Fundamental Subsets of Edges of Hypercubes

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1 Introduction

By Q_n we mean the n-cube, i.e. the graph whose vertices $V(Q_n)$ are the 2^n binary n-tuples and whose edges $E(Q_n)$ are those unordered pairs $< x, y >= e$ of vertices which differ in precisely 1 coordinate. If that coordinate is the i^{th}, we say that the direction of e is i. Edges with the same direction are called parallel. We recall a definition from [2].

Definition 1 *Let E be a subset of $E(Q_n)$. If G is a subgroup of $Aut(Q_n)$ such that $\{g(E) \mid g \in G\}$ is an edge decomposition of Q_n, we say that E is a fundamental set for Q_n with group G. We call E a fundamental set if some such subgroup G exists.*

In [2] we proved (Corollary 5.13) that if E has n edges, no three of which have the same direction, and if the subgraph of Q_n induced by E is connected, then E is a fundamental set. We also showed [2, Theorem 5.17] that for a connected graph Γ on n edges, the following two statements are equivalent: (1) Each edge of Γ belongs to at most one cycle. (2) There is an embedding $\varphi : \Gamma \hookrightarrow Q_n$ such that no three edges of $\varphi(\Gamma)$ have the same direction (and hence $E(\varphi(\Gamma))$ is a fundamental set for Q_n).

In this paper we study subsets E of $E(Q_n)$ with $2n$ edges, with the aim of discovering which are fundamental sets. We are also interested in the question of which graphs Γ (and especially, which trees) on $2n$ edges have embeddings $\varphi : \Gamma \hookrightarrow Q_n$ such that $E(\varphi(\Gamma))$ is a fundamental set for Q_n. We call such a φ a fundamental embedding. As we shall see, one natural way to try to obtain a fundamental embedding is to define a 2-1 labelling $\lambda : E(\Gamma) \to \{1, 2, \dots, n\}$. This yields a subgraph of Q_n with exactly 2 edges in each direction. One of our results (Theorem 1) is a characterization of the trees of diameter 4 with a $2-1$ edge labelling λ which yields a fundamental set. In section 3 we first characterize the trees on $2n$ edges which can be embedded in Q_n (Proposition 4). We then characterize those for which there is an embedding which comes from a 2-1 edge labelling (Propositions

5 and 6). In particular, this is true for every tree which is not a path, has maximum degree at most n and has diameter at least 5.

In section 4 we define the notion of a good 2-1 edge-labelling of a graph on $2n$ edges, and then state Theorem 1, described earlier.

In section 5 we raise the question of whether there is any path on $2n$ edges in Q_n which is a fundamental set. For n odd, the answer is no (Proposition 8). We give an example of such a path for $n = 4$, and conjecture that such paths exist whenever $n \geq 4$ is even.

2 Counter-examples and Necessary Conditions.

Example 1 *$\Gamma = Q_3 - v$ is a subgraph of Q_3 such that for every n, Q_n has no edge decomposition by isomorphic copies of $E(\Gamma)$.*

Proof: Any embedding of Γ in Q_n will yield a subgraph with exactly 3 edges in each of exactly 3 directions. Thus if Q_n had such an edge decomposition then for each i, the number of edges in the i^{th} direction would be a multiple of 3. On the other hand, we know that this number is 2^{n-1}. Contradiction. □

The next result is quite easy.

Proposition 1 *If Q_n has an edge decomposition by a family $\{\Gamma_\alpha\}$ of d-regular subgraphs, then d is a divisor of n.*

Proof: For any vertex v of Q_n, $n = \sum_{v \in \Gamma_\alpha} deg_{\Gamma_\alpha}(v) \equiv 0 \pmod{d}$. □

In a similar vein we have the following proposition.

Proposition 2 *Let Γ be a graph, with $\mid V(\Gamma) \mid = m \leq 2^n$, and $\mid E(\Gamma) \mid = q$, where q divides $n \cdot 2^{n-1}$. Let $k = n \cdot 2^{n-1}/q$. Let $d_1 \geq d_2 \geq \ldots \geq d_m$ be the degree sequence of Γ. For $m+1 \leq j \leq 2^n$ let $d_j = 0$. The following is a necessary condition for the existence of a family of embeddings $K = \{\varphi_i \mid 1 \leq i \leq k\}$ of Γ into Q_n such that $\{\varphi_i(E(\Gamma) \mid 1 \leq i \leq k\}$ is an edge decomposition of Q_n : there exists a $k \times 2^n$ matrix M whose first row is $(d_1 \ldots d_{2^n})$, every other row of M is a permutation of the first, and every column sum is n.*

Proof: First order the vertices of Q_n so that for $1 \leq j \leq 2^n$, $\deg_{\varphi_1(\Gamma)}(v_j) = d_j$. For $1 \leq i \leq k = \mid K \mid$, let $d_i(v_j) = deg_{\varphi_i(\Gamma)}(v_j)$. Form the $k \times 2^n$ matrix $M = (M_{ij})$ where $M_{ij} = d_i(v_j)$. Since $\{\varphi_i(E(\Gamma) \mid 1 \leq i \leq k\}$ is an edge decomposition of the n-regular graph Q_n, for each j, the column sum $\sum_i d_i(v_j)$ is n. Furthermore, since φ_i is a graph isomorphism, the i^{th} row is a permutation of the first. □

Example 2 *There is no edge decomposition of Q_3 by isomorphic copies of the tree T, if T is either of the trees shown in Figure 1.*

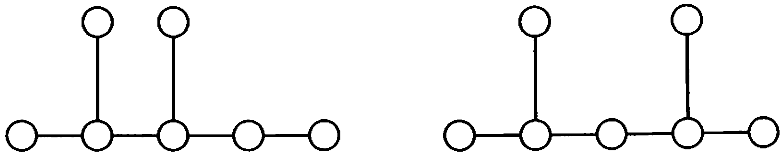


Figure 1

For the degree sequence of T is $3, 3, 2, 1, 1, 1, 1$. Since $| E(T) |= 6$ and $| E(Q_3) |= 12$, we must have a 2×8 matrix M whose first row is 33211110, and each of whose column sums is 3. But for this we would need $M_{12} = M_{22} = 0$, which is impossible since there is only one 0 in the first row. □

We shall restrict our attention to those automorphisms of Q_n known as complementations. For a subset A of $\{1, 2, \ldots, n\}$, $\sigma_A \in \text{Aut}(Q_n)$ is defined by

$$\sigma_A(\vec{x}) = \vec{x} + \vec{z}$$

where $\vec{z} = \sum_{i \in A} \vec{e}_i$, with e_i denoting the i^{th} standard basis vector of $\mathcal{Z}_2^n$. The set of complementations, Σ_n, is a subgroup of AutQ_n, isomorphic to $\mathcal{Z}_2^n$. An important feature of complementations is that they preserve edge directions.

