes a theorem originally given by V. Bergström [4] and later generalized [16] concerning rearrangement of vector sums.[1] The work of this section also provides the tools for an envy free algorithm described in Section 4.

Theorem 3.1. *Given a set of vectors $V = \{v_i\}_{i=1}^t$ in E^d such that for all i, $\|v_i\| \le M$, and $\sum_{i=1}^t v_i = 0$, there is a permutation Π of $\{1, 2, \dots, t\}$ such that all of the vectors $w_r = \sum_{i=1}^r v_{\Pi(i)}$ have magnitude $\|w_r\| \le Md$.*

For a proof see [16] pages 15 and 16.

With Bergström's theorem we can now describe a finite algorithm accomplishing near exact division. We can also observe that for a fixed ϵ there is an absolute bound on the number of steps required by the algorithm.

Theorem 3.2. *Let $\mu_1, \mu_2, \dots, \mu_n$ be probability measures on a cake X. Given $\epsilon > 0$ and positive numbers $\alpha_1, \dots, \alpha_n$ with $\sum \alpha_i = 1$, there is a bounded finite algorithm which produces a partition $X = X_1 \cup X_2 \cup \dots \cup X_n$ which is ϵ-near exact.*

Proof: Assume $\mu_1(X) = \dots = \mu_n(X) = 1$. First have P_1 cut X into k pieces with equal μ_1 measure, where k will be chosen later, then have P_2 reduce (by no more than $k-1$ cuts) those pieces so that no piece has a μ_2 measure exceeding $1/k$. Repeat for all n players to produce the partition $X = A_1 \cup A_2 \cup \dots \cup A_t$, $t \le nk$, with $\mu_i(A_j) \le 1/k$ for all i, j. Set $x_j = \frac{1}{n}\sum_{i=1}^n \mu_i(A_j)$ and define ϵ_{ij} by $\mu_i(A_j) = x_j + \epsilon_{ij}$. We see that $\sum_{j=1}^t x_i = \frac{1}{n}\sum_{j=1}^t \sum_{i=1}^n \mu_i(A_j) = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^t \mu_i(A_j) = 1$ and $\sum_{j=1}^t \epsilon_{ij} = \sum_{j=1}^t (\mu_i(A_j) - x_j) = 0$. Also, $|\epsilon_{ij}| \le \frac{1}{k}$ since both x_j and $\mu_i(A_j)$ are bounded by $1/k$.

Then whenever $1 \le j \le t$, the vector $v_j = (\epsilon_{1j}, \dots, \epsilon_{nj})$ satisfies $\|v_j\| \le \frac{\sqrt{n}}{k}$. Assume the vectors v_j are ordered (which can be done using a number of steps depending on nk) so that 3.1 applies. Then $\|\sum_{j=0}^r v_j\| \le \frac{n^{3/2}}{k}$ whenever $1 \le r \le t$. Since for all p, $0 \le x_p \le \frac{1}{k}$, there is an increasing sequence $0 = t_0 < t_1 < t_2 < \dots < t_n = t$ so that $|\sum_{p=t_{i-1}+1}^{t_i} x_p - \alpha_i| < \frac{1}{k}$, $1 \le i \le n$. Set $X_j = \bigcup_{p=t_{j-1}+1}^{t_j} A_p$. Then $|\mu_i(X_j) - \alpha_j| = |\sum_{p=t_{j-1}+1}^{t_j}(x_p +$

[1] The authors are indebted to Paul Erdös for calling this theorem to our attention.

$\epsilon_{ip}) - \alpha_j| \leq \frac{1}{k} + \frac{2n^{3/2}}{k} = \frac{1}{k}(1 + 2n^{3/2})$. The proof is complete by choosing $k > \frac{1}{\epsilon}(1 + 2n^{3/2})$.

