Is There a Matroid Theory of Signed Graph Embedding?

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Abstract. A graph with signed edges is *orientation embedded* in a surface when it is topologically embedded and a polygon preserves or reverses orientation depending on whether its sign product is positive or negative. We study orientation-embedding analogs of three facts about unsigned graph embedding: planarity is equivalent to having cographic polygon matroid, the polygon matroid of a graph determines the surfaces in which it embeds, and contraction preserves embeddability of a graph in a surface.

We treat three matroids of a signed graph. Our main results: For a signed graph which is orientation embeddable in the projective plane, the bias and lift matroids (which coincide) are cographic. Neither the bias nor lift nor complete lift matroid determines the surfaces in which a signed graph orientation embeds. Of the two associated contractions of signed edges, the bias contraction does not preserve orientation embeddability but the lift contraction does. Thus the signed graphs which orientation embed in a particular surface are characterizable by forbidden lift minors.

1 Introduction

A *signed graph* $\Sigma = (\Gamma, \sigma)$ is a graph Γ together with a mapping $\sigma\colon E(\Gamma) \to \{+, -\}$ that labels each edge with a sign. A graph Γ embedded in a surface

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S is naturally signed in the following way. First each polygon C (the edge set of a simple closed path) receives a sign, + or − according as going once around C preserves or reverses orientation. Then σ is chosen so that the product of edge signs in each polygon matches its orientation-induced sign. We say the resulting signed graph is *orientation embedded* in S. There is a close relationship between the surface embeddability of an ordinary, unsigned graph Γ and its polygon matroid $G(\Gamma)$. In this article we discuss how that relationship does and does not extend to signed graphs.

The facts we wish to extend are these:

(G1) Determinacy. The polygon matroid determines exactly which surfaces a graph embeds in (see [17]; [10] for planarity).

(G2) Monotonicity. If Γ embeds in a surface S, then every minor of Γ also embeds in S. Explanation: A *minor* of a signed or unsigned graph is any result of successively deleting edges and vertices and contracting edges. A minor of a matroid is any result of deleting and contracting points. For unsigned graphs, graph and matroid minors are compatible: the matroid of a minor of Γ is the corresponding matroid minor of $G(\Gamma)$.

(G3) Duality. The matroid $G(\Gamma)$ is cographic (dual to the matroid of a graph) if and only if Γ is planar [10].

By (G2) there exists a list of graphs which are *forbidden (graph) minors* for S: a graph embeds in S if and only if none of its minors is in the forbidden list (up to isomorphism) and the list is minimal with respect to this property. The list of forbidden minors has recently been found to be finite for every surface [1, 2, 4]. In view of (G1), Γ embeds in S if and only if no minor of $G(\Gamma)$ is the matroid of a forbidden graph minor for S. It follows that finiteness of the list of forbidden minors for each surface is a consequence of Robertson's conjecture that binary matroids are well-quasi-ordered: that means that if matroids are quasi-ordered by $M_1 \leq M_2$ if M_1 is isomorphic to a minor of M_2 (the *minor ordering*), then there is no infinite antichain of binary matroids.

In seeking signed analogs of these facts the first problem is to decide which matroid to use. Certainly there are many matroids on the edge set $E(\Sigma)$ of a signed graph Σ, but two, the 'bias' and 'lift' matroids, seem especially close in spirit to the polygon matroid. These matroids were originally motivated by geometry (see [12, Section 8] and [14]); here we apply them to orientation embedding.

The *bias matroid* $G(\Sigma)$ of a signed graph Σ can be defined in terms of its circuits as follows [12, 13]. First, a polygon is *positive* or *balanced* if its sign product is positive. A *handcuff* is (the edge set of) a graph consisting of two polygons meeting at a single vertex or two vertex-disjoint polygons and a simple path meeting each polygon at an endpoint and nowhere else. A circuit of $G(\Sigma)$is a balanced polygon or a handcuff whose polygons are both unbalanced. This matroid is ternary. If all polygons in $\Sigma = (\Gamma, \sigma)$ are balanced, $G(\Sigma)$ is the polygon matroid $G(\Gamma)$.