Proposition 3 *Let $E \subset E(Q_n)$ with $| E |= n$. Suppose that E is a fundamental set for Q_n with group G, where G is a subgroup of Σ_n of index 4. Then no three edges of E have the same direction. Hence E has exactly two edges in each direction, and each edge of E belongs to at most one cycle in E. Furthermore, the length of each cycle in E is a multiple of 4.*

Proof: If $g \in G$ then g is a complementation, and so for all $e \in E$, $g(e)$ is parallel to e. We shall denote by E_i the set of edges of E whose direction is i. Then for $1 \leq i \leq n$ and for all $g \in G$, $(g(E))_i = g(E_i)$. Thus $| E_i |=| g(E_i) |$, i.e. $g(E)$ and E have the same number of edges in the i^{th} direction. Since E is fundamental for Q_n with group G, $\cup_{g \in G} g(E) = E(Q_n)$, and it follows that $\cup_{g \in G} g(E_i) = (E(Q_n))_i$. Hence

$$| G | \cdot | E_i |=| G | \cdot | g(E)_i |=| G | \cdot | g(E_i) |=| (E(Q_n))_i |= 2^{n-1}.$$

Thus $| G | \cdot | E_i |= 2^{n-1}$. Since $| G |= 2^{n-2}$, we have, for all i, $| E_i |= 2$. So E has exactly two edges in each direction, and no three edges are mutually parallel. By [2, Lemma 5.16] each edge of E belongs to at most one cycle in E.

Finally, let C be a cycle contained in E. Then since the number of edges of C in each direction must be even, and the number in E is exactly two, in each direction C has either 0 or 2 edges. Let $| C |= 2$. With no loss of generality we may assume that the edges of C have directions $1, 2, \ldots, k$, and that $\vec{0}$ is a vertex of C. Since $G \subset \Sigma_n$ and index_$\Sigma_n G = 4$, then by an argument very similar to the one given in the proof of [2, Lemma 5.1, part (2)] we see that there are subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$ such that

$$G = \{\sigma_A \in \Sigma_n \mid\mid A \cap T_1 \mid \equiv \mid A \cap T_2 \mid \equiv 0 \pmod 2\}.$$

Let $G_k = G \cap \sum_k$. Since G is E-good, so is G_k, and since $C \subset E$, G_k is C-good. Now

$$G_k = \{\sigma_A \in \Sigma_k \mid\mid A \cap T_1 \mid \equiv \mid A \cap T_2 \mid \equiv 0 \pmod 2\},$$

so the index of G_k in Σ_k is 4, and hence $| G_k |= 2^{k-2}$. Therefore

$$| G_k | \cdot | C |= 2^{k-2} \cdot 2k = k \cdot 2^{k-1}.$$

Thus C is a fundamental set for Q_k (with group G_k). Since C is 2-regular and Q_k is k-regular, k must be even. Hence $| C |= 2k \equiv 0 \pmod 4$. □

3 Embedding Trees into Q_n

We are interested in seeing which trees on $2n$ edges can be embedded in Q_n, and which of these embeddings yield fundamental sets. Note that since Q_n has Hamiltonian paths and cycles any path on $\geq 2^{n-1}$ edges can be embedded in Q_n. In particular, for $n \geq 3$ a path on $2n$ edges can be embedded. Thus it will be sufficient to consider the question for a tree on $2n$ edges which is not a path.

If Γ is a tree, $v_0 \in V(\Gamma)$ and $\lambda : E(\Gamma) \rightarrow \{1, 2, \ldots, n\}$ is a mapping, then there is a unique map $\varphi : V(\Gamma) \hookrightarrow V(Q_n)$ such that (i) $\varphi(v_0) = \vec{0}$ and (ii) for every $\langle u, v \rangle \in E(\Gamma)$, $\langle \varphi(u), \varphi(v) \rangle \in V(Q_n)$ and direction$(\langle \varphi(u), \varphi(v) \rangle) = \lambda(\langle u, v \rangle)$. For if z is any other vertex of Γ and P is the unique $v_0 - z$ path, define $\varphi(z) = x_1 x_2 \cdots x_n$, where $x_i = 1$ if $i \in \Delta_{e \in P}\{\lambda(e)\}$ and $x_i = 0$ otherwise. It is easy to check that this φ has the stated properties. A necessary and sufficient condition for φ to be an embedding (i.e. for φ to be $1-1$) is that for each path P' in Γ we have $\Delta_{e \in P'}\{\lambda(e)\} \neq \emptyset$, i.e. that on each path P' some edge label occurs an odd number of times [1, Theorem I]. (In particular, it is necessary that incident edges of Γ receive different λ-values.

Definition 2 *We call λ an edge-labelling of the tree Γ if λ satisfies conditions (i) and (ii) given above. If, in addition $| E(\Gamma) |= 2n$ and the mapping λ is $2-1$, we call λ a $2-1$ edge-labelling, and any associated embedding $\varphi : \Gamma \hookrightarrow Q_n$ a $2-1$ embedding.*

Proposition 4 *Let T be a tree on $2n$ edges which is not a path, and suppose that $\Delta(T) \leq n$. Then there exists an embedding of T into $Q_n \iff T \neq T_n$, where T_n is the tree obtained from two n-stars by identifying a leaf of each (see Figure 2).*

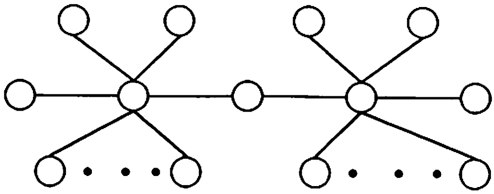


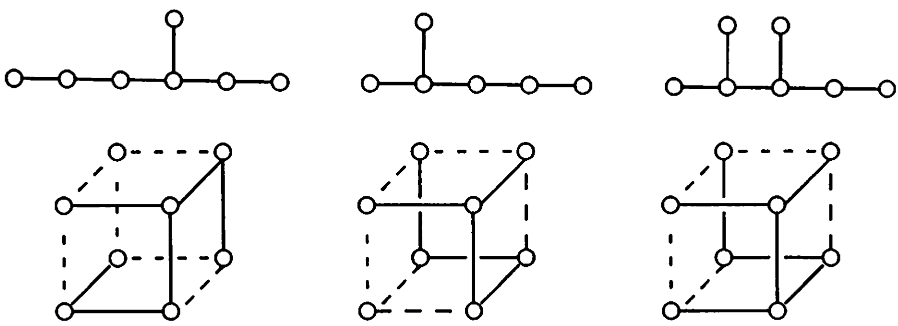
Figure 2. T_n

Proof: First we show that the diameter of any tree satisfying the initial hypotheses of the propostion must be at least For since T has $2n$ edges and $\Delta(T) \leq n$, T is not a star and so its diameter is at least 3. Suppose its diameter is 3. Then T consists of 2 stars, together with an edge joining their centers, x and y. But then $\deg(x) + deg(y) = 1 + \mid E(T) \mid = 2n + 1$, which contradicts the assumption that $\Delta(T) \leq n$.

