A coloring problem on chordal rings

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ABSTRACT. A graph G having n vertices is called a chordal ring if its vertices can be arranged in a Hamiltonian cycle $0, 1, 2, \ldots, n-1$ and there is a proper divisor d of n such that for all vertices i and j, i adjacent to j in G if and only if $i+d$ is adjacent to $j+d$. (addition modulo n) We consider here the efficacy of coloring the vertices of such a graph by the greedy algorithm applied to the vertices in the order of their appearance on the cycle. For any positive integer n let Σ_n denote the set of all permutations of the set $\{1, 2, \ldots, n\}$ together with the adjacency relation $\sim$ defined as follows: for σ and τ in Σ_n, $\sigma \sim \tau \leftrightarrow$ there is an integer i such that $\sigma - i = \tau - i$. (here $\sigma - i$ denotes the permutation of length $n-1$ obtained by deleting i from σ.) In this paper we study some of the properties of the graph $(\Sigma_n, \sim)$. Two of the results obtained are the following: **Theorem** (i) $(\Sigma_n, \sim)$ is a chordal ring for every positive integer n; (ii) the chromatic number of Σ_5 is 5 but the greedy algorithm colors Σ_5 in 9 colors.

1 Introduction

The classes of graphs known as *chordal rings* and *generalized chordal rings* have been studied in [1] and [2] with respect to the construction of graphs having a large number of vertices for a given degree and diameter. In this paper we will consider a natural coloring problem associated with chordal rings. The *chromatic number* of a graph is an important parameter which occurs in many contexts and applications involving graphs such as scheduling. Since the determination of the chromatic number of a graph can be, in

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general, a large and difficult problem computationally (and is well-known to be NP-complete [4]), it is important to attempt to find efficient methods for such calculations for commonly occuring, significant classes of graphs. A common and useful way to color the vertices of a graph is to list the vertices in some natural linear order and then color them one by one in this order by the "greedy algorithm" in which a vertex is assigned the first available color not already assigned to any of its neighbors which precede it in the ordering. The efficacy of this method depends completely on the particular ordering chosen, and as is well-known, the number of colors used by this method may far exceed the actual chromatic number of the graph. As a result, one might try to find, for particular graphs or classes of graphs, "good" orderings of the vertices for which the greedy algorithm does color the graph in the minimum (or at least a small) number of colors. This point of view has been amplified and pursued by the work of many other authors; we cite [3], [7], and [8] as useful references in this context. Now, since the defining structure of chordal rings presents the vertices in a natural order (in a cycle), and because of the underlying symmetry with respect to this order, it is an obvious idea to color such graphs using the greedy method in conjunction with this natural order. The family of graphs studied in this paper shows that this method will not always color a chordal ring in the minimum number of colors. These graphs, which are permutation graphs, also have many other interesting properties, and are the main focus of this paper.

We will now set down the basic concepts, notation, and terminology to be employed in the sequel, for which our basic reference is [5].

Throughout this paper the term *graph* refers to a finite, undirected graph with no loops and no multiple edges. We often denote a graph G by writing $G = (V, E)$, where V denotes the set of vertices of G and E denotes the set of edges of G. If $W \subseteq V$, when we wish to regard W itself as a graph, it is always as an induced subgraph of G. Following [2] we say that a graph G is a *chordal ring* if the vertices of G can be ordered as $v_0, v_1, v_2, \dots, v_{n-1}$, where $n = |V|$, such that v_i is adjacent to v_{i+1} for all $i = 0, 1, 2, \dots, n-1$, and there is a proper divisor d of n such that, for all $0 \leq i, j \leq n-1$, v_i is adjacent to v_j in G if and only if v_{i+d} is adjacent to v_{j+d} in G. (here addition is modulo n) Such a number d will be called a *period* for G (with respect to the given ordering of the vertices of G).

Let $G = (V, E)$ be a graph and let k be a positive integer. A *k-coloring* of G is an assignment, to each of the vertices of G, of one of k given colors, so that no two adjacent vertices of G are assigned the same color. Such a coloring is equivalently described as a function $f : V \to \{1, 2, \dots, k\}$ having the property that, for any vertices v, w of G, if v is adjacent to w in G then $f(v) \neq f(w)$. In this latter formulation, $f(v)$ is the "color" assigned to v. When it is more appropriate. we will use letters like $a, b, c, \dots,$ instead of numbers, to denote colors. The smallest positive integer k for which G has a k-coloring is called the *chromatic number* of G and is denoted as usual by $\chi(G)$. A subset S of V is *clique* in G (also called a *complete subgraph* of G), if every pair of vertices of S are adjacent in G. The largest size of a clique in G is denoted by $\omega(G)$ and is called the *clique number* of G. The *clique cover number* of G is the smallest positive integer r such that V is the union of r cliques; this number is denoted by $k(G)$. A subset S of V is called an *independent set* in G (also called a *stable set* in G) if no two vertices of S are adjacent in G. The largest size of an independent set in G is denoted by $\alpha(G)$ and is called the *stability number* of G. Note that $\chi(G)$ can also be described as the smallest positive integer k such that V is the union of k independent sets. Two obvious inequalities concerning these notions are that $\omega(G) \leq \chi(G)$ and $|V| \leq \chi(G)\alpha(G)$.

In a graph $G = (V, E)$, for any vertex v of G, we let $N_G(v) = \{w \in V : w$ is adjacent to v in $G\}$. $N_G(v)$ is known as the set of *neighbors* of v in G, or as the *neighborhood* of v in G. The number $|N_G(v)|$ is called the *degree* of v in G and is denoted by $d_G(v)$. When the graph G is understood from the context, these will be denoted by $N(v)$ and $d(v)$ respectively. The largest of the numbers $d(v)$, for v in G, is denoted by $\Delta(G)$. If all the vertices of G have the same degree k, we say that the graph G is *regular* (of degree k). The well-known inequality $\chi(G) \leq \Delta(G) + 1$ is an elementary, useful connection between the concepts of degree and chromatic number.

Let l be a positive integer with $l \geq 3$. A *cycle* of length l in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_l$ of G such that v_i is adjacent for all $i = 1, 2, \ldots l-1$ and v_l is adjacent to v_1. If these are the only adjacencies among the vertices $v_1, v_2, \ldots, v_l$, the cycle is called a *chordless* cycle. We recall *Brooks' theorem*, which states that, for any connected graph G which is not equal to a complete graph or a chordless cycle having an odd number of vertices, we have $\chi(G) \leq \Delta(G)$.

We now recall the following simple procedure for coloring the vertices of a graph G. Suppose we have a given listing or ordering of the vertices of $G : v_1, v_2, \ldots, v_i, \ldots$ and that we have a given list of (sufficiently many) colors $1, 2, 3, \ldots$. We assign colors to the vertices of G as follows: color 1 is assigned to v_1. We then inductively assign to v_i the first color in the list of colors which has not already been assigned to any of the neighbors of v_i which precede v_i in the given listing of the vertices. This procedure is usually referred to as the *greedy algorithm* (with respect to the given ordering of the vertices).