In the *lift matroid* $L(\Sigma)$ a circuit is a balanced polygon or a pair of unbalanced polygons meeting, if at all, at a single vertex. (This is the standard lift of matroid theory; see [13].) The *complete lift matroid* $L_0(\Sigma)$ is a one-point extension on the set $E(\Sigma) \cup \{e_0\}$, where e_0 is an *extra point* not in any graph, whose circuits are those of the lift and also the union with e_0 of any unbalanced polygon. (Thus e_0 behaves like a negative loop.) These matroids are binary. If Σ is balanced, then $L(\Sigma) = G(\Gamma)$ and $L_0(\Sigma) = G(\Gamma) \oplus$ (matroid coloop).

Each matroid, bias or lift, implies a compatible definition of contraction and therefore minors for signed graphs. (See Section 2.) The definitions agree for contraction of an edge which is not a negative loop, so we call a *link contraction* any result of successively contracting links (nonloop edges) and a *link minor* any link contraction of a subgraph. If we contract negative loops by the bias rule we get *bias minors*, which are always signed graphs; if we adopt the lift rule we get *lift minors*, which can be signed or unsigned graphs.

If we could associate to every signed graph a matroid $M(\Sigma)$ which (S1) determines the surfaces in which Σ can orientation embed and (S2) implies a definition of signed-graph contraction which is monotone (that is, if Σ orientation embeds in S, so does every minor), and if furthermore all matroids $M(\Sigma)$ belong to a class of matroids which is known to be well-quasi-ordered in the minor ordering, one could conclude that forbidden minors for orientation-embeddability in any given surface exist and are finitely many. The importance of this deduction makes it worthwhile to examine possible functions M. But our results on the bias and lift matroids are largely negative, though they do not entirely preclude the possibility of deducing finiteness of forbidden lift minors for orientation embeddability from binary matroid theory. (See the end of Section 4.)

Now here is the tenor of our results.

(S1) None of the three matroids determines the orientation embeddability properties of signed graphs. However, it remains possible that the lift or complete lift might determine the orientation embeddability of 3-connected signed graphs.

(S2) If Σ orientation embeds in a surface S, then so does every lift minor; consequently there exist forbidden lift minors characterizing orientation embeddability in S. (Note again that L and L_0 are binary.)

The same does not hold true for bias minors, not even if one can replace a bias minor by an arbitrary signed graph having the same bias matroid.

(S3) The closest signed analog of the plane is the projective plane. Here the bias and lift matroids coincide and are cographic. Conversely however, having a cographic bias or lift matroid does not make a signal graph projective planar. (It does if the matroid is not graphic but we do not prove that here.)

The implication of these results is that there is not a matroid theory *per se* of orientation embedding, at least none based on the bias or lift matroids. But the lift-inspired definition of contraction does yield forbidden-minor characterizations like those in ordinary graph theory. It remains to learn the properties of these characterizations.

2 Preliminaries

Our graphs (and matroids) are all finite. They may have multiple edges and loops. The symbols Σ and Γ will always denote respectively signed and unsigned graphs. The vertex and edge sets of a graph or signed graph are denoted by V and E. A signed graph, subgraph, or edge set is *balanced* if all its polygons are positive. Particular signed graphs of importance are: $+\Gamma$, or Γ with all edges signed $+$; $-\Gamma$; and $\pm\Gamma$, or Γ with all edges doubled, one of each pair signed $+$ and one $-$. If Σ is a signed graph, Σ° denotes Σ with a negative loop at each vertex: for instance, $+K_2^\circ$ or $\pm K_3^\circ$. By $|\Sigma|$ we mean the underlying graph of the signed graph Σ. We call Σ *k-connected* when $|\Sigma|$ is k-connected.

Switching Σ means reversing the signs of all edges between a vertex set X and its complement. Two signed graphs can be switched one to the other if and only if they have the same underlying graph and the same balanced polygons (and consequently the same matroids). In particular a balanced graph can be treated as all positive ($\sigma \equiv +$). An *isomorphism* between Σ_1 and Σ_2 is an isomorphism of their underlying graphs which preserves the signs of polygons. We consider isomorphic signed graphs to be the same.