($\Rightarrow$): Suppose that φ is an embedding of T_n into Q_n, and λ is the associated edge labelling, given by $\lambda(e) =$ the direction of e. Let u and w be the two vertices of T_n of degree n, and let v be the vertex (of degree 2) which is adjacent to both u and w. Since edge labels of incident edges must be distinct, all n edge labels must occur on the n edges which meet u, and also on the n edges which meet w. Say $\lambda(\langle u, v\rangle) = i$ and $\lambda(\langle v, w\rangle) = j$. Then i must be the edge label of some edge meeting v and j must be the label of some edge meeting u. Thus T_n has a path of length 4 whose image under φ has two sets of parallel edges, and is therefore a 4-cycle. But since T_n is acyclic, this contradicts the assumption that φ is an embedding.

($\Leftarrow$): We argue by induction on n. A tree with maximum degree 2 is a path, and so the statement is trivially true in the case $n = 2$. Let $n = 3$. Up to isomorphism, there are three trees on 6 edges which are not paths and which are not T_3. Each tree, together with an embedding in Q_3, is shown in Figure 3.

So assume $n \geq 4$. Suppose first that $\Delta(T) \leq n - 2$. Choose vertices x and y such that $dist(x, y) = diameter(T) \geq 3$. Then x and y are leaves of T and therefore lie on two independent pendant edges, say e_x and e_y . Deleting x and y we obtain a tree T' with $2(n-1)$ edges and for which $\Delta(T') \leq n - 2 < n - 1 = \Delta(T_{n-1})$. Hence $T' \neq T$. Since T is not a path, neither is T'. So by induction, T' can be embedded in Q_{n-1}. By labelling both e_x and e_y n, we obtain an embedding of T in Q_n.



3a. 3b. 3c.

Figure 3

Next, suppose $\Delta(T) = n-1$. If $T \not\supset T_{n-1}$ then form T' as above. $T' \neq T_{n-1}$, so by the preceding argument T embeds in Q_n. So suppose that $T \subset T_{n-1}$. First let $n = 4$. Then $T \subset T_3$, and so $T = T_3 + 2$ pendant edges e and f. Now e and f are incident with at most 2 of the 4 leaves of T_3. Let u and w be the two vertices of T_3 of degree 3. If both of them are adjacent to leaves of T, delete one leaf adjacent to u and one leaf adjacent to w to form T'. Then T' is a path on 6 edges and can therefore be embedded in Q_3. Assigning the label 4 to each of the two edges of $T - T'$ we obtain an embedding of T in Q_4. Now suppose that one of u and w, say u, is not adjacent to a leaf of T. Delete one leaf adjacent to w and one leaf at distance 2 from u (see Figure 4) to form T'. An embedding of T' in Q_3 is shown in Figure 3c. As before, this extends to an embedding of T in Q_4.

T T'

Figure 4

Now suppose $n \geq 5$. If $T_{n-1} \not\subset T$, let $T' = T - x - y$, where x and y are leaves of T such that $dist(x, y) \geq 3$. Then $\Delta(T') \leq \Delta(T) \leq n-1$ and T' has $2(n-1)$ edges. Since, in addition, $T' \neq T_{n-1}$, it follows by induction that there is an embedding φ' of T' in Q_{n-1}. Hence labelling each of the two edges of $T - T'$ with n yields an extension of φ' to an embedding φ of T in Q_n. Now asssume that $T_{n-1} \subset T$. Let u and w be the 2 vertices of degree $n-1$ and let v be the vertex which is adjacent to both u and w. Let x and y be the two vertices of $T - T_{n-1}$. Neither of them is adjacent to u or w since $\Delta(T) = n-1$. Let $T' = T - x - a$, where a is a leaf of T and $a \neq x$, $a \neq y$. Then $T' \neq T_{n-1}$ and by induction, there is an embedding φ'

of T' in Q_{n-1}. Since the two edges of $T' - T_{n-1}$ are independent, we may assign both of them the label n and thereby extend φ' to an embedding φ of T in Q_n. This completes the proof in the case $\Delta(T) = n - 1$.

Finally, suppose $\Delta(T) = n$. T has at most two vertices of degree n. Case 1: T has two vertices of degree n. Then since $T \neq T_n$, T is unique up to isomorphism and the labelling shown in Figure 5 yields an embedding of T in Q_n. Case 2: T has exactly one vertex of degree n, say u. Subcase 2a: No vertex adjacent to u is a leaf. Then T is the "stretched star" shown in Figure 6, and the edge labelling shown there yields an embedding of T in Q_n. Subcase 2b: Some vertex z adjacent to u is a leaf. There must be another leaf y which is not adjacent to u. Let $T' = T - z - y$. Then T' has $2(n-1)$ edges, and since $deg_{T'}(u) = n - 1$, $\Delta(T') = n - 1$. As we observed at the beginning of the proof of this proposition, a tree with $2k$ edges and $\Delta \leq k$ must have diameter≥ 4. Again, the two edges of $T - T'$ are independent and so if $T' \neq T_{n-1}$, there is a natural extension of the inductively guaranteed embedding of T' in Q_{n-1} to an embedding of T in Q_n. Therefore, assume that $T' = T_{n-1}$. Then let $T^* = T - y$, and let λ^* be the edge labelling indicated in Figure 7.

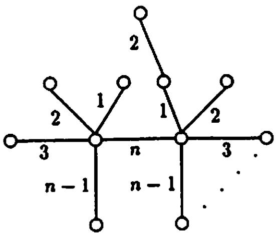


Figure 5

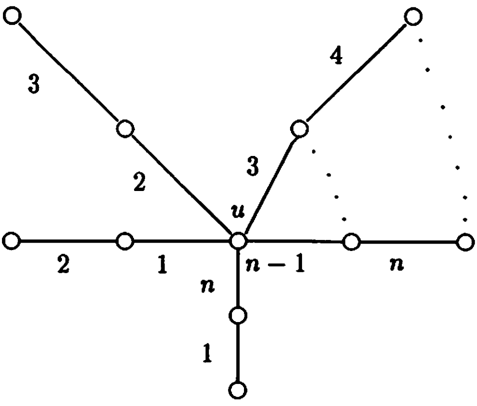


Figure 6. The "stretched n star"

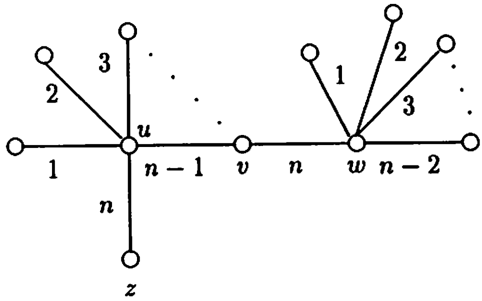


Figure 7. $T^* = T - y$.