Suppose G is a chordal ring having n vertices. Thus the vertices of G can be labeled as $v_0, v_1, v_2, \ldots, v_{n-1}$ such that v_i is adjacent to v_{i+1} for all i (addition mod n), and there is a proper divisor d of n such that, for any two vertices v_i and v_j of G, v_i is adjacent to v_j if, and only if, v_{i+d} is adjacent to v_{j+d} (again, additions are modulo n). Suppose we apply the above greedy algorithm to color the vertices of G in the cyclical order in which they are presented $v_0, v_1, v_2, \ldots, v_{n-1}$. Will this color G in the minimum number of colors? Under what conditions? If not, to what extent can the number of colors used by the greedy method differ from the chromatic number of G? We will see how the permutation graphs Σ_n, described in the next section, shed some light on these questions.

2 The graph Σ_n as a chordal ring

For any positive integer n we let Σ_n denote the set of all permutations of the set $\{1, 2, \ldots, n\}$. The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$. We think of a permutation of a set as simply being an ordered list, from left to right, of the elements of the set. If σ is a permutation of $[n]$ we will exhibit σ as $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. When there is no danger of ambiguity we will sometimes omit the brackets and commas in using this notation. (thus, for example, $(5, 3, 4, 1, 2)$ and 53412 denote the same permutation of the set $\{1, 2, 3, 4, 5\}$) The element σ_i is referred to as the *element in position i of* σ. If x and y are elements of $[n]$ and if $\sigma \in \Sigma_n$, we will sometimes write $x < y$ *in* σ to indicate that x occurs to the left of y in σ.

If $\sigma \in \Sigma_n$ and if $I \subset [n]$ then $\sigma - I$ denotes the permutation of the set $[n] - I$ obtained by deleting the elements of I from σ and considering the remaining elements, in the order they appear in σ. For example, if $\sigma = 536412$, then $\sigma - \{1, 6\} = 5342$. In the case of a singleton set, we will write $\sigma - x$ instead of $\sigma - \{x\}$. The notation $\sigma \mid S$ (the restriction of σ to S) denotes the permutation of the set S obtained by considering the elements of S in the order they appear within σ. (thus $\sigma \mid S = \sigma - ([n] - S)$.) If σ is a permutation of the set $[n]$ and if τ is a permutation of a subset S of $[n]$, we will write $\tau \prec \sigma$ if $\tau = \sigma \mid S$.

We will consider Σ_n as a graph with the adjacency relation $\sim$ defined as follows. Let $\sigma, \tau \in \Sigma_n$ with $\sigma \neq \tau$. We define $\sigma \sim \tau \leftrightarrow$ there exists an element x of $[n]$ such that $\sigma - x = \tau - x$. When we refer to *the graph* Σ_n, it is always with respect to this adjacency relation $\sim$.

The ordering of Σ_n relative to which $(\Sigma_n, \sim)$ is a chordal ring will now be described. This ordering has the property that each permutation can be obtained from its predecessor by an interchange of two consecutive symbols (including the first from the last). This ordering is due to Johnson [6] and Trotter [10], and is described and studied in detail in 1.1 of [9]. As in [9], we will refer to this ordering as the *Johnson-Trotter ordering of* Σ_n. The permutations of $[n]$ are listed as $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(n!-1)}$. This listing is described recursively, beginning with the list 12,21 of Σ_2, as follows. Suppose we have the list of all permutations of $[n]$: $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(n!-1)}$. To get the list for $[n+1]$, each $\sigma^{(i)}$ is replaced by a consecutive group of $n+1$ permutations of $[n+1]$, namely the ones obtained by inserting the number

$n+1$ into $\sigma^{(i)}$ into each of the possible $n+1$ positions. These insertions are done from left to right if i is odd and from right to left if i is even. This procedure is illustrated in Figure 1. The permutation which appears r'th in this list, $\sigma^{(r)}$, is said to have *rank* r. The rank of a permutation τ will be denoted by $\rho(\tau)$. Thus, for any $\tau \in \Sigma_n$, $\rho(\tau)$ is the unique integer $r \in \{0, 1, 2, \ldots, n! - 1\}$ such that $\tau = \sigma^{(r)}$. Note that, if $\tau \in \Sigma_n$, and if $\tau' = \tau - n$, then, assuming that the number n appears in position j of τ, we have that $\rho(\tau) = n\rho(\tau') + j - 1$ if $\rho(\tau')$ is odd, and $\rho(\tau) = n\rho(\tau') + n - j$ if $\rho(\tau')$ is even. These latter facts serve as the basis for a simple algorithm for finding the rank of a given permutation of $[n]$, and for finding (given n and r) the permutation of $[n]$ whose rank is r. (see [9])

We note that if $\sigma, \tau \in \Sigma_n$ and if σ can be obtained from τ by interchanging two consecutive symbols, then σ is adjacent to τ in $(\Sigma_n, \sim)$. Therefore the Johnson-Trotter ordering lists all the elements of Σ_n in a Hamilton cycle with respect to this graph.

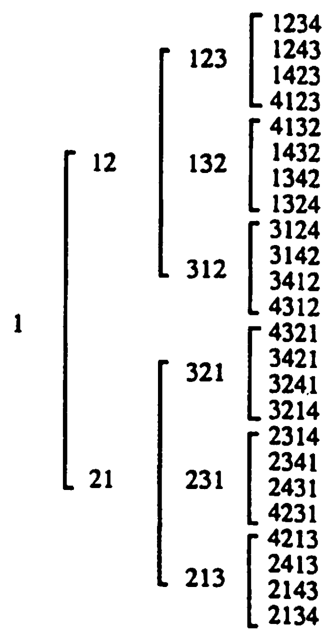

Figure 1: Generating permutations in the Johnson-Trotter order

Before proceeding further we would like to indicate diagrams for the graph Σ_n in the cases $n = 3$ and $n = 4$. These are shown in Figure 2 and Figure 3 below. In each case we have listed the vertices clockwise in a cycle in the Johnson-Trotter ordering.