The bias and lift *contractions* of Σ by an edge e, denoted by Σ/e and $\Sigma/_L e$, agree on all but negative loops. A positive loop e is simply deleted. If e is a link, Σ is switched so e is positive; then the endpoints of e are coalesced and e is deleted. In the *bias contraction* of a negative loop, it and its vertex v are deleted, all other loops at v are deleted, and each link from v becomes a negative loop at its other endpoint. With these definitions we have $G(\Sigma \setminus e) = G(\Sigma) \setminus e$, and also $G(\Sigma/e) = G(\Sigma)/e$ with perhaps some matroid loops deleted [12, Theorem 5.2; 13, Theorem 2.5]. The *lift contraction* by a negative loop e is the unsigned graph $|\Sigma \setminus e|$. Then we have

$L_0(\Sigma \setminus e) = L_0(\Sigma) \setminus e$ and $L_0(\Sigma/e) = L_0(\Sigma)/e$ and similar equations for the lift matroid [13, Theorem 3.6]. As for contraction by an arbitrary edge set $S \subseteq E(\Sigma)$, written Σ/S or $\Sigma/_L S$ as appropriate, it means contracting the edges of S successively; the result is independent of the order of operations.

Graphs and signed graphs, more precisely their isomorphism classes, are partially ordered by minors in several ways. The *minor ordering* of graphs is given by $\Gamma_1 \leq \Gamma_2$ if Γ_1 is (isomorphic to) a minor of Γ_2. In the *link-minor ordering* of signed graphs, $\Sigma_1 \leq \Sigma_2$ if Σ_1 is a link minor of Σ_2. In the *lift-minor ordering* of signed and unsigned graphs, $A \leq_L \Sigma$ if A is a signed or unsigned lift minor of Σ, and $\Gamma_1 \leq_L \Gamma_2$ if $\Gamma_1 \leq \Gamma_2$. (A 'lift minor' of Γ therefore means just a minor.)

For basic matroid theory we refer the reader to [3, 7, 8]. An n-point uniform matroid of rank r is denoted by $U_r(n)$. A *line* is a uniform matroid of rank two.

Let $L_0'(\Sigma)$ denote the basepointed matroid $(L_0(\Sigma), e_0)$. It is clear that $L_0'(\Sigma)$, hence also $L(\Sigma)$, is not altered by Whitney's *2-isomorphism* operations [11]: adding or deleting isolated vertices, changing the attachment points of blocks (including a loop as a block all by itself), and twisting across a 2-separation. Conversely, if $L_0'(\Sigma_1) \cong L_0'(\Sigma_2)$ then Σ_1 and Σ_2 are *2-isomorphic*, that is, related by these operations.

Here are some examples. The bias and lift matroids of an unbalanced polygon are independent (hence they are the same matroid); the complete lift matroid is a circuit. $G(\pm K_2^\circ)$ is the four-point line, but $L(\pm K_2^\circ)$ is a three-point line with one point doubled in parallel and $L_0(\pm K_2^\circ)$ is the same with e_0 added to the double point, making a triple point.

Lemma 2.1. *If $G(\Sigma)$ is the four-point line $U_2(4)$, then $\Sigma = \pm K_2^\circ$.*

Proof: Σ is connected because $G(\Sigma)$ is [13, Theorem 2.8]. If Σ were balanced then $G(\Sigma) = G(|\Sigma|)$ would be graphic, which $U_2(4)$ is not. Since Σ is unbalanced it has two vertices [12, Theorem 5.1(j); 13, Theorem 2.1(j)]. In order to avoid loops and parallel elements in $G(\Sigma)$, Σ can have at most one loop at each vertex – and that negative – and at most two links, signed differently. Thus it is $\pm K_2^\circ$. □

The surfaces important for embedding are the compact surfaces. These are the orientable surfaces T_g for $g \geq 0$, having Euler characteristic $\chi(T_g) = 2-2g$, and the nonorientable surfaces U_h for $h \geq 1$, for which $\chi(U_h) = 2-h$. It is customary to call g the *genus* of T_g and h the *crosscap number* or *nonorientable genus* of U_h. To have a compatible unit of measurement I define the *demigenus* $d(S)$ to be $2 - \chi(S)$; thus $d(T_g) = 2g$ and $d(U_h) = h$. All surfaces herein will be compact.

An embedding of an unsigned graph Γ in a surface S is a topological homeomorphism of Γ with a closed subset of S. We call this *topological* *embedding* to emphasize that orientation embedding of a signed graph is something different; nevertheless it is true that we always embed unsigned graphs ignoring orientation while signed graph embedding respects it. The *demigenus* $d(\Gamma)$ is the smallest demigenus of any surface in which Γ embeds topologically. The *demigenus* $d(\Sigma)$ of a signed graph is the smallest demigenus of a surface in which Σ orientation embeds. For each signed graph Σ there is a unique *minimal surface* $S(\Sigma)$ in which Σ orientation embeds, such that if Σ orientation embeds in S then S is obtained from $S(\Sigma)$ by adding handles or crosscaps. The minimal surface is orientable if Σ is balanced and nonorientable otherwise. (For these basic properties and appropriate references see [16].)