4 Cutting Envy Free Portions

The following theorem produces a division which is both envy free and near exact. It is proved by induction on the number of players, which introduces a complicating factor that as a piece is being shared in an envy free manner by a subset of the players, the remaining players who don't get a share of this piece must remain satisfied with the overall scheme. This is accomplished by proving an even stronger result which allows for observers who don't get any cake but must view the division as near exact. Although we may assume that the measures are normalized so that $\mu_i(X) = 1$ on the whole cake, we will not do so for the piece X in the theorem below in order to be able to apply the induction hypothesis to smaller pieces. We may assume however that X and consequently all subpieces of X have measure at most one.

Theorem 4.1. *Given any piece of cake X, any $\epsilon > 0$, players $P_1, P_2, \ldots, P_n$, $Q_1, \ldots, Q_m$, and any ratio of positive numbers $\alpha_1 : \alpha_2 : \cdots : \alpha_n$, there is a finite algorithm which assigns pieces $X_1, \ldots, X_n$ to the players $P_1, \ldots, P_n$ respectively which is*

(1) *envy free among $P_1, \ldots, P_n$ and*

(2) *ϵ-near exact among $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_m$.*

Proof: The theorem is trivial for $n = 1$ and any m, P_1 gets all of X. Assume the theorem is true any time less than n players share X.

Using Theorem 3.2 we can produce an ϵ-near exact partition $X_1, \ldots, X_n$ in the ratios $\alpha_1 : \alpha_2 : \ldots, : \alpha_n$ where all $n+m$ players $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ view the division as ϵ-near exact. (The proof of Theorem 3.2 is carried out in E^{n+m} but only n pieces are cut.) Moreover, by allowing P_1 to make a last series of cuts we may assume $\mu_1(X_i) = \alpha_i \mu_1(X)$ for $1 \leq i \leq n$. If all of the players $P_2, \ldots, P_n$ agree with P_1, that is if $\mu_j(X_i) = \alpha_i \mu_j(X)$ whenever $1 \leq i \leq n$ and $2 \leq j \leq n$, then the partition of X is also envy free and we are done.

Otherwise there is a piece A which has been cut, on which there is disagreement among some of $P_1, \ldots, P_n$. But cutting A into smaller pieces if necessary, we may assume that A is such that $\mu(A)/\mu(X) < \epsilon/4$ where μ without a subscript denotes the measure of any of the players $P_1, \ldots, P_n$ or $Q_1, \ldots, Q_m$ throughout the proof. Also, since there is some disagreement, there is a k, $1 \leq k < n$ such that:

$$\frac{\mu_1(A)}{\mu_1(X)} = \cdots = \frac{\mu_k(A)}{\mu_k(X)} = a > b = b_{k+1} = \frac{\mu_{k+1}(A)}{\mu_{k+1}(X)} \geq \cdots \geq \frac{\mu_n(A)}{\mu_n(X)} = b_n.$$

In particular we may also assume $a/(1-a) < \epsilon/4$.

We then divide the remaining cake $X - A$ into pieces B and C in a near exact manner, have $P_1, \dots, P_k$ share $A \cup B$ in an envy free manner, have $P_{k+1}, \dots, P_n$ share C in an envy free manner and maintain a close enough degree of near exactness to guarantee envy freeness among all of $P_1, \dots, P_n$ as well as ϵ-near exactness. This is possible since $P_1, \dots, P_k$ view A as larger than any of $P_{k+1}, \dots, P_n$.

To this end, scale the α_i so that $\alpha_1 + \cdots + \alpha_n = 1$, and let $\alpha_1 + \cdots + \alpha_k = \alpha$. If we set $m_1 = \min\{\alpha_i\}$ and $m_2 = \min\{\mu(X)\}$, $\Delta = \min\{\epsilon/4, m_1(a-b)/2\}$ we can choose ϵ_1 so that $0 < \epsilon_1 < \min\{m_2\epsilon/8, m_1 m_2 \Delta\ (1-a)/2\}$.

We will write $x = y + \Omega(\epsilon_1)$ only if $|x - y| < \epsilon$. In particular, if $x = y + \Omega(\epsilon_1)$ and $|\beta| \leq 1$ then $\beta x = \beta y + \Omega(\epsilon_1)$.