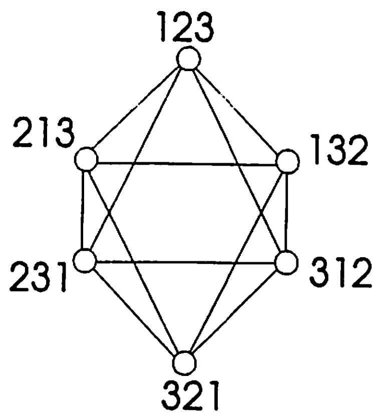

Figure 2: The graph Σ_3

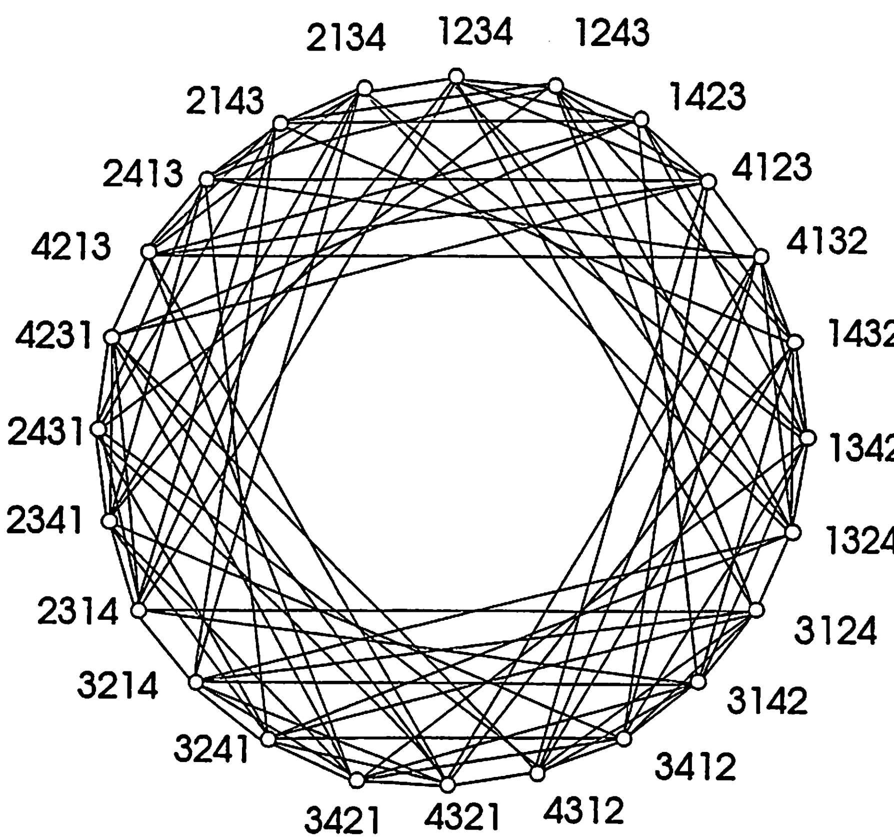

Figure 3: The graph Σ_4

For convenience, we will refer to the Johnson-Trotter ordering of Σ_n as the *JT-ordering* of Σ_n.

It is clear that, for any n, the graph Σ_n is *vertex transitive*; that is, for any $\sigma, \tau \in \Sigma_n$ there is an automorphism from Σ_n onto itself which maps σ to τ. Two other elementary properties of the graph Σ_n are contained in our first proposition.

Theorem 2.1. *Let n be any positive integer with $n \geq 3$. Then*

(i) for all $\sigma \in \Sigma_n$ we have $d(\sigma) = (n-1)^2$, and

(ii) $n \leq \omega(\Sigma_n) \leq \chi(\Sigma_n) \leq (n-1)^2$.

Proof: All the neighbors of σ can be described as follows. Let $i \in [n]$ and $j \in [n]$ such that $i \neq j$. Let $\sigma(i \to j)$ be the permutation obtained from σ by selecting the element which lies in position i of σ, and moving it so that it occupies position j, leaving the other elements of σ in the same relative order as they were in σ. Of the $n(n-1)$ permutations so described, it is easy to see that the only duplications that occur are that $\sigma(i \to i+1) = \sigma(i+1 \to i)$, so there are exactly $n(n-1)-(n-1)$ distinct neighbors of σ in Σ_n. This proves (i). The first inequality in (ii) follows from the fact that, for any $\tau \in \Sigma_{n-1}$, the set $\{\sigma \in \Sigma_n : \tau \prec \sigma\}$ is a clique in Σ_n of size n. The second inequality is standard and the third follows from (i) and Brooks' theorem. □

The property of the JT-ordering which enables us to prove that Σ_n is a chordal ring involves the following concept.

Definition: Let $\{\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(n!-1)}\}$ be an ordering of the elements of Σ_n, and let d be a positive integer such that $d \leq n$. This ordering is said to be *d-periodic* if, for all k, l such that $0 \leq k,\ l \leq n!-1$, and for all $i, j \in [n]$, if $\sigma_i^{(k)} = \sigma_j^{(l)}$, then $\sigma_i^{(k+d)} = \sigma_j^{(l+d)}$. (here addition is interpreted as modulo $n!$.)

Lemma 2.2. *Let n and d be positive integers such that d divides n and let $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(n!-1)}$ be an ordering of the elements of Σ_n which is d-periodic. Then d is a period for the graph $(\Sigma_n, \sim)$ with respect to this ordering.*

Proof: Suppose $\sigma^{(k)} \sim \sigma^{(l)}$ in Σ_n. There exists $x \in [n]$ such that $\sigma^{(k)} - x = \sigma^{(l)} - x$. Suppose that x occupies positions i and j in $\sigma^{(k)}$ and $\sigma^{(l)}$ respectively. We may assume that $i < j$. Then we must have that $\sigma_j^{(l+d)} = y$. The first three of the above equations for $k+d$ and $l+d$ then imply that $\sigma_r^{(k)} = \sigma_r^{(l)}$ for all $r < i$, $\sigma_r^{(k)} = \sigma_{r-1}^{(l)}$ for all r such that $i < r \leq j$, and $\sigma_r^{(k)} = \sigma_r^{(l)}$ for all $r > j$. Of course we also have that $\sigma_i^{(k)} = \sigma_j^{(l)}$ $(= x)$. Because the ordering is d-periodic these last four equations all hold with k

and l replaced by $k+d$ and $l+d$ respectively. Now, let $y = \sigma_i^{(k+d)}$. So we also have that $\sigma^{(k+d)} - y = \sigma^{(l+d)} - y$, and hence that $\sigma^{(k+d)} \sim \sigma^{(l+d)}$ in Σ_n. Since d divides n, the adjacency of $\sigma^{(k+d)}$ and $\sigma^{(l+d)}$ for all k and l also implies that of $\sigma^{(k)}$ and $\sigma^{(l)}$. □

Lemma 2.3. *Let $n \geq 3$, and let $d = \frac{n!}{3}$. Then the JT-ordering of Σ_n is d-periodic.*

Proof: We use induction on n. For $n = 3$, the JT-ordering is $\sigma^{(0)} = 123, \sigma^{(1)} = 132, \sigma^{(2)} = 312, \sigma^{(3)} = 321, \sigma^{(4)} = 231$, and $\sigma^{(5)} = 213$. To check that the JT-ordering is 2-periodic, we must check, for each of the 15 pairs of elements $\{k, l\}$ from the set $\{0, 1, 2, 3, 4, 5\}$, that whenever $i, j \in [3]$ and $\sigma_i^{(k)} = \sigma_j^{(l)}$, we also have that $\sigma_i^{(k+2)} = \sigma_j^{(l+2)}$. For any given k and l, there are three such occurrences i, j to check. To illustrate, consider $k = 2$, $l = 5$. We have that $\sigma_1^{(2)} = 3 = \sigma_3^{(5)}$, $\sigma_2^{(2)} = 1 = \sigma_2^{(5)}$ and $\sigma_3^{(2)} = 2 = \sigma_1^{(5)}$. We then note that (addition modulo 3!), $\sigma_1^{(2+2)} = 2 = \sigma_3^{(5+2)}$, $\sigma_2^{(2+2)} = 3 = \sigma_2^{(5+2)}$, and $\sigma_3^{(2+2)} = 1 = \sigma_1^{(5+2)}$. The other 14 cases are checked similarly.

Let $n \geq 4$, and assume that the lemma is true for $n-1$. We prove it for n. Let k and l be any two integers such that $0 \leq k,\ l \leq n! - 1$, and let i and j be any numbers in $[n]$ such that $\sigma_i^{(k)} = \sigma_j^{(l)}$. We must show that

$$\sigma_i^{(k+d)} = \sigma_j^{(l+d)}. \qquad (\bullet)$$

Note that, in the JT-ordering of Σ_n, the position that the number n occupies in each of the permutations $\sigma^{(0)}, \sigma^{(1)}, \sigma^{(2)}, \ldots$ repeats with a period of $2n$, as it sweeps alternately from right to left to right through all the positions from 1 through n. Since d is an even multiple of n, ($d = n(\frac{(n-1)!}{3})$), it follows that n occupies the same position in $\sigma^{(k)}$ that it does in $\sigma^{(k+d)}$, and the same position in $\sigma^{(l)}$ that it does in $\sigma^{(l+d)}$. In particular,if $\sigma_i^{(k)} = \sigma_j^{(l)} = n$, then $\sigma_i^{(k+d)} = \sigma_j^{(l+d)} = n$. So ($\bullet$) holds if $\sigma_i^{(k)} = \sigma_j^{(l)} = n$.