A signed graph is *planar* if it orientation embeds in the plane. Obviously Σ is planar if and only if it is balanced and $|\Sigma|$ is planar. We call Σ *projective planar* if it orientation embeds in the projective plane U_1.

Example 2.2. Here are three examples. A simple proof for $-W_4$ appears in [16, Example 9.3] and a complicated proof for all three is contained in [18]. Direct proofs are not hard to construct.

The all-negative 4-spoke wheel, written $-W_4$, has demigenus 2.

Let Φ_4 be the signed graph consisting of $+K_4$ with all edges at one vertex (say v) doubled by negative edges to the other vertices (x_1, x_2, and x_3). Then $d(\Phi_4) = 2$.

Let Ψ_5 be the signed graph consisting of $+K_{2,3}$ (with left vertices v, w and right vertices x_1, x_2, x_3) and negative edges $-vx_1, -vx_2, -vx_3$. Then $d(\Psi_5) = 2$.

3 The bias matroid and bias contraction

We show that $G(\Sigma)$ does not determine $d(\Sigma)$.

Example 3.1(a). For $n \geq 2$ let Υ_n be the signed graph consisting of a polygon C_n on the vertices $v_1, \ldots, v_n$ (in circular order; subscripts modulo n) with all edges e_i (endpoints v_i and v_{i+1}) positive except e_n between v_n and v_1, and also another vertex v supporting a negative loop e and positively linked to every v_i. (See Figure 3.1.) Then $d(\Upsilon_n/e) = n$ because Υ_n/e has n disjoint unbalanced polygons, the f_i at v_i for $i = 1, \ldots, n$.

Let Φ_n consist of C_n with e_n replaced by a negative loop e'_n at v_1 and with negative edges f'_i having endpoints v_i and v_n for $i = 1, \ldots, n$ (so f'_n is a negative loop). We see in Figure 3.2 that Φ_n orientation embeds in the Klein bottle U_2. It is not projective planar because it has two disjoint unbalanced polygons. Thus $d(\Phi_n) = 2$.

Now we have $d(\Upsilon_n/e) = n > d(\Phi_n) = 2$ for $n \geq 3$. Yet the correspondence of Υ_n/e to Φ_n indicated by the edge names is a matroid isomorphism. This is obvious for $n = 2$. To prove it for $n \geq 3$ we use induction.

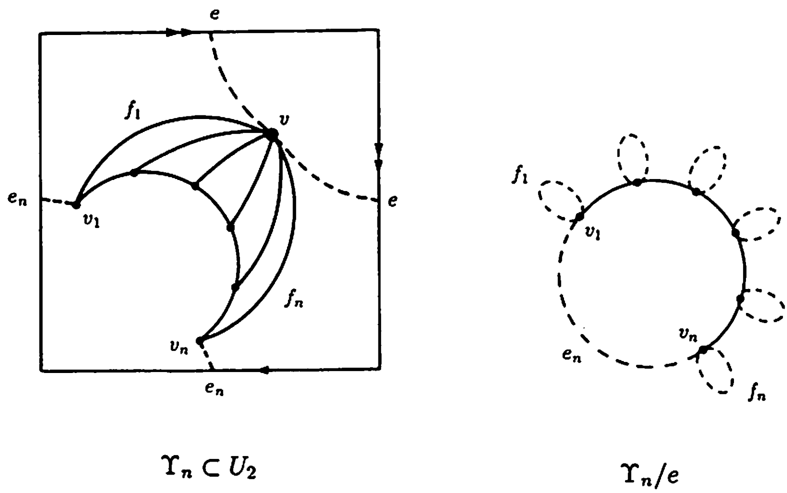

Figure 3.1. The signed graphs Υ_n and Υ_n/e of Example 3.1. Positive edges are solid; negative ones are dashed.