Let $e = \langle x, y \rangle$ be the unique edge in $E(T) - E(T^*)$. To define an extension λ of λ^* to $E(T)$ we only need to define $\lambda(e)$. If $x = v$, let $\lambda(e) = 1$. If x is adjacent to w, let $\lambda(e) = n - 1$, and if x is adjacent to u let $\lambda(e) = n$. It is easy to see that λ yields an embedding of T in Q_n. The proof is now complete. □

Proposition 5 *Let $n \geq 3$ and suppose Γ is a tree on $2n$ edges with $\Delta(\Gamma) \leq n$ and diameter$(\Gamma) = 4$. Then Γ has a $2-1$ embedding in Q_n $\Longleftrightarrow$ (1) every non-central vertex has degree $\leq n - 1$ and (2) $\Gamma \subset \Gamma_0$, the stretched 3-star, formed from 3 paths on 2 edges by identifying a terminal vertex of each (see Figure 8).*

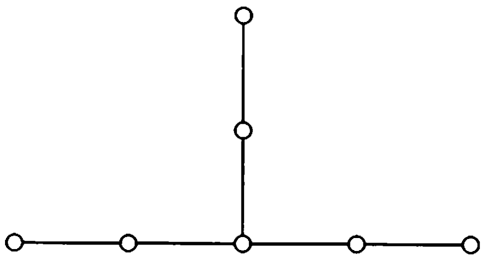


Figure 8. Γ_0.

Proof: ($\Longrightarrow$) First note that diameter$(\Gamma) = 4 \Leftrightarrow \Gamma$ is the ammalgamation of stars at a common leaf y (see Figure 9). Call y the center of Γ.

If Γ has any embedding in Q_n then $\Delta(\Gamma) \leq n$. But if some non-central vertex x has degree n then Γ has no $2-1$ embedding in Q_n. For let $e = \langle x, y \rangle$ and suppose $\lambda(e) = n$. Then, assuming that λ: $E(\Gamma) \to 1, 2, \ldots, n$ is $2-1$, $\lambda(f) = n$ for some pendant edge f. Let w be the end of f adjacent to y. Let $\lambda(\langle w, y \rangle) = j < n$. Since $\deg(x) = n$, all n labels must occur on the edges meeting x. Thus some edge h at x has $\lambda(h) = j$. So we get a path with labels j, n, j, n. Its image in Q_n must be a 4-cycle, which is a contradiction. Thus every vertex adjacent to a leaf has degree $\leq n-1$. This proves (1).

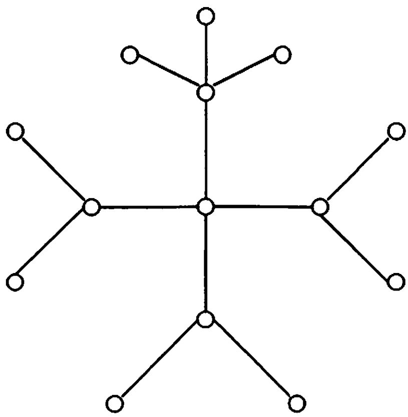


Figure 9. A typical tree of diameter 4.

Now suppose that $\Gamma \not\supset \Gamma_0$. Then Γ consists of a path of length 4, whose internal vertices are x, y, and z, and $2n-4$ additional pendant edges incident with these internal vertices (see Figure 10). Suppose that λ is a $2-1$ edge labelling of Γ which induces an embedding in Q_n. We may assume that $\lambda(\langle x,y\rangle) = 1$ and $\lambda(\langle y,z\rangle) = 2$. No other edge meeting y or z can have label 2. Thus some edge meeting x has label 2. Similarly, some edge meeting z must have label 1. But then there is a path with labels $2,1,2,1$ which, as we have seen, is impossible.

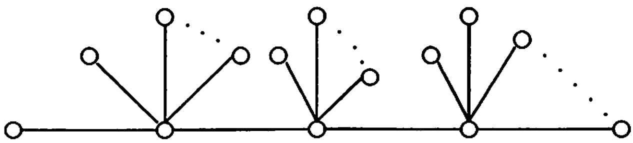


Figure 10

($\Longleftarrow$) By induction on n. If $n = 3$, then $\Gamma = \Gamma_0$, which has the labelling shown in Figure 11a. For $n = 4$, the only possibilities for Γ are Γ_1 and Γ_2, shown in Figures 11b and 11c together with their $2-1$ labellings.

Now let $n \geq 5$ and assume the result is true for $n-1$. First suppose $\Delta(\Gamma) \leq n-2$. Let $\Gamma' = \Gamma - e - f$, where e and f are independent pendant edges. If $\Gamma' \subset \Gamma_0$ then by induction there is a $2-1$ embedding of Γ' in Q_{n-1}, and this extends, in the obvious way, to a $2-1$ embedding of Γ in

Q_n. So assume that $\Gamma' \not\subset \Gamma_0$. Then Γ' is of the form shown in Figure 10, and so $\Gamma = \Gamma' + e + f$ must be as in either Figure 12a or Figure 12b.

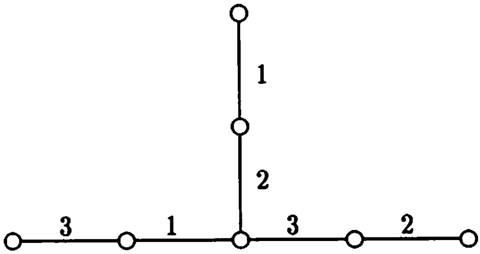


Figure 11a. Γ_0.

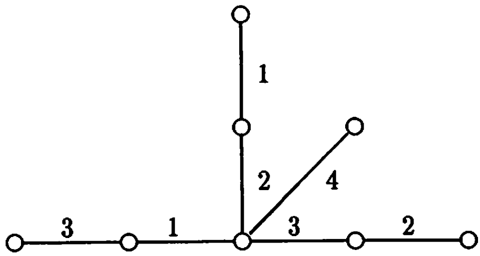


Figure 11b.

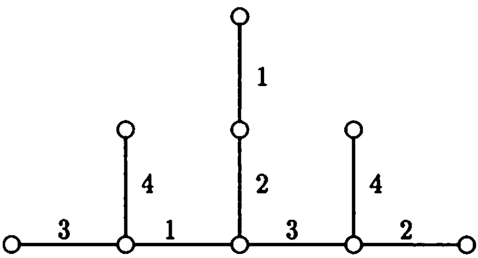


Figure 11c.