So we may assume that $\sigma_i^{(k)} = \sigma_j^{(l)} = x$, for some $x \neq n$. There are 4 possibilities for the positions of n relative to x in $\sigma^{(k)}$ and $\sigma^{(l)}$:

Case 1: $x < n$ in both $\sigma^{(k)}$ and $\sigma^{(l)}$.

Case 2: $x < n$ in $\sigma^{(k)}$ and $x > n$ in $\sigma^{(l)}$.

Case 3: $x > n$ in both $\sigma^{(k)}$ and $\sigma^{(l)}$.

Case 4: $x > n$ in $\sigma^{(k)}$ and $x < n$ in $\sigma^{(l)}$.

Since Cases 3 and 4 are similar to Cases 1 and 2, we will only verify that ($\bullet$) holds for the first two cases. For notational convenience, for any

$\sigma \in \Sigma_n$, we will let σ' denote the permutation $\sigma - n$ in Σ_{n-1}. Dividing by n we can write $k = k_1 n + r$, where $0 \leq r < n$, and $l = l_1 n + s$, where $0 \leq s < n$. Note that, because of the way the JT-ordering is defined, we have that $(\sigma^{(k)})' = \sigma^{(k_1)}$, where the latter denotes the element of Σ_{n-1} of rank k_1 in the JT-ordering of Σ_{n-1}. (This equation holds even when $k \geq n!$, where the integers k and k_1 are interpreted modulo $n!$ and $(n-1)!$ respectively, as the reader can easily verify). Similarly $(\sigma^{(l)})' = \sigma^{(l_1)}$ in Σ_{n-1}.

Case 1: Here $(\sigma^{l)})'_i = (\sigma^{(l)})'_j = x$. So, in Σ_{n-1} we have that $\sigma_i^{(k_1)} = \sigma_j^{(l_1)}$.. By the inductive hypothesis it follows that, in Σ_{n-1}, $\sigma_i^{(k_1+d_1)} = \sigma_j^{(l_1+d_1)}$, where $d_1 = \frac{(n-1)!}{3}$. Now $k + d = k + nd_1 = k_1 n + d_1 n + r = (k_1 + d_1)n + r$ and similarly $l + d = (l_1 + d_1)n + s$. It follows that the permutation $\sigma^{(k+d)}$ of Σ_n is obtained by inserting the number n into the permutation $\sigma^{(k_1+d_1)}$ of Σ_{n-1}, and similarly for $\sigma^{(l+d)}$. Since n has the same position in $\sigma^{(k+d)}$ and $\sigma^{(l+d)}$ as it does in $\sigma^{(k)}$ and $\sigma^{(l)}$ respectively, it follows that n occurs to the right of position i in $\sigma^{(k+d)}$, and to the right of position j in $\sigma^{(l+d)}$. Therefore $\sigma_i^{(k+d)} = \sigma_i^{(k_1+d_1)}$, and $\sigma_j^{(l+d)} = \sigma_j^{(l_1+d_1)}$. (where the right-hand sides refer to the JT-ordering in Σ_{n-1}). So the statement ($\bullet$) follows directly from the inductive hypothesis.

Case 2: In this case n is in a position to the right of position i in $\sigma^{(k)}$, and to the left of position j in $\sigma^{(l)}$. In this case, we have that, in Σ_{n-1}, $\sigma_i^{(k_1)} = \sigma_{j-1}^{(l_1)}$. So, by the inductive hypothesis, $\sigma_i^{(k_1+d_1)} = \sigma_{j-1}^{(l_1+d_1)}$. Now, as noted above, $\sigma^{(k+d)}$ (respectively $\sigma^{(l+d)}$) is equal to $\sigma^{(k_1+d_1)}(\sigma^{(l_1+d_1)})$ with n inserted. Since n has the same position in $\sigma^{(k+d)}$ as it does in $\sigma^{(k)}$, (to the right of position i), and the same position in $\sigma^{(l+d)}$ as it has in $\sigma^{(l)}$, (to the left of position j), it follows that $\sigma_i^{(k+d)} = \sigma_i^{(k_1+d_1)}$ and that $\sigma_j^{(l+d)} = \sigma_{j-1}^{(l_1+d_1)}$. Since the right-hand sides of these last two equations are equal by the inductive hypothesis, we see that ($\bullet$) holds in case 2. □

An immediate consequence of 2.2 and 2.3 is the fact that Σ_n is a chordal ring.

Theorem 2.4. *Let $n \geq 3$, and let $d = \frac{n!}{3}$. Then the graph Σ_n is a chordal ring of period d with respect to the JT-ordering.*

By examining Figures 2 and 3 it becomes apparent that Theorem 2.4 does not give the *smallest* period for Σ_n with respect to the JT-ordering when n is either 3 or 4. In fact 1 is a period for Σ_3, and 4 is a period for Σ_4, both of which are half the size of the period given by 2.4. It is of interest to note however, that 2.3 *does* give the smallest d for which the JT-ordering is d-periodic in these two cases. For example, in Σ_3, we have $\sigma_1^{(0)} = \sigma_1^{(1)}$ whereas $\sigma_1^{(1)} \neq \sigma_1^{(2)}$, showing that the ordering is not 1-periodic for $n = 3$. In particular, the converse of 2.2 above does not hold. For $n = 5$, Theorem 2.4 does give the smallest period for the graph Σ_5: In this case, $\frac{n!}{3} = 40$. Clearly the smallest positive period for Σ_5 must divide 40, so it is enough to check that neither 20 nor 8 is a period. We observe that, in Σ_5, $\sigma^{(0)} = 12345$ is adjacent to $\sigma^{(8)} = 12453$, but $\sigma^{(20)} = 41325$ is not adjacent to $\sigma^{(28)} = 14352$, and $\sigma^{(8)} = 12453$ is not adjacent to $\sigma^{(16)} = 45123$. We have been unable to determine the smallest period for general n.

3 The graph Σ_n as an example for the coloring problem

In this section we will see that the graph Σ_n furnishes us with with an example, in the case $n = 5$, of a chordal ring for which the greedy algorithm does not use the minimum number of colors. While the existence of such an example may not be so surprising, the fact that there is a graph having this property which is vertex transitive and which has the symmetry and structure of Σ_5 is somewhat more interesting. It is clear that any chordal ring on which the the greedy method does not color the vertices in a minimum number of colors must have chromatic number at least 3. Our example, Σ_5, has chromatic number 5.