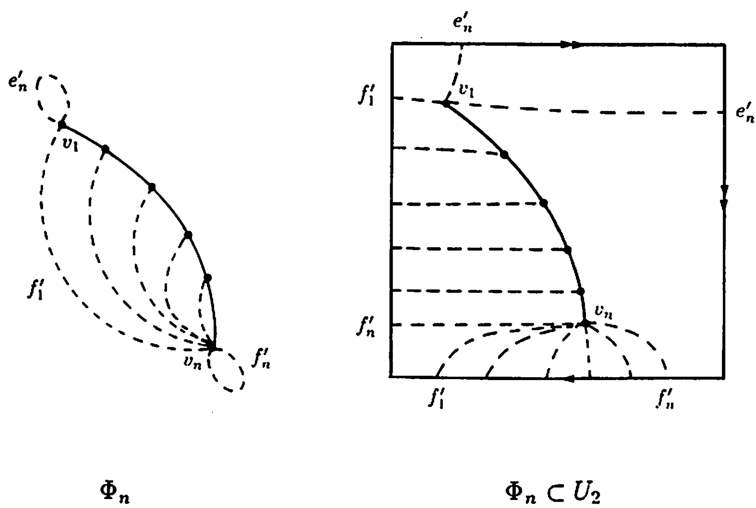

Figure 3.2. The signed graph Φ_n of Example 3.1(a).

The conclusion follows if we show that e_n and e'_n have the same rank in $G(\Upsilon_n/e)$ and $G(\Phi_n)$, which is obvious, that $G((\Upsilon_n/e)/e_n) \cong G(\Phi_n/e'_n)$ by the edge-name correspondence, and that $G((\Upsilon_n/e) \setminus e_n) \cong G(\Phi_n \setminus e'_n)$ similarly. The former isomorphism follows by induction because $(\Upsilon_n/e)/e_n$ is isomorphic to Υ_n/e'_n with f_n added as a negative loop doubling f_1 and

Φ_n/e'_n is isomorphic to Φ_{n-1} with f'_n similarly doubling f'_1. These isomorphisms are specified not by the edge-name correspondence but by the rule

$$e_1, f_n, e_i, f_i \in E((\Upsilon_n/e)/e_n) \leftrightarrow$$
$$e_{n-1}, f_{n-1}, e_{i-1}, f_{i-1} \in E(\Upsilon_{n-1}/e)$$

where $2 \le i < n$, and a similar rule for the primed edges. The second isomorphism is demonstrated by the following argument. Let Σ_n be $\Upsilon_n \setminus e_n$ with e replaced by a negative link from v to v_n. Then $\Upsilon_n \setminus e_n$ and Σ_n have the same matroid, for in each case e is a matroid isthmus and the remainder is the same (all-positive) graph. It follows that $G((\Upsilon_n \setminus e_n)/e) \cong G(\Sigma_n/e)$. We now have the desired isomorphism, because $(\Upsilon_n \setminus e_n)/e = (\Upsilon_n/e) \setminus e_n$ and $\Sigma_n/e = \Phi_n$. □

The next two examples show that bias contraction of a negative loop can increase the demigenus, hence that the signed graphs orientation embeddable in a particular nonorientable surface (possibly excepting the projective plane) have no characterization by forbidden bias minors.

Example 3.1(b). $d(\Upsilon_n) = 2$ (see Figure 3.1). Yet if $n \ge 3$, $d(\Upsilon_n/e) = n > 2$.

Example 3.2(a). The signed graph M_n, $n \ge 3$, is shown in Figure 3.3 together with an orientation embedding in the Klein bottle, which is its minimal surface. The bias contraction M_n/e (Figure 3.3), which has n disjoint unbalanced polygons, cannot orientation embed in U_{n-1}. It does orientation embed in U_n (Figure 3.3).

So far bias matroids and minors are not working out. There still remains the possibility that the bias matroids of U_h-embeddable signed graphs are closed under taking of minors. But even this fails for $h \ge 2$, according to the continuation of Example 3.2.

Example 3.2(b). No other signed graph has the same matroid as M_n/e. We omit the proof.

For an example of signed graphs Σ and $+\Gamma$ which are 3-connected and have the same bias matroid but not demigenus, see Example 4.2.

4 The lift matroids and lift contraction

Neither $L(\Sigma)$ nor $L_0(\Sigma)$ nor even $L'_0(\Sigma)$ determines $d(\Sigma)$, for 2-isomorphism operations can change the demigenus. We give a 2-connected example.