Next, suppose $\Delta(\Gamma) = n$. Since every non-central vertex, by hypothesis, has degree at most $n-1$, the vertex of degree n must be the central vertex y. If each vertex adjacent to y has degree 2, then Γ is the "stretched star" of Figure 6, and the labelling shown there is $2-1$. Otherwise, there is at least one leaf adjacent to y. Call it u. Since exactly n of the $2n$ edges of Γ meet y, there must also be n leaves at distance 2 from y. Deleting one of these, as well as u, yields a Γ' which satisfies $\Delta(\Gamma') = n-1$ and $\Gamma' \subset \Gamma_0$. In addition, every non-central vertex of Γ' has degree at most $n-2$. Thus by induction there is a $2-1$ embedding of Γ' in Q_{n-1}. Since the two edges in $E(\Gamma)-E(\Gamma')$ are independent, this embedding extends to a $2-1$ embedding of Γ in Q_n.

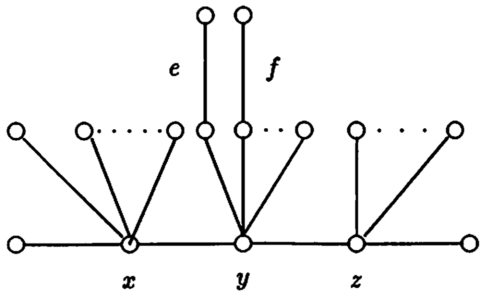


Figure 12a.

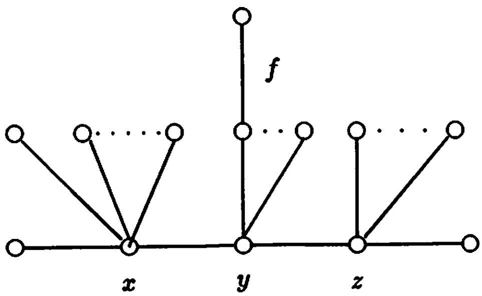


Figure 12b.

Finally, suppose $\Delta(\Gamma) = n-1$. As before, at most two vertices of Γ have degree $n-1$. Suppose there are two such. If possible, delete a leaf adjacent to each to form Γ'. Then $\Gamma' \subset \Gamma_0$ and $\Delta(\Gamma') = n-2 = (n-1)-1$. Furthermore, every non-central vertex of Γ' has degree $\leq n-2$. Thus by induction, Γ' has a $2-1$ embedding in Q_{n-1}, and this extends to a $2-1$ embedding of Γ in Q_n. On the other hand, suppose one of the vertices of degree $n-1$ is not adjacent to any leaf. The only possible candidate for such a vertex is y. In that case each neighbor of y has degree≥ 2, and Γ consists of the stretched $(n-1)$-star centered at y, together with two additional edges meeting neighbors of y. But then if $z \neq y$, $deg\ z \leq 4$. Thus $n-1 \leq 4$. Since we are assuming that $n \geq 5$, we must have $n = 5$. Hence Γ must be the tree shown in Figure 13, and the $2-1$ labelling shown there yields the desired embedding in Q_5.

Now suppose that only one vertex of Γ has degree $n-1$. First assume that this vertex is a non-central vertex, say x. Then $\deg(y) \leq n-2$. Γ has $2n-(n-1) = n+1 \geq 6$ edges not meeting x. Suppose that of these, at most two did not meet y. Then $\deg(y) \geq (n+1)-2 = n-1$, contradicting the assumption that $\deg(y) \leq n-2$. So at least three edges meet neither x nor y. By deleting a leaf on one of these edges and a leaf adjacent to x we obtain a $\Gamma' \subset \Gamma_0$ with $|E(\Gamma')| = 2(n-1)$ and $\Delta(\Gamma') = n-2$. As before, Γ' has the desired embedding which extends to a $2-1$ embedding of Γ.

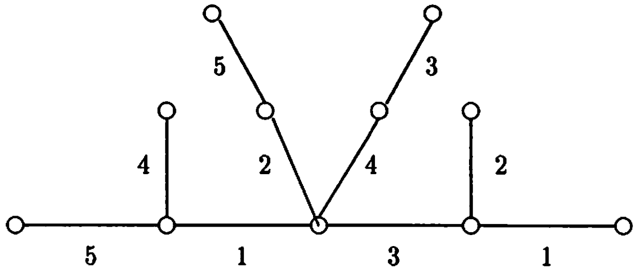


Figure 13. A 2-1 labelling of Γ.

Now assume that the unique vertex of degree $n-1$ is the central vertex y. Then there are $n+1$ edges which do not meet y, all of which must be pendant. Excluding three of them which are pendant edges of Γ_0, we have $n-2$ more. Since y is the only vertex of degree $n-1$, these $n-2$ pendant edges can not all meet at the same vertex, and so at least 2 of them are independent. Thus we may delete 2 leaves which are ends of these independent edges and thereby obtain a Γ' which provides the extendable $2-1$ embedding. □

Example 3 *For any $n \geq 3$, the tree T_n shown in Figure* 14 *has a unique (up to an automorphism of Q_n) embedding into Q_n, which comes from a labelling, such as the one shown in Figure* 14, *that is* not $1-1$. *And the image of $E(T_n)$ in Q_n is* not *a fundamental set for Q_n. In fact, Q_n has no edge decomposition into isomorphic copies of $E(T_n)$.*

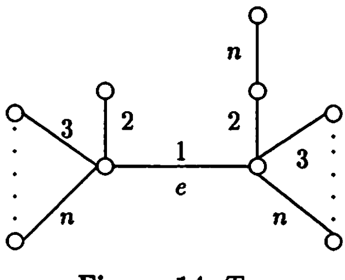


Figure 14. T_n.

Proof: Suppose the contrary. Then $E(Q_n)$ is the disjoint union of 2^{n-2} copies of the n-set $E(T_n)$. Now each end of edge e (see Figure 14) has degree n. If e' and e'' are the edges corresponding to e in two isomorphic copies T' and T'' of $E(\mathcal{T}_n)$ in the edge decomposition, then no end of e' can be adjacent to an end of e''. For if so, then we would have an edge $\langle x', x'' \rangle$ with x' an end of e' and x'' an end of e'', and since Q_n is n-regular and $deg_{T'}(x') = deg_{T''}(x'') = n$, this edge would belong to $E(T') \cap E(T'') = \emptyset$. So we have a family of 2^{n-2} edges, such that any two endpoints chosen from different edges are not adjacent. In other words, these 2^{n-2} edges form a matching which is an induced subgraph of Q_n. In Lemma 1 we shall show that these edges must all be parallel. So say they all have direction j. Then each copy of $E(\mathcal{T}_n)$ has only one edge with direction j. But the union of the copies is $E(Q_n)$, which has 2^{n-1} edges with direction j. Contradiction. Hence there is no such edge decomposition of Q_n. □

Lemma 1 *Let M be an induced matching in Q_n with 2^{n-2} edges. Then all the edges of M are parallel.*