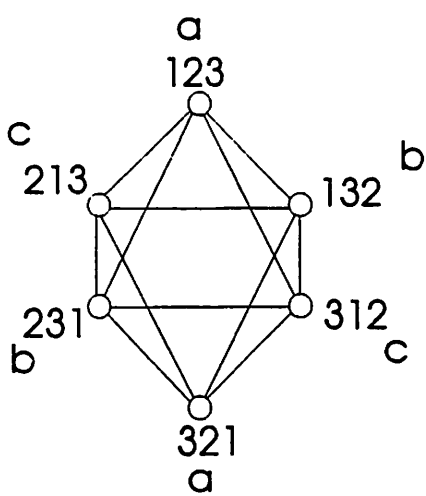

Figure 4:
The result of the greedy method of coloring Σ_3 in the JT-order

If we apply the greedy algorithm to color the vertices of the graphs Σ_3 and Σ_4, in the JT-ordering, we see that 3 and 4 colors respectively are required. The results are shown in Figures 4 and 5 where the colors used are denoted by a, b, c, d. Together with the first inequality in 2.1(ii) above, this shows that $\chi(\Sigma_3) = 3$ and $\chi(\Sigma_4) = 4$. The case $n = 5$ proves to be a more interesting one. It is a routine matter to determine that the greedy algorithm, in the case $n = 5$, requires 9 colors. We explicitly list the 16 neighbors of $\sigma^{(0)} = 12345$. Then, while generating the remaining elements $\sigma^{(k)}$ for $k = 1, 2, \ldots, 119$ of Σ_5 by the Johnson-Trotter method, we generate the neighbors of $\sigma^{(k)}$ by applying the natural automorphism of Σ_5 which takes $\sigma^{(0)}$ to $\sigma^{(k)}$ to the neighbors of $\sigma^{(0)}$. We then assign a color to $\sigma^{(k)}$. Denoting the colors used by $a, b, c, d, \ldots$, the reader can check that the greedy method colors the vertices as follows in the JT-order:

$a, b, c, d, e, a, b, d, e, c, d, a, c, e, b, c, d, a, b, e, a, b, c, e, d, c, a, b, e, f, b,$
$d, c, f, a, b, e, a, g, h, b, a, c, d, f, d, e, b, a, c, d, e, b, a, g, b, a, c, d, g, b, d,$
$a, f, e, b, d, c, a, f, a, e, b, f, c, a, b, c, d, g, d, e, a, b, c, a, d, c, f, b, f, b, c,$
$a, d, f, b, e, g, a, c, d, f, g, a, e, g, a, h, d, a, f, c, e, h, f, h, a, i, e.$

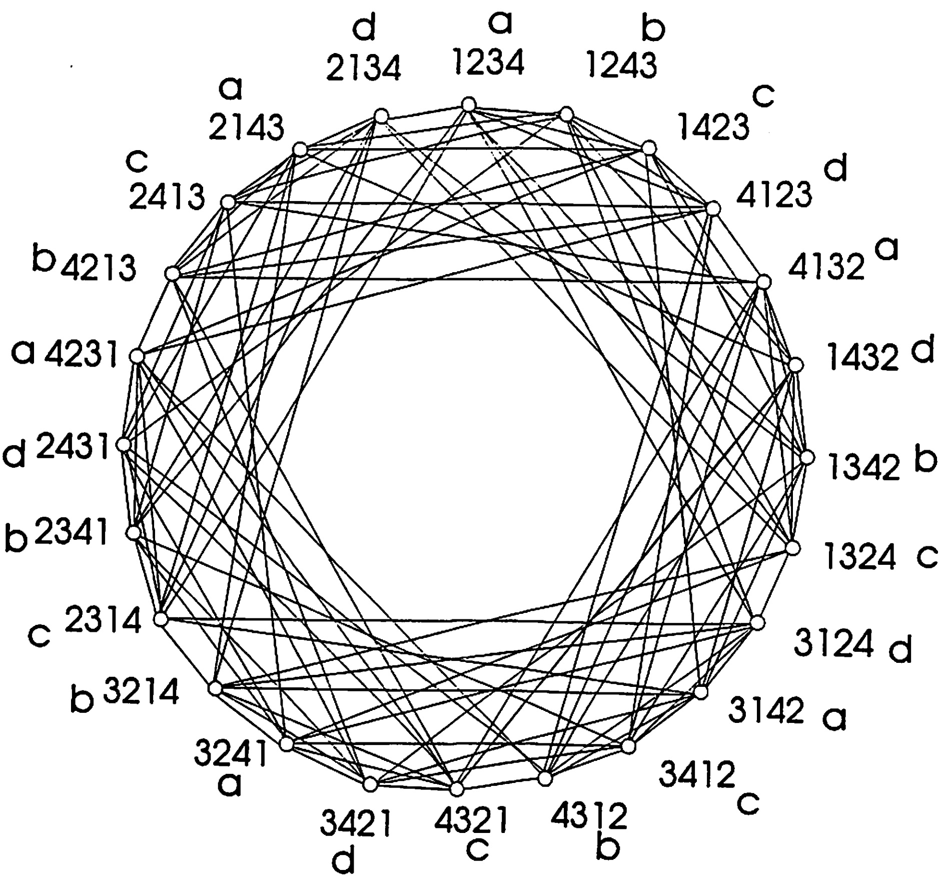

Figure 5:
The result of the greedy method of coloring Σ_4 in the JT-order

To show that Σ_5 is the desired example,we must show that it has chromatic number less than 9. We originally did this by applying the general deductions found in 3.1 - 3.4 below, which imply, among other things, that $\chi(\Sigma_5) \leq 8$. We subsequently found, by a rather ad hoc method, that, in fact, $\chi(\Sigma_5) = 5$. Because the deductions contained in 3.1 - 3.4 seem to be interesting in their own right, and because they provide a possible lead to finding an optimal way to color Σ_n in general, we have included this material (as well as the ad hoc coloring).

In the following lemma, we employ the following concept of homomorphism. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. A *homomorphism* from G_1 to G_2 is a function $f\colon V_1 \to V_2$ which has the property that, for all $v, w \in G$, if v is adjacent to w in G_1, then $f(v)$ is adjacent to $f(w)$ in G_2. (note that, in particular, adjacent vertices cannot have the same image)

Lemma 3.1. *Let $G = (V, E)$ be a graph, and let m and n be positive integers. Let $V_1, V_2, \ldots, V_m$ be disjoint subsets of V such that $V = \bigcup_{i=1}^{m} V_i$. Let $H = (W, F)$ be a graph with $\chi(H) = n \geq m$. Suppose that, for each $i = 1, 2, \ldots, m$ there is a homomorphism $f_i\colon V_i \to H$ such that, for all $i, j \leq m$ with $i \neq j$, we have, for any $x \in V_i$ and for any $y \in V_j$, if x is adjacent to y in G then $f_i(x) = f_j(y)$. Then $\chi(G) \leq n$.*

Proof: It is no loss of generality to assume that $n = m$. Otherwise, we could add $n - m$ isolated vertices $v_{m+1}, v_{m+2}, \ldots, v_n$ to G to form a larger graph G', and let $V_i = \{v_i\}$ for $i = m+1, \ldots, n$. Let f_i be any map from V_i to H for $i > m$. Then $\chi(G) \leq \chi(G')$, so the result applied to G' implies the result for G.

Let $c\colon W \to \{0, 1, 2, \ldots, n-1\}$ be an n-coloring of H. Addition in the following is modulo n. We define a function $\gamma\colon V \to \{0, 1, 2, \ldots, n-1\}$ as follows: $\gamma(x) = c(f_i(x)) + i$ if $x \in V_i$.