Example 4.1. Let Σ_1 consist of $+C_4$ with two adjacent edges doubled by negative edges; Σ_2 is similar but the doubled edges are nonadjacent. It is easy to see that $d(\Sigma_1) = 1$ and $d(\Sigma_2) = 2$, although $L'_0(\Sigma_1) = L'_0(\Sigma_2)$. □

Signed graphs which are not 2-isomorphic may nonetheless have the same lift matroid and at the same time different demigenus.

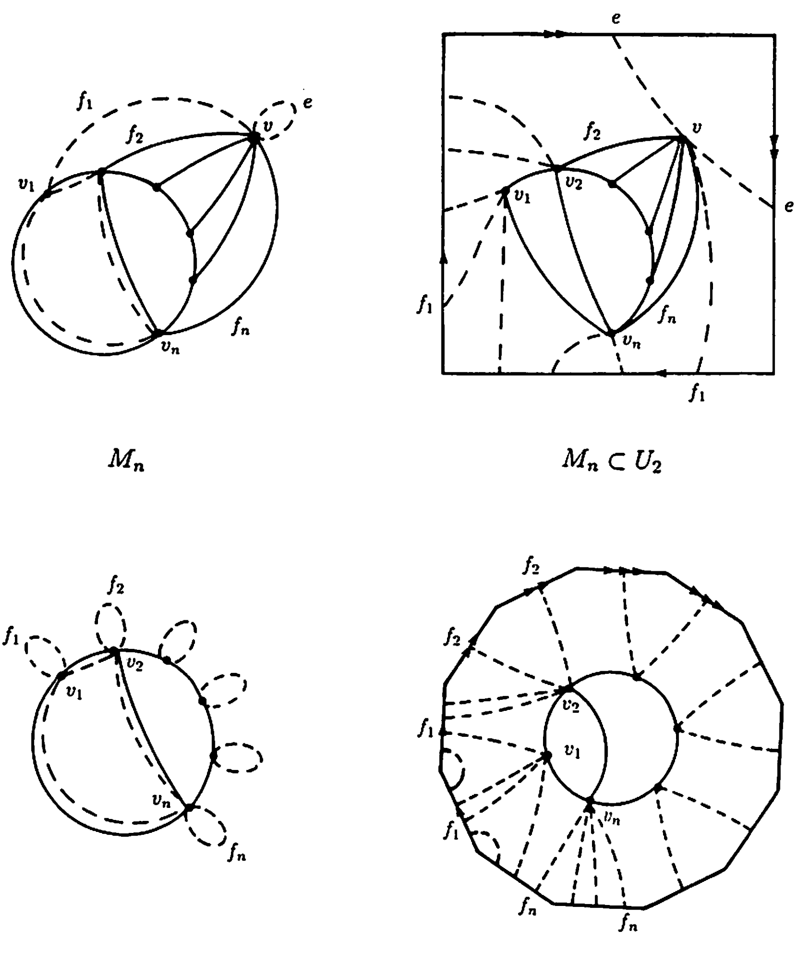

Figure 3.3. Signed graphs in Example 3.2.

Example 4.2. Let Γ_0 be a 2-connected graph and e a link in Γ_0. Set $\Gamma = \Gamma_0 \setminus e$ and $\Sigma = (+\Gamma \cup -e)/e$. Then $G(\Sigma) = L(\Sigma) = L(+\Gamma) = G(+\Gamma) = G(\Gamma)$ but $d(\Sigma)$ may or may not equal $d(+\Gamma)$.

For instance (cf. Example 2.2): Let $\Gamma = K_{3,3} \setminus e$; then $\Sigma = -W_4$, so $d(+\Gamma) = 0 \neq d(\Sigma) = 2$. Or let $\Gamma = K_5 \setminus e$; then $\Sigma = \Phi_4$ and again we have $d(+\Gamma) = 0 \neq d(\Sigma) = 2$. (In this case both graphs are 3-connected.) On the other hand if $\Gamma = K_{3,3}$ and e is an edge not parallel to any in Γ, then $\Sigma = \Psi_5$ and we have $d(+\Gamma) = 2 = d(\Sigma)$.