We now partition $X - A = B \cup C$ in ϵ_1-near exact ratio $\frac{\alpha-a}{1-a} + \Delta : \frac{1-\alpha}{1-a} - \Delta$ so we know;

$$\frac{\mu(B)}{\mu(B \cup C)} = \frac{\alpha - a}{1-a} + \Delta + \Omega(\epsilon_1)$$
$$\frac{\mu(C)}{\mu(B \cup C)} = \frac{1-\alpha}{1-a} - \Delta + \Omega(\epsilon_1).$$

Fianlly, by induction $P_1, \dots, P_k$ can share $A \cup B = X_1 \cup \cdots \cup X_k$ envy free and ϵ_1-near exact in the ratio $\alpha_1 : \dots : \alpha_k$, and $P_{k+1}, \dots, P_n$ can share $C = X_{k+1} \cup \cdots \cup X_n$ envy free and ϵ_1-near exact in the ratio $\alpha_{k+1} : \dots : \alpha_n$. Thus, $\frac{\mu(X_i)}{\mu(A \cup B)} = \frac{\alpha_i}{\alpha} + \Omega(\epsilon_1)$ for $1 \leq i \leq k$, and $\frac{\mu(X_h)}{\mu(C)} = \frac{\alpha_h}{1-\alpha} + \Omega(\epsilon_1)$ for $k < h \leq n$.

It must be shown that no player $P_1, \dots, P_k$ envys a piece given to any of the players $P_{k+1}, \dots, P_n$ or vise versa, and that the division is ϵ-near exact.

Whenever $1 \leq i,\ j \leq k < h,\ l \leq n$

$$\begin{aligned}
&\mu_i(X_j) \frac{\alpha_j}{\alpha} \mu_i(A \cup B) + \Omega(\epsilon_1) \\
&\quad = \frac{\alpha_j}{\alpha}[a\mu_i(X) + \mu_i(B \cup C)\left(\frac{\alpha - a}{1-a} + \Delta\right) + \Omega(\epsilon_1)] + \Omega(\epsilon_1) \\
&\quad = \frac{\alpha_j}{\alpha}[a\mu_i(X) + \mu_i(X)(1-a)\left(\frac{\alpha - a}{1-a} + \Delta\right)] + 2\Omega(\epsilon_1)
\end{aligned}$$

and so

$$\mu_i(X_j) - \alpha_j \mu_i(X) = \frac{\alpha_j}{\alpha} \mu_i(X)(1-a)\Delta + 2\Omega(\epsilon_1). \qquad (1)$$

Similarly,

$$\mu_i(X_h) - \alpha_h \mu_i(X) = -\frac{\alpha_h}{1-\alpha} \mu_i(X)(1-a)\Delta + 2\Omega(\epsilon_1), \qquad (2)$$

$$\mu_l(X_h) - \alpha_h\mu_l(X) = \alpha_h\mu_l(X)[\frac{a-b_l}{1-a} - \Delta\left(\frac{1-b_l}{1-\alpha}\right)] + 2\Omega(\epsilon_1), \quad (3)$$

$$\mu_l(X_j) - \alpha_j\mu_l(X) = -\frac{\alpha_j}{\alpha}\mu_l(X)\frac{(1-\alpha)(a-b_l)}{1-a} + \frac{\alpha_j}{\alpha}(1-b_l)\mu_l(X)\Delta + 2\Omega(\epsilon_1). \quad (4)$$

Also, for any measure μ, including those associated with the Q_i, and any j, $1 \le j \le n$:

$$\mu(X_j) - \alpha_j\mu(X) = (\Omega\left(\frac{a}{1-a}\right) + \Omega(\Delta))\mu(X) + \Omega(\mu(A)) + 2\Omega(\epsilon_1). \quad (5)$$

The proof is completed by verifying that with the choices of a, Δ and ϵ_1 above that the right hand sides of (1) and (3) are positive, (2) and (4) are negative making the division envy free, and the right hand side of (5) has magnatude less than $\epsilon\mu(X)$ guaranteeing ϵ-near exactness.