Claim: γ is a proper coloring of G:

Let x and y be adjacent vertices of G. If there is a set V_i such that $\{x, y\} \subseteq V_i$, then $f_i(x)$ is adjacent to $f_i(y)$ in H and so $c(f_i(x)) \neq c(f_i(y))$. Hence $c(f_i(x)) + i \neq c(f_i(y)) + i$. Otherwise, there are distinct i and j such that $x \in V_i$ and $y \in V_j$. This implies that $f_i(x) = f_j(y)$, and hence that $c(f_i(x)) = c(f_j(y))$. Since $i \neq j$, the latter equation implies that $\gamma(x) \neq \gamma(y)$. □

While it is not directly related to our main object in this section, we would like to point out an interesting consequence of 3.1.

Corollary 3.2. *Let $G = (V, E)$ be a graph. Suppose that there are disjoint subsets A and B of V such that $V = A \cup B$, and that there exists a homomorphism $f\colon A \to B$ such that, for all $x \in A$ and $y \in B$, if x is adjacent to y, then $f(x) = y$. Then $\chi(G) = \chi(B)$.*

The assumptions in 3.2 obviously imply that, for each $x \in A$ there is at most one vertex $y \in B$ which is adjacent to x, namely $y = f(x)$. We note, however, that even when this latter condition holds for B relative to A as well as for A relative to B, we can have $\chi(G)$ larger than $\chi(A)$ and $\chi(B)$. An example of this is shown in Figure 6, in which we can let $A = \{a, b, c\}$ and $B = \{d, e, f\}$.

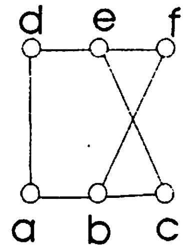

Figure 6

The consequence of 3.1 which we will employ is the following.

Corollary 3.3. *Let $n \geq 3$. For $i = 1, 2, \ldots, n$, let $V_i = \{\sigma \in \Sigma_n : n$ has position i in $\sigma\}$. Then $\chi(\bigcup_{i\ \text{even}} V_i) \leq \chi(\Sigma_{n-1})$, and $\chi(\bigcup_{i\ \text{odd}} V_i) \leq \chi(\Sigma_{n-1})$.*

Proof: Define $f_i : V_i \to \Sigma_{n-1}$ by $f_i(\sigma) = \sigma - n$. Clearly f_i is a homomorphism. (note that $\sigma \neq \tau \to \sigma - n \neq \tau - n$ when n has the same position in σ and τ.)

Now, if $| i - j | \geq 2$, and if $\sigma \in V_i$ and $\tau \in V_j$, and if $\sigma \sim \tau$ in Σ_n, then we must have $\sigma - n = \tau - n$. This holds because, under the assumptions on i and j, the position of n in σ differs by at least two places from its position in τ. Deleting any one number, other than n, from both σ and τ can bring the position of the n in σ at most one place closer to the position of the n in τ. Since there is an $x \in [n]$ for which $\sigma - x = \tau - x$, this x must be n. This implies that $f_i(\sigma) = f_j(\tau)$. Since $\lceil \frac{n}{2} \rceil \leq n - 1$, we can apply Lemma 3.1 above to each of the subgraphs $G_1 = \bigcup_{i\ \text{even}} V_i$, and $G_2 = \bigcup_{i\ \text{odd}} V_i$ of Σ_n. □

One could attempt to somehow 'merge' a coloring of the even V_i's with the odd V_i's in conjunction with 3.3 above to obtain an estimate of $\chi(\Sigma_n)$. For our purposes, we can get away with allowing totally different colors on the two.

Corollary 3.4. *Let $n \geq 3$. Then $\chi(\Sigma_n) \leq 2\chi(\Sigma_{n-1})$.*

Note that Corollary 3.4 will not lead to good estimates for $\chi(\Sigma_n)$, because it implies an exponential bound, as compared with the simple polynomial bound found in 2.1(ii) above. However, it does enable us to deduce that the graph Σ_5 provides us with our desired example.

Example 3.5: $(\Sigma_5, \sim)$ *is a chordal ring in the JT-ordering whose chromatic number is less than the number of colors required by the greedy algorithm applied to the vertices in that order.*

Proof: We have seen above that Σ_5 is a chordal ring in the JT-order, and that the greedy method uses 9 colors to color Σ_5 in the JT-order. Since, as shown at the beginning of this section, $\chi(\Sigma_4) = 4$, it follows from 3.4 that $\chi(\Sigma_5) \leq 8$. □

Although it is not necessary to determine $\chi(\Sigma_5)$ any more exactly for the purpose of Example 3.5, we have subsequently determined that, in fact $\chi(\Sigma_5) = 5$. Since we feel that the determination of $\chi(\Sigma_n)$ is an interesting problem in its own right, this fact deserves some additional comment. Rather than simply exhibit a 5-coloring of Σ_5, we would like to indicate how this coloring was obtained. While we have been unable to generalize our approach to apply to arbitrary n, certain elements of the method hold some promise toward finding $\chi(\Sigma_n)$ for arbitrary n.

We employ the notation V_i as in the statement of 3.3 above. We proceed in Σ_5 as follows. We consider the subgraph $H = V_1 \cup V_2$. We color H by the greedy algorithm using the inherited JT-ordering on H as the ordering of its vertices. This turns out to require 5 colors. Let us denote by γ the resulting coloring. We next attempt to extend this to a coloring of the subgraph $H_1 = V_1 \cup V_2 \cup V_4 \cup V_5$. Since the mapping $\sigma \to \sigma^{opp}$ is an automorphism of the graph Σ_5, (where σ^{opp} denotes the permutation σ written in reverse order), a natural way to do this is to assign to each $\sigma \in V_4 \cup V_5$ the color $\gamma(\sigma^{opp})$. This turns out to *be* a proper coloring because the greedy coloring of H so happens to have the property that $\gamma(5,a,b,c,d) \neq \gamma(5,d,c,b,a)$, and that $\gamma(5,a,b,c,d) \neq \gamma(d,5,c,b,a)$, and that $\gamma(a,5,b,c,d) \neq \gamma(d,5,c,b,a)$. So this produces a 5-coloring of H_1, which we also refer to as γ. Finally, we attempt to extend the coloring to all of $\Sigma_5 = H_1 \cup V_3$, by assigning colors to the vertices of V_3, in a manner compatible with the coloring of γ on H_1. We assign colors to the vertices of V_3 in the inherited JT-ordering in a greedy way, choosing the first color possible at each vertex σ which would provide a proper coloring of $H_1 \cup \{\tau \in V_3 : \tau \leq \sigma\}$, where $\leq$ denotes the JT-ordering. This turns out to require just 2 colors for the vertices of V_3, the last 2 colors of the 5 used in the greedy method on H. So this produces a 5-coloring of Σ_5.

Theorem 3.6. $\chi(\Sigma_5) = 5$.