Proof: Let H be the subgraph of Q_n obtained by deleting the 2^{n-1} vertices of M. The number of edges which are incident with exactly one vertex of M is $(n-1) \cdot 2^{n-1}$, while the number incident with two vertices of M is, of course, $|E(M)| = 2^{n-2}$. Hence

$$\begin{aligned} |E(H)| &= |E(Q_n)| - ((n-1) \cdot 2^{n-1} + 2^{n-2}) \\ &= n \cdot 2^{n-1} - (n-1) \cdot 2^{n-1} - 2^{n-2} \\ &= 2^{n-1} - 2^{n-2} = 2^{n-2}. \end{aligned}$$

Since H has 2^{n-1} vertices, the average degree of a vertex of H is 1. We shall show that H is 1-regular by showing that no vertex of H can have degree 0. Suppose $x \in H$ and $deg_H(x) = 0$. Then every Q_n-neighbor of x belongs to M. So let y be a vertex adjacent to x, with $\langle y, z \rangle \in M$, for some z. There is a unique $w \in Q_n$ such that x, y, z, w is a 4-cycle. Since w is adjacent to x, $w \in M$ by our assumption. But the adjacency of w and z contradicts the fact that the matching M is an induced subgraph of Q_n. Hence for all $x \in H$, $deg_H(x) \geq 1$, and so H is 1-regular, i.e. a matching. Furthermore, since H was obtained from Q_n by deletion of vertices, H is also an induced subgraph of Q_n. In addition, $M \cup H$ is a perfect matching of Q_n.

It suffices now to show that all edges of $M \cup H$ are parallel. Let $x \in Q_n$. Without loss of generality we may assume $x \in e_x$, where $e_x \in M$ and $direction(e_x) = 1$. Let F be the $(n-1)$-dimensional subcube of vertices whose first coordinate agrees with that of x. It suffices to show that for every $z \in F$, if e_z is the edge of $M \cup H$ containing z, then $direction(e_z) = 1$. This is proved by induction on the Hamming distance between x and z. It is easy to see that it suffices to establish the result when x and z are adjacent.

Let $e_x = \langle x, y\rangle$, and let $\langle y, w\rangle$ be parallel to $\langle x, z\rangle$. So x, y, z, w is a 4-cycle, and $\langle z, w\rangle$ is parallel to $e = \langle x, y\rangle$. Suppose $\langle z, w\rangle \notin M \cup H$, i.e. $\langle z, w\rangle \neq e_z$. Then $e_z \neq e_w$. Since z is adjacent to x and $e_x \in M$ and M is induced, it follows that $e_z \notin M$. Hence $e_z \in H$. But w is adjacent to both y and z, and $e_y = e_x \in M$, while $e_z \in H$. Since both M and H are induced, $e_w \notin M$ and $e_w \notin H$, which contradicts the fact that $e_w \in M \cup H$. Hence $\langle z, w\rangle \in M \cup H$ (in fact, $\langle z, w\rangle \in H$) and so e_z is parallel to $\langle x, y\rangle = e_x$. □

Lemma 2 *Let $n \geq 4$ and let Γ be a tree on $2n$ edges, with $\Delta(\Gamma) = n$ and diameter$(\Gamma) \geq 5$. Then there exist independent pendant edges e and f such that if $\Gamma' = \Gamma - v_e - v_f$, where v_e and v_f are the leaves incident with e and f respectively, then $\Delta(\Gamma') = n - 1$ and diameter$(\Gamma') \geq 5$.*

Proof: If Γ had more than one vertex of degree n then Γ would be either T_n (see Figure 2) or $\mathcal{T}_n$ (see Figure 14), and so diameter(Γ) would be 4, contrary to hypothesis. Let x be the unique vertex of degree n. Γ must have at least one leaf adjacent to x, since otherwise Γ would be a "stretched n-star" and thus have diameter 4. First suppose that Γ has only one leaf, x_n, adjacent to x. Then since *diameter*$(\Gamma) \geq 5$, Γ must be the tree shown in Figure 15.

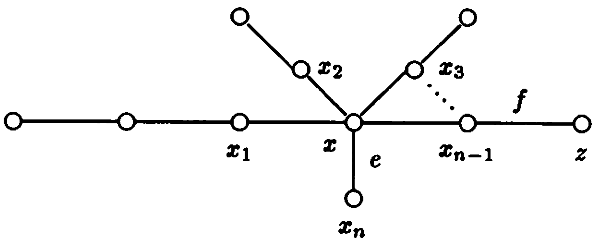


Figure 15

Since $n \geq 4$ there is a pendant edge $f = \langle x_{n-1}, z\rangle$ where $H(x, z) = 2$. Letting $e = \langle x, x_n\rangle$, $v_e = x_n$, and $v_f = z$, we see that $\Gamma' = \Gamma - x_n - z$ works.

Now suppose that Γ has at least two leaves, x_{n-1} and x_n adjacent to x. Let P be a longest path. First assume that $x \in P$. Since *length*$(P) \geq 5$, at least one of x_{n-1} and x_n does not belong to P. With no loss of generality, say $x_n \notin P$. Let $e = \langle x_n, x\rangle$. Let y be an end vertex of P, not adjacent to x. If *length*$(P) \geq 6$, length$(P - y) \geq 5$ and $P - y$ is a path in $\Gamma - x - y = \Gamma'$. So diameter$(\Gamma') \geq 5$, $\Delta(\Gamma') = n - 1$ and $E(\Gamma')$ has $2(n-1)$ edges. So suppose now that *length*$(P) = 5$. If $n = 4$ then there are exactly two x_j on P and exactly two x_k not on P. (The latter are the two leaves x_3 and x_4 adjacent to x.) P has exactly three other edges. This accounts for 7 of the 8 edges of Γ. The 8^{th} edge f must be a pendant edge not incident with x.

Thus we can delete the leaf x_4 adjacent to x and the leaf at the end of f, to obtain the desired Γ'. So suppose $n \geq 5$. Then $\Gamma \supset \Gamma^*$, where Γ^* is the tree on 8 edges and maximum degree 5 shown in Figure 16.

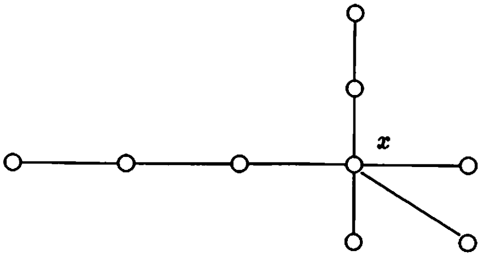


Figure 16. Γ^*.

$E(\Gamma) - E(\Gamma^*)$ has $2n - 8$ edges, of which exactly $n - 5$ are incident with x. Thus there are $n - 3$ edges in $E(\Gamma) - E(\Gamma^*)$, none of which meets x. At least one of these is a pendant edge. Call it f, and let the leaf at its end be v. Then f and $\langle x_n, x\rangle$ are independent, and $\Gamma' = \Gamma - x_n - v$ works.