To accomplish envy free fair division in ratios $\alpha_1 : \ldots : \alpha_n$ among $P_1, \ldots, P_n$ apply the theorem with $m = 0$. Thus we have:

Theorem 4.2. *There is a finite algorithm which accomplishes envy free and ϵ-near exact division among n players in given ratios $\alpha_1 : \alpha_2 : \ldots : \alpha_n$.*

In [33] Woodall showed how to use a cut piece of cake on which two players disagree to accomplish a division of the cake in which each of n players felt they received strictly more than $1/n^{\text{th}}$ of the cake. We give a similar result in the spirit of the serendipity of disagreement. Contrast this result with that of Theorem 2.2 where we know there are two measures where the following hypothesis cannot be satisfied.

Theorem 4.3. *Given a piece A of a cake X such that $\mu_1(A) = 0$ and $\mu_2(A) = a$, $0 < a < 1$, there is a finite algorithm which produces exact fair division of X in the ratio $\alpha_1 : \alpha_2$, $\alpha_i > 0$, $\alpha_1 + \alpha_2 = 1$.*

Proof: Set aside A and divide $X - A = Y_1 \cup Y_2$ in ϵ-near exact portions in the ratio $\alpha_1 - 2\epsilon : \alpha_2 + 2\epsilon$ (where ϵ will be chosen sufficiently small as described below). Then

$$\left|\frac{\mu_1(Y_1)}{\mu_1(X-A)} - (\alpha_1 - 2\epsilon)\right| = |\mu_1(Y_1) - (\alpha_1 - 2\epsilon)| < \epsilon$$

and

$$\alpha_1 - 3\epsilon < \mu_1(Y_1) < \alpha_1 - \epsilon.$$

It follows that $\mu_1(Y_2) > \alpha_2 + \epsilon$. Also $\left|\frac{\mu_2(Y_2)}{\mu_2(X-A)} - (\alpha_2 + 2\epsilon)\right| < \epsilon$ and since $\mu_2(X - A) = 1 - a$ we have $(\alpha_2 + \epsilon)(1 - a) < \mu_2(Y_2) < (\alpha_2 + 3\epsilon)(1 - a)$. Thus for ϵ sufficiently small we know $6\epsilon < \mu_2(Y_2) < \alpha_2$.

Now P_2 can divide Y_2 into 2 or more pieces all of which have μ_2 measure between 3ϵ and 6ϵ. At least one of the pieces, say Y_2', satisfies $\mu_1(Y_2') \geq 3\epsilon$ since $\mu_1(Y_2) > \alpha_2 > \mu_2(Y_2)$. Let P_1 cut a piece Y_2'' from Y_2' so that $\mu_1(Y_1 \cup Y_2'') = \alpha_1$. Furthermore, $\alpha_2 > \mu_2(Y_2 - Y_2'') > (\alpha_2 + \epsilon)(1-a) - 6\epsilon = \alpha_2 - a(\epsilon + \alpha_2) - 5\epsilon > \alpha_2 - a$ for ϵ sufficiently small. This insures that P_2 can cut $A = A_1 \cup A_2$ so that $\mu_2((Y_2 - Y_2'') \cup A_1) = \alpha_2$ while $\mu_1(Y_1 \cup Y_2'' \cup A_2) = \alpha_1$. □

References

[1] N. Alon. Splitting necklaces, *Advances in Mathematics*, **63** (1987), 247–253.

[2] A.K. Austin, Sharing a cake, *Mathematical Gazette* **66**, No. 437 (1982), 212–215.

[3] A. Beck, Constructing a fair border, *Am. Math. Monthly* **94** (1987), 157–162.

[4] V. Bergström, Zwei Sätze über ebene Vectorpolygone, *Hamburgische Abhandlungen* **8** (1930), 205–219.

[5] S.J. Brams and A.D. Taylor. A note on envy-free cake division, *Journal of Combinatorial Theory A* (to appear).