Notice that $L_0(\Sigma) \not\cong L_0(+\Gamma)$ because e_0 is a matroid isthmus in $L_0(+\Gamma)$ but there are no isthmi in $L_0(\Sigma)$. Also notice that $L(\Sigma)$ is graphic; for such signed graphs it appears to be exceptionally difficult to determine all Σ_1, and therefore all $d(\Sigma_1)$, such that $L(\Sigma_1) \cong L(\Sigma)$. □

Despite the inadequacy of the lift matroids themselves, the lift contraction turns out to be suitable for the demigenus problem. The question is whether $d(\Sigma/_L e) \leq d(\Sigma)$ for every edge e. If e is a negative loop we have the slight complication that $\Sigma/_L e$ is unsigned; consequently we have to order by lift minors the whole class of signed and unsigned graphs, whence the definition given in Section 2 of the lift-minor ordering.

The exposition of our results is greatly simplified by the somewhat forbidding notations

$$\mathbf{O}(U_h) = \{\Sigma\colon \Sigma \text{ orientation embeds in } U_h\},$$
$$\mathbf{T}(h-1) = \{\Gamma\colon d(\Gamma) \leq h-1\}.$$

Theorem 4.3. *The class of lift minors of signed graphs whose minimal surface is U_h equals $\mathbf{O}(U_h) \cup \mathbf{T}(h-1)$.*

Proof: Suppose Σ has minimal surface U_h. The only minor-forming operation which requires attention here is contraction of a negative loop. Suppose e is a negative loop in Σ. Orientation embed Σ in U_h. Pinching e to a point, forming the space U_h/e, amounts to replacing a 1-sphere by a point; thus the Euler characteristic rises by one. The following argument shows that U_h/e is a surface. The closed edge e has a neighborhood that is a Möbius band with an odd number of half-twists, which is homeomorphic to $U_1 \setminus$ (disk), e being a noncontractible simple closed curve in U_1. Shrinking e in U_1 yields the sphere. Thus the neighborhood of e remains a surface in U_h/e. We conclude that $U_h/e = U_{h-1}$ or $T_{(h-1)/2}$. Hence the class of lift minors is contained in $\mathbf{O}(U_h) \cup \mathbf{T}(h-1)$.

The converse containment is obvious from a fact we formulate as Lemma 4.4. □

Lemma 4.4. *An unsigned graph Γ has $d(\Gamma) \leq h-1$ if and only if there is a signed graph of the form $(\Gamma,\sigma)^{(v)}$ which has demigenus $\leq h$.* □

Here we mean by $(\Gamma,\sigma)^{(v)}$ a signing of Γ together with an extra negative loop at a vertex v which may be in Γ or may be a new vertex.

It follows from Theorem 4.3 that the signed graphs orientation embeddable in U_h can be characterized by forbidden lift minors. The minimal nonmembers of $\mathbf{O}(U_h)$ in the link minor ordering are called the *forbidden link minors* for $\mathbf{O}(U_h)$. The minimal nonmembers of the class $\overline{\mathbf{O}}(U_h) = \mathbf{O}(U_h) \cup \{\Gamma\colon \Gamma$ is a lift minor of a signed graph which orientation embeds in $U_h\}$ in the lift minor ordering are the *forbidden lift minors* for $\mathbf{O}(U_h)$. By Theorem 4.3, the unsigned forbidden lift minors for $\mathbf{O}(U_h)$ are precisely the forbidden minors for $\mathbf{T}(h-1)$, and the signed forbidden lift minors for $\mathbf{O}(U_h)$ are precisely the (signed) forbidden link minors Σ for $\mathbf{O}(U_h)$ such that, for every negative loop e in Σ, $d(|\Sigma \setminus e|) < h$. It follows from the former fact and the finiteness of forbidden minors for $\mathbf{T}(h-1)$ [1, 6] that the forbidden link and lift minors are both finite or both infinite in number.

I conjecture they are finite. Possibly this can be deduced from Neil Robertson's conjecture that binary matroids are well-quasi-ordered by minor inclusion. A small step has been taken already:

Proposition 4.5. *Let $h \geq 1$. If the number of signed, 2-connected forbidden lift minors for $\mathbf{O}(U_k)$ is finite for each $k = 1, 2, \ldots, h$, then the number of forbidden link minors for $\mathbf{O}(U_h)$ is finite.*

Proof: Finiteness of 2-connected forbidden minors for $\mathbf{O}(U_k)$ for all $k \leq h$ implies the number of all forbidden minors is finite by [16, Corollary 10.9]. □

One needs a 3-connected analog of this result, which should be a consequence of solving

Problem 4.6. Calculate $d(\Sigma_1 \cup \Sigma_2)$ in terms of Σ_1 and Σ_2, when $\Sigma_1 \cap \Sigma_2$ consists of two vertices.