Proof: We know by 2.1(ii) that $\chi(\Sigma_5) \geq 5$. The method outlined in the preceding discussion produces the following 5-coloring of the vertices of Σ_5, where the colors are denoted by a, b, c, d, e and the vertices are listed in the

JT-ordering:

$e, c, d, a, b, a, b, e, c, d, a, b, e, c, d, e, c, d, a, b, d, c, e, a, b, c, a, d, b, e, c, a,$
$d, b, e, d, c, e, a, b, a, b, d, c, e, a, b, e, c, d, e, b, d, a, c, d, b, e, a, c, b, a, d, c,$
$e, d, c, e, b, a, d, c, e, b, a, b, a, d, c, e, b, a, e, c, d, e, b, d, a, c, e, b, d, a, c, b,$
$a, e, c, d, e, c, d, b, a, d, c, e, b, a, c, a, d, b, e, c, a, e, b, d$

□

The information in this section suggests an interesting conjecture, namely that $\chi(\Sigma_n) = n$, for all $n \geq 3$. We have, as yet, been unable to settle this conjecture. We will return to this problem in the next section.

4 Some other properties of the graph Σ_n

We will show that the largest size of a clique in Σ_n is n. First we will establish a lemma which is useful in discussing adjacency in Σ_n. if $\sigma, \tau \in \Sigma_n$, and if $\{x, y\}$ is a pair of elements in $[n]$, we say that $\{x, y\}$ *is opposite in* σ *and* τ if $x < y$ in one of the two permutations and $y < x$ in the other. Similarly, if $\{x_1, x_2, x_3\}$ is a triple of elements of $[n]$, we say that $\{x_1, x_2, x_3\}$ *is opposite in* σ *and* τ if, for some permutation i_1, i_2, i_3 of 1,2,3 we have $x_{i_1} < x_{i_2} < x_{i_3}$ in one of the two permutations and $x_{i_3} < x_{i_2} < x_{i_1}$ in the other.

Lemma 4.1. *Let $n \geq 3$, and let $\sigma, \tau \in \Sigma_n$ with $\sigma \neq \tau$. Then σ is* not *adjacent to τ in Σ_n if, and only if, one of the following two conditions holds: Either there are two disjoint pairs $\{x_1, y_1\}$, $\{x_2, y_2\}$ in $[n]$ both of which are opposite in σ and τ, or there is a triple of elements $\{x, y, z\}$ in $[n]$ which is opposite in σ and τ.*

Proof: To prove the 'if' implication, note that if either of the two stated conditions holds, then, for any $i \in [n]$, σ and τ cannot agree on the set $[n] - \{i\}$, because, in the first case, one of the two pairs does not involve i, and in the second, two of the three elements are different from i and are opposite in the two permutations.

Conversely, suppose that σ is distinct from and not adjacent to τ. There are elements x, y such that $x < y$ in σ and $y < x$ in τ. Since $\sigma - x \neq \tau - x$, there are two elements u, v in $[n] - \{x\}$ such that $\{u, v\}$ is opposite in σ and τ. If $y \notin \{u, v\}$, then the first condition holds. So let us suppose that $y = v$. Now, if $y < u$ in σ, then we would have $x < y < u$ in σ, and $u < y < x$ in τ, so the second condition would hold. So we may as well assume that $u < y$ in σ. So both u and x precede y in σ, and both follow y in τ. If x and u are in the opposite order in σ and τ, then again we would have a triple satisfying the second condition. So let us suppose that $x < u < y$ in σ and $y < x < u$ in τ. Since σ and τ are not adjacent, we have $\sigma - y \neq \tau - y$.

Therefore there exist two elements a, b in $[n] - \{y\}$ such that $a < b$ in σ and $b < a$ in τ. At least one of the elements x, u does not belong to the set $\{a, b\}$. This element, together with y, and the pair $\{a, b\}$, are two disjoint pairs which satisfy the first condition in the statement of the lemma. □

If $\sigma, \tau \in \Sigma_n$ and $\sigma \sim \tau$, we will say, for an element $x \in [n]$, that x *witnesses* $\sigma \sim \tau$ if $\sigma - x = \tau - x$. (by definition, some such x exists.)

Theorem 4.2. *Let $n \geq 3$. Then $\omega(\Sigma_n) = n$.*

Proof: Because of the first inequality in 2.1(ii) above, what has to be shown is that no clique in Σ_n has more than n elements. We prove this by induction on n. For $n = 3$ the result is straightforward to check. (it also follows from the fact that $\chi(\Sigma_3) = 3$) So let us assume that $n \geq 4$, and that the theorem is true for $n - 1$. For the sake of contradiction, let K be a clique in Σ_n with $|K| > n$.

We first show that there can be at most 2 elements which occur as the first entry in any of the permutations which belong to the set K. Otherwise, (without loss of generality), there are $\sigma_1, \sigma_2, \sigma_3$ in K whose first entries are 1, 2,and 3 respectively. Now, the adjacency $\sigma_1 \sim \sigma_2$ has to be witnessed by either 1 or 2. (for any other x, $\sigma_1 - x$ and $\sigma_2 - x$ have different first component) We may assume that it is witnessed by 2. This implies that 1 is the second entry of σ_2. This implies that 2 cannot witness the adjacency between σ_2 and σ_3. Therefore 3 does. This implies that the first three entries of σ_3 are 321. Therefore $\sigma_1 \sim \sigma_3$ must be witnessed by 1. Therefore the first three entries of σ_1 are 132. Also, because 2 is a witness for $\sigma_1 \sim \sigma_2$, 3 must immediately follow 1 in σ_2, so its first three entries are 213. Now, let τ be any other element of K. Since $\{1, 2, 3\}$ cannot be opposite in τ and any of the σ_i (by Lemma 4.1), τ must agree with one of the σ_i on the set $\{1, 2, 3\}$. We may assume that $\tau \mid \{1, 2, 3\} = 132$. Therefore $\tau \sim \sigma_2$ must be witnessed by 2. So the first two entries in τ are 13. Therefore $\tau \sim \sigma_3$ must be witnessed by 1. This implies that the third entry of τ is 2. So we have $\sigma_1 = 132\alpha_1$, $\tau = 132\alpha_2$ where α_1 and α_2 are two different permutations of the set $\{4, , 5, \dots, n\}$. Hence there are elements $a, b > 3$ such that $\{a, b\}$ is opposite in α_1 and α_2. Since $\sigma_1 - 2 = \sigma_2 - 2$, $\{a, b\}$ is also opposite in τ and σ_2. But then there are two disjoint pairs opposite in τ and σ_2, which, by Lemma 4.1, is contrary to $\tau \sim \sigma_2$.

So there are at most 2 elements which occur as the first entry of any of the elements of K. Since deleting the first entry i, from all the members of K which have i as their first entry, produces (up to labeling), a clique in Σ_{n-1}, the inductive hypothesis implies that there are, in fact, two such first entries, and that each is the first entry of at least 2 members of K. Let us suppose that these two 'first entries' are 1 and 2. Let $\sigma_1, \sigma_2, \dots, \sigma_k$ denote the members of K whose first entry is 1, and let $\tau_1, \tau_2, \dots, \tau_l$ denote the members of K whose first entry is 2. Note that each of the adjacencies