[6] S.J. Brams and A.D. Taylor, Fair Division: Procedures for Allocating Divisible and Indivisible Goods, (to appear).

[7] S.J. Brams, A.D. Taylor and W.S. Zwicker, Old and new moving-knife schemes, *Mathematical Intelligencer* **17** No. 4 (1995), 30–35.

[8] S.J. Brams, A.D. Taylor and W.S. Zwicker, A moving-knife solution to the four-person envy-free cake division problem, *Proc. Am. Math. Soc.*, (to appear).

[9] S.J. Brams and A.D. Taylor, An envy free cake division protocol, *Am. Math. Monthly* **102** (1995), 9–18.

[10] L.E. Dubins and E.H. Spanier. How to cut a cake fairly, *Am. Math. Monthly,* **68** (1961), 1–17.

[11] S. Even and A. Paz, A note on cake cutting, *Discrete Applied Mathematics* **7** (1984), 285–296.

[12] A.M. Fink, A note on the fair division problem, *Mathematics Magazine* (1964), 341–342.

[13] D. Gale, Mathematical entertainments, *Mathematical Intelligencer* **15**No. 1 (1993), 48–52.

[14] M. Gardner, aha! Insight, *Scientific American Inc.*, W.H. Freeman and Company, 1978, 123–124.

[15] T.P. Hill, Determining a fair border, *Am. Math. Monthly* **90** (1983), 438–442.

[16] V.M. Kadets, M.I., Kadets, Rearrangements of Series in Banach Spaces, *Trans. of Math. Monographs*, (86), A.M.S., Providence 1991.

[17] B. Knaster, Sur le problème du partage pragmatique de H. Steinhaus, *Ann. de la Soc. Polonaise de Math.* **19** (1946), 228–231.

[18] H.W. Kuhn, On games of fair division, *Essays in Mathematical Economics in honor of Oskar Morgenstern* (Martin Shubik, ed.) Princeton University Press, 1967.

[19] A.A. Lewis, Aspects of fair division, *Rand Corporation* **P-6475** (1980).

[20] K. McAvaney, J. Robertson and W. Webb, Ramsey partitions of integers and fair division, *Combinatorica* **12**No. 2 (1992), 193–201.

[21] D. Olivastro, Preferred shares, *The Sciences* Mar/April (1992), 52–54.

[22] K. Rebman, How to get (at least) a fair share of the cake, *Mathematical Plums* (Ross Honsberged, ed.) The Mathematical Association of America (1979), 22–37.

[23] J.M. Robertson and W.A. Webb, Minimal number of cuts for fair division, *Ars Combinatoria* **31** (1991), 191–197.

[24] J.M. Robertson and W.A. Webb, Approximating fair division with a limited number of cuts, *Journal of Combinatorial Theory - Series A* (to appear).

[25] E. Singer, Extensions of the classical rule "Divide and Choose", *Southern Economic Journal* **28** (1962), 391–394.

[26] L. Steen ed., *For all practical purposes*, W.H. Freeman, New York, NY, 1988.

[27] H. Steinhaus. The problem of fair division. *Econometrica*, **16** (1948), 101–104.

[28] H. Steinhaus, Sur la division pragmatique, *Econometrica (supplement)* **17** (1949), 315–319.

[29] W. Stromquist. How to cut a cake fairly, *Am.Math. Monthly*, **87** (1980), 640–644.

[30] W. Stromquist and D.R. Woodall, Sets on which several measures agree, *Journal of Math. Analysis and Applications*, **108**, (1985), 241–248.

[31] W. Webb, A combinatorial algorithm to establish a fair border, *Europ. J. Combinatorics* **11** (1990), 301–304.

[32] D.R. Woodall. Dividing a cake fairly, *Journal of Math. Analysis and Applications*, **78** (1980), 233–247.

[33] D.R. Woodall. A note on the cake-division problem, *Journal of Combinatorial Theory*, Ser. A **42** (1986), 300–301.