Finally, suppose that $x \notin P$. Since P has at least 5 edges, none meeting x, and $\deg(x) = n = 1/2 \mid E(\Gamma) \mid$, it follows that $deg(x) \geq 5$. So Γ must contain as a subtree the tree $\hat{\Gamma}$ shown in Figure 17 .

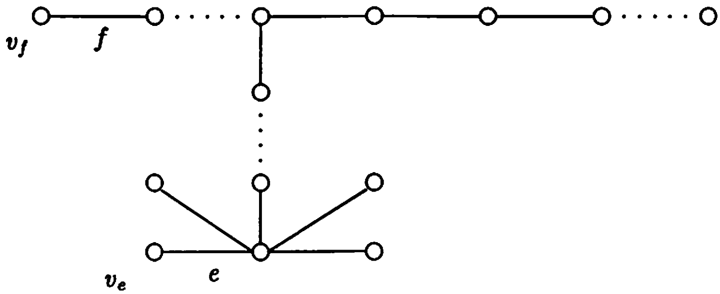


Figure 17. $\hat{\Gamma}$.

The edge f which is indicated is pendant and independent of e, and $\Gamma' = \Gamma - v_e - v_f$ satisfies $\Delta(\Gamma') = n - 1$ and $diameter(\Gamma') \geq 5$. □

Proposition 6 *Let $n \geq 3$ and let Γ be a tree on $2n$ edges which is not a path. Suppose $\Delta(\Gamma) \leq n$ and $diameter(\Gamma) \geq 5$. Then there exists a $2-1$ embedding of Γ into Q_n.*

Proof: By induction on n. Suppose $n = 3$. There are only two possibilities (up to isomorphism) for Γ, and these are shown in Figures 18a and 18b, together with $2 - 1$ edge labellings.

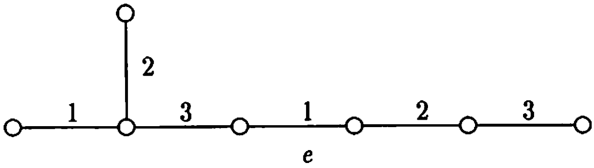


Figure 18a

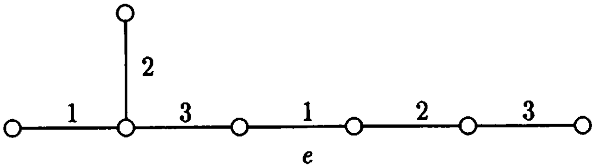


Figure 18b

Now let $n \geq 4$ and assume the proposition is true for $n-1$. By Lemma 2, there exist independent pendant edges e and f in Γ which end in leaves x_e and x_f so that $\Gamma' = \Gamma - x_e - x_f$ is a tree (on $2n-1$ edges) which is not a path, has a path of length ≥ 5, and satisfies $\Delta(\Gamma') \leq n-1$. By our induction hypothesis, there is a $2-1$ edge labelling λ' of Γ', λ': $E(\Gamma') \rightarrow \{1, 2, \ldots, n-1\}$. Since e and f are independent, we can define λ: $E(\Gamma) \rightarrow \{1, 2, \ldots, n\}$ by $\lambda(e') = \lambda'(e')$ if $e' \in E(\Gamma')$, and $\lambda(e) = \lambda(f) = n$. □

4 Trees with Good 2-1 Labellings.

Definition 3 *Let Γ be a graph on $2n$ edges and suppose*

$$\lambda : E(\Gamma) \rightarrow \{1, 2, \ldots, n\}$$

is a $2-1$ labelling which yields an embedding φ of Γ into Q_n. We say that λ and φ are good *if there is a subgroup G of Σ_n of index 4 such that $E(\varphi(\Gamma))$ is a fundamental set for Q_n with group G.*

Let E be a subset of $E(Q_n)$ and let $g \in Aut(Q_n)$, $g \neq identity$. As in [2, Definition 4.1] we say that g is E-good if $g(E) \cap E = \emptyset$ and E-bad otherwise. Furthermore, a subgroup G of $Aut(Q_n)$ is called E-good if all its non-identity elements are E-good. In [2, Section 5, Remark (preceding Lemma 5.1)] we showed that if edges e and e' are parallel, there is a unique subset B of $\{1, 2, \ldots, n\}$ with $| B |$ even and $\sigma_B(e) = e'$ (and since $(\sigma_B)^2 = identity$, $\sigma_B(e') = e$). Moreover, if direction$(e) = j$ then $\sigma_A(e) = e \iff A = \{j\}$ or $A = \emptyset$ (so $\sigma_A = identity$). Thus $\sigma_A(e) = e' \iff A = B$ or $A = B \Delta \{j\}$.

Now suppose that $| E |= 2n$ and for $1 \leq j \leq n$, E has exactly two edges, e_j and e'_j, in direction j. Let B_j denote the unique subset with $| B_j |$ even and $\sigma_{B_j}(e_j) = e'_j$. Then $\sigma_A \in \Sigma_n$ is E–*bad* $\Longleftrightarrow A = \{j\}$ or $A = B_j$ or $A = B_j \Delta \{j\}$, for some $1 \leq j \leq n$. Thus if $\mathcal{F} = \cup_{j=0}^{n} \{j\}, B_j, B_j\Delta\{j\}$ then a subgroup G of Σ_n is E-good$\Longleftrightarrow G \cap \sigma(\mathcal{F}) = \emptyset$, where $\sigma(\mathcal{F}) = \{\sigma_A \mid A \in \mathcal{F}\}$.

Lemma 3 *Let (Γ', λ') be a 2–1 edge-labelled graph with $| E(\Gamma') |= 2(n-1)$. Suppose $\lambda'(e') = \lambda'(f') = n-1$. Let $\mathcal{P}$ be the set of internal edges of some $e' - f'$ path in Γ'. (Note we include the special case where $e' = f'$ and $\mathcal{P} = \emptyset$.) Let (Γ, λ) be the $2-1$ labelled graph on $2n$ edges obtained from (Γ', λ') by adjoining edges e and f, with $\lambda(e) = \lambda(f) = n$, where e is incident with e' and f with f'(see Figure 19). Then*

$$B_n = \begin{cases} B_{n-1} \\ \text{or } B_{n-1}\Delta\{n-1, n\} \end{cases}$$

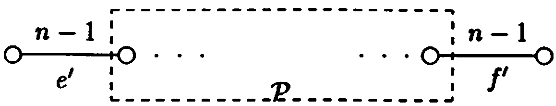


Figure 19

Proof: If exactly one of e and f is incident with an edge of $\mathcal{P}$ then $n - 1 \in B_n$. If $n-1 \in B_{n-1}$, then $B_n = B_{n-1}$. If $n-1 \notin B_{n-1}$, then $B_n = B_{n-1} \cup \{n-1, n\} = B_{n-1}\Delta\{n-1, n\}$. On the other hand, if either both or neither of e and f is incident with an edge of $\mathcal{P}$ then $n-1 \notin B_n$. Hence if $n-1 \notin B_{n-1}$, then $B_n = B_{n-1}$, while if $n-1 \in B_n - 1$, then $B_n = B_{n-1} - \{n-1\} \cup \{n\} = B_{n-1}\Delta\{n-1, n\}$. □