$\sigma_i \sim \tau_j$ must be witnessed by either 1 or 2. In particular, each σ_i agrees with each τ_j on the set $[n] - \{1,2\}$. Therefore, *all* the members of K agree in the way they order the set $[n] - \{1,2\}$. Without loss of generality, suppose that 1 witnesses $\sigma_1 \sim \tau_1$. Thus, there is a permutation α on the set $[n] - \{1,2\}$ such that $\sigma_1 = 12\alpha$. All the members of K agree with α on the set $[n] - \{1,2\}$. Let x be the first entry of α. Since $\sigma_2 \neq \sigma_1$, we must have $x < 2$ in σ_2. Therefore $1 < x < 2$ in σ_2. Since $\tau_1 \neq \tau_2$, there must be a pair of elements which are opposite in these two permutations. This pair cannot contain 2, and must contain 1, since all the τ_j agree on $[n] - \{1,2\}$. So there is an entry y in α such that the pair $\{1,y\}$ is opposite in τ_1 and τ_2. In one of these latter two, we have $y < 1$, and hence $x < 1$, since x is the first entry of α. In this particular τ_j, we have $2 < x < 1$. But this means that the triple $\{1,x,2\}$ is opposite in σ_2 and τ_j, contrary (by lemma 4.1) to their adjacency. □

Corollary 4.3. $k(\Sigma_n) = (n-1)!$

Proof: The collection of cliques $K_\tau = \{\sigma \in \Sigma_n : \tau \prec \sigma\}$, for $\tau \in \Sigma_{n-1}$, covers Σ_n. Since, by 4.2, no clique in Σ_n has more than n elements, at least $(n-1)!$ cliques must be present in any cover of Σ_n by cliques. □

The question raised at the end of section 3 above, whether the equality $\chi(\Sigma_n) = n$ holds for arbitrary n, can now be seen to be equivalent to asking whether $\chi(\Sigma_n) = \omega(\Sigma_n)$. We recall that a graph G is called a *perfect* graph if $\chi(H) = \omega(H)$ for every induced subgraph H of G. We can easily see that, while Σ_3 is perfect, Σ_n is not perfect for $n > 3$. We can see this by noting that Σ_4 contains a chordless 5-cycle (and hence so does Σ_n for all $n > 3$); one such cycle being 1423, 3142, 4312, 2431, 2143. (the author thanks James Currie for his rapid sighting of an odd cycle from the diagram of Σ_4 in Figure 3 above.) Some authors have used the term *good* to refer to a graph G for which $\chi(G) = \omega(G)$ (and so a graph is perfect when all of its subgraphs are good). Using this term, our question is whether or not Σ_n is a good graph for all $n \geq 3$. While we have been unable to answer this question,we have been able to show, at least, that the graph Σ_n,is *locally good*; that is $\chi(N(v)) = \omega(N(v))$ holds for the neighborhood $N(v)$ of every vertex v of the graph. Before proving this, we would like to note that one consequence of the equality $\chi(\Sigma_n) = n$ is that $\alpha(\Sigma_n) = (n-1)!$: If Σ_n is the union of n independent sets, then at least one of them has at least $(n-1)!$ elements, and, since Σ_n is the union of $(n-1)!$ cliques, no independent set can be any larger than this. Unfortunately, we have not been able to settle this question either. Does Σ_n contain an independent set of size $(n-1)!$ for all $n \geq 3$? In more explicit terms, this asks for a set S of permutations of $[n]$, with $|S| = (n-1)!$, such that, for all $i \leq n$, $\{\sigma - i : \sigma \in S\}$ is the set of all permutations of the set $[n] - \{i\}$.

Theorem 4.4. *Let $n \geq 3$. Then $\chi(N(\sigma)) = n-1 = \omega(N(\sigma))$ for all $\sigma \in \Sigma_n$.*

Proof: Note that, since Σ_n is vertex transitive, it is really enough to show this for any one σ in Σ_n. The second equality follows from the fact that $\omega(\Sigma_n) = n$.

Let $\sigma \in \Sigma_n$. As in 2.1 above, for any i, j with $1 \leq i, j \leq n$ and $i \neq j$, we let $\sigma(i \to j)$ denote the permutation obtained from σ by selecting the element which lies in position i of σ and moving it so that it occupies position j, leaving the other $n-1$ elements in the same order among themselves. As noted in 2.1 above, we have $\sigma(i \to i+1) = \sigma(i+1 \to i)$, and aside from this, the elements $\sigma(i \to j)$ are all distinct. Together they are all the neighbors of σ in Σ_n. It is straightforward to check (with the help of Lemma 4.1) that all the adjacencies among these neighbors can be described as follows: $\sigma(i \to i+1) \sim \sigma(j \to j+1) \leftrightarrow |i-j| \leq 1$, if $|i-j| > 1$, then $\sigma(i \to j) \sim \sigma(k \to k+1) \leftrightarrow i = k$ or $i = k+1$, and if (i,j), (k,l) are distinct pairs such that $|i-j| > 1$ and $|k-l| > 1$, then $\sigma(i \to j) \sim \sigma(k \to l) \leftrightarrow$ either $i = k$ or $\{(i,j),(k,l)\} = \{(a, a+2), (a+2, a)\}$ for some $a \leq n-2$.

Now, for each $j = 1, 2, \ldots, n-1$, let

$$S_j = \{\sigma(i \to j+1) : 1 \leq i \leq j\} \cup \{\sigma(i \to j) : j+2 \leq i \leq n\}.$$

From the description of the adjacencies, it is clear that each of the sets S_j is an independent set in Σ_n. Since $N(\sigma) = \bigcup_{j=1}^{n-1} S_j$, this shows that $\chi(N(\sigma)) \leq n-1$. Since $N(\sigma)$ contains a clique of size $n-1$, the result follows. □

It is also possible to determine the *local* stability number for the graph Σ_n, and this leads to our last result.

Theorem 4.5. *Let $n \geq 4$, and let $\sigma \in \Sigma_n$. Then*

(i) $\alpha(N(\sigma)) = n$ and

(ii) $k(N(\sigma)) = n$.

Proof: Select any number k with $2 \leq k \leq n-2$ and keep it fixed. Let $L = \{1, 2, \ldots, k\}$, and let $R = \{k+1, k+2, \ldots, n\}$. For $i \in L$, let $\tau_i = \sigma(i \to n)$, and for $i \in R$, let $\tau_i = \sigma(i \to 1)$. We have used the same notation as in 4.4 above. We will show that the set $T = \{\tau_i : i = 1, 2, \ldots, n\}$ is independent in Σ_n: If i_1 and i_2 both belong to L and $i_1 < i_2$, then we have, in τ_{i_1}, that $\sigma_{i_2} < x < \sigma_{i_1}$ where x is any element of R, whereas in τ_{i_2} we have $\sigma_{i_1} < x < \sigma_{i_2}$. The non-adjancecy of τ_{i_1} and τ_{i_2} follows from 4.1 above. A similar argument applies when both i_1 and i_2 are in R. Now, if $i_1 \in L$ and $i_2 \in R$, let x and y be elements of L and R respectively, which are distinct from i_1 and i_2. We then observe that the pairs $\{x, i_2\}$ and $\{y, i_1\}$ are both opposite in τ_{i_1} and τ_{i_2}. Therefore, by 4.1, τ_{i_1} and τ_{i_2} are not adjacent. We see that T is an independent set in $N(\sigma)$ of size n.

The preceding argument implies that $n \leq \alpha(N(\sigma))$. Since $\alpha(N(\sigma)) \leq k(N(\sigma))$, the proof of 4.5 will be complete once we show that $k(N(\sigma)) \leq n$, in other words, that $N(\sigma)$ is equal to the union of n cliques. This is clear: for each $i \in [n]$, let $K_i = \{\tau \in \Sigma_n : \sigma - i \prec \tau\}$. Then $\{K_i : i \leq n\}$ is a suitable family of cliques. □

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