The deduction of finiteness from Robertson's conjecture would be complete if it could be proved that $L_0(\Sigma)$ determines $d(\Sigma)$ when Σ is 3-connected. Even if this is false, there might be a way to use the fact that $L_0'(\Sigma)$ determines such a Σ.

5 The projective plane

Protective-planar signed graphs are especially nice.

Lemma 5.1. *If Σ is projective planar, then $G(\Sigma) = L(\Sigma)$ and $L(\Sigma)$ is a regular matroid.*

Proof: Since Σ can have no two vertex-disjoint unbalanced polygons $G(\Sigma) = L(\Sigma)$ (obviously) and this matroid is regular [15, Corollary 3D]. □

In higher surfaces $G(\Sigma)$ need not even be binary, since $d(\pm K_2^\circ) = 2$ and $G(\pm K_2^\circ)$ is the four-point line, which is not binary. $L(\Sigma)$ need not be regular because, setting $\Sigma = (C_4)\setminus$ edge, $d(\Sigma) = 2$ and $L(\Sigma) = F_7^\perp$, the dual of the Fano matroid F_7.

Lemma 5.2. *If Σ is projective planar, then $L_0(\Sigma) \not\geq G(K_5), G(K_{3,3})$.*

Proof: $L_0(\Sigma) \geq G(\Gamma)$ means either $\Sigma \geq \Sigma'$ where $L_0(\Sigma')$ or $L(\Sigma') = G(\Gamma)$, or $\Sigma \geq_L \Gamma'$ where $G(\Gamma') = G(\Gamma)$. In the latter case, setting $\Gamma = K_5$ or $K_{3,3}$ we have $\Gamma' = \Gamma$ by Whitney's 2-isomorphism theorem [11] because Γ is 3-connected. But Γ is nonplanar, hence when Σ is projective planar $\Sigma \not\geq_L \Gamma$ (clearly, or see Theorem 4.3).

Suppose that $L_0(\Sigma') = G(\Gamma)$ where again $\Gamma = K_5$ or $K_{3,3}$. By [15, Proposition 5A], $\Sigma' = \Phi_4$ or $-W_4$ (Example 2.2). Therefore, Σ' would not be projective planar, a contradiction.

Suppose that $L(\Sigma') = G(\Gamma)$. The several possible Σ' are classified by [15, Proposition 5A]. Either $\Sigma' = +\Gamma$ (but then [16, Lemma 3.3] would imply $+\Gamma$ is planar, which is ridiculous) or Σ' is unbalanced. Such a Σ' either is $-W_4$ or Φ_4 with a negative loop at the vertex of highest degree, or Ψ_5 or $-(K_5\setminus$ edge), or contains two disjoint unbalanced polygons. In no case is Σ' projective planar. This contradicts the hypothesis. □

Notice that $L_0(\Sigma)$ need not be cographic, since $\pm K_3$ and $-K_4$ are projective planar and $L_0(\pm K_3) = F_7$, $L_0(-K_4) = F_7^{\perp}$ [15, Proposition 3A].

Theorem 5.3. *If Σ is projective planar, $L(\Sigma)$ [$= G(\Sigma)$] is cographic.*

Proof: $L(\Sigma)$ is regular by Lemma 5.1. By Lemma 5.2 it has no minor isomorphic to $G(K_5)$ or $G(K_{3,3})$. Thus by Tutte's famous theorem ([9, Theorem 9.42 et. seq.]; what Tutte called "graphic" is now called "cographic") $L(\Sigma)$ is cographic. □

The converse is false since $\overline{K_2^{\circ}}$, consisting of two vertices each supporting a negative loop, is not projective planar but $G(\overline{K_2^{\circ}})$ and $L(\overline{K_2^{\circ}})$ are cographic. Still it seems that there is a kind of converse: for a signed 2-connected graph Σ, if $G(\Sigma)$ is cographic then either Σ is projective planar or $G(\Sigma)$ is graphic. (My unverified "proof", which depends on knowing the forbidden link minors for projective planarity [18], consists of a long and probably unreliable case-by-case analysis.)

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