Proposition 7 *Let (Γ', λ') and (Γ, λ) be as in Lemma 3. Suppose $E(\varphi'(\Gamma'))$ is fundamental for Q_{n-1} with associated subgroup G' of Σ_{n-1}. Then $E(\varphi(\Gamma))$ is fundamental for Q_n with an associated subgroup G of Σ_n.*

Proof: Let $G' = \{\sigma_A \in \Sigma_{n-1} \mid\mid A \cap T'_i \mid\equiv 0 \pmod 2 \; for \; i = 0, 1\}$. For $1 \leq j \leq n-1$, $B_j(\Gamma) = B_j(\Gamma') = B_j$. By Lemma 3, $B_n = B_n(\Gamma) =$ either B_{n-1} or $B_{n-1}\Delta\{n-1, n\}$.

Case 1: $B_n = B_{n-1}$. Since $\sigma_{B_{n-1}} \notin G'$, at least one of $| B_{n-1} \cap T'_0 |$ and $| B_{n-1} \cap T'_1 |$ is odd. With no loss of generality we may assume $| B_{n-1} \cap T'_0 |$ is odd. Let $T_0 = T'_0$ and $T_1 = T'_1 \cup \{n\}$. So by Lemma 3, $T_0 \cup T_1 = \{1, 2, \ldots, n\}$. For $k < n$ and $j = 0, 1$, $B_k \cap T_j = B_k \cap T'_j$ and $(B_k\Delta\{k\}) \cap T_j = (B_k\Delta\{k\}) \cap T'_j$, so in each case the cardinalities are equal. Since G' is $E(\Gamma')$-good, and thus $\sigma_{B_k} \notin G'$, for each k, $| B_k \cap T_j |$ is odd for at least one value of j, and the same is true for $| (B_k\Delta\{k\}) \cap T_j |$.

Finally, $|B_n \cap T_0| = |B_{n-1} \cap T_0'|$ is odd, by our assumption, and since $(B_n \Delta \{n\}) \cap T_0 = B_n \cap T_0$, $|(B_n \Delta \{n\}) \cap T_0|$ is odd. Thus if $G = \{\sigma_A \in \Sigma_n \,\|\, A \cap T_j \equiv 0 \pmod 2 \; for \; j = 0, 1\}$, then G is $E(\varphi(\Gamma))$-good.

Case 2: $B_n = B_{n-1} \Delta \{n-1, n\}$. For at least one $j \in \{0,1\}$, $|(B_{n-1})\Delta\{n-1\} \cap T_j'|$ is odd, since $\sigma_{B_{n-1}\Delta\{n-1\}} \notin G'$. Without loss of generality, assume that $|(B_{n-1}\Delta\{n-1\}) \cap T_1'|$ is odd. Let $T_0 = T_0' \cup \{n\}$, and $T_1 = T_1'$. Let $G = \{\sigma_A \in \Sigma_n \,\|\, A \cap T_j \,|\!\equiv 0 \pmod 2 \; for \; j = 0, 1\}$. Then $T_0 \cup T_1 = \{1, 2, \ldots, n\}$. As before, for $k < n$, $B_k \cap T_j = B_k \cap T_j'$ and $(B_k \Delta \{k\}) \cap T_j = (B_k \Delta \{k\}) \cap T_j'$, and so $\sigma_{B_k} \notin G$ and $\sigma_{(B_k \Delta \{k\}} \notin G$.

Finally,

$$|(B_n \Delta \{n\}) \cap T_1| = |(B_n - 1\Delta\{n-1\})\Delta\{n\}) \cap T_1|$$

$$= |(B_{n-1}\Delta\{n-1\}) \cap T_1 \,\|\, (B_{n-1}\Delta\{n-1\}) \cap T_1'|$$

which, by assumption, is odd. Also,

$$|B_n \cap T_1| = |(B_{n-1}\Delta\{n-1, n\}) \cap T_1'|$$

$$\equiv$$

$$|(B_{n-1}\Delta\{n-1\}) \cap T_1'| + |\{n\} \cap T_1'| \pmod 2.$$

But $n \notin T_1'$, so $|B_n \cap T_1| = |(B_{n-1}\Delta\{n-1\}) \cap T_1'|$ which is odd. Thus $\sigma_{B_n} \notin G$ and $\sigma_{B_n \Delta\{n-1\}} \notin G$. Hence G is $E(\varphi(\Gamma))$-good and $E(\varphi(\Gamma))$ is fundamental for Q_n. □

We now give some examples of $2-1$ labellings of trees, both bad and good.

Example 4 *Let (T, λ) be the labelled tree of Figure 20.*

Then $B_1 = \{2,3\}$,$B_2 = \{1,3\}$,$B_3 = \{1,3\}$,$B_4 = \{1,2\}$ and $B_1\Delta\{1\} = B_2\Delta\{2\} = B_3\Delta\{3\} = \{1,2,3\}$. Thus $\mathcal{F} \subset \Sigma_3 - \{\emptyset\}$, and so there is no subgroup G of index 4 in Σ_4 with $G \cap \mathcal{F} = \emptyset$. So the $2-1$ labelling λ of T is bad.

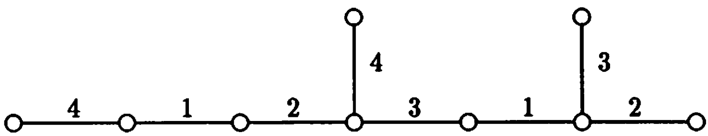


Figure 20. (T, λ)

On the other hand, the following $2-1$ labelling $\hat{\lambda}$ of T, shown in Figure 21, is good.

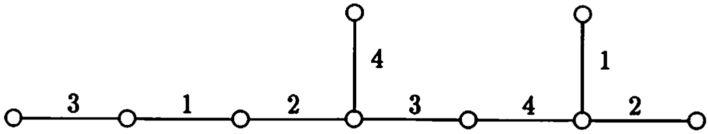


Figure 21. $(T, \hat{\lambda})$

For $B_1 = \{1,2,3,4\}$, $B_2 = B_4 = \{3,4\}$, $B_3 = \{1,2\}$, $B_1\Delta\{1\} = B_2\Delta\{2\} = \{2,3,4\}$, $B_3\Delta\{3\} = 1,2,3$, and $B_4\Delta\{4\} = \{3\}$. Let $S_1 = \{1,2,3\}$ and $S_2 = \{2,3,4\}$, and let $G = \{\sigma_A \in \Sigma_4 \mid\mid A \cap S_j \mid\equiv 0 \pmod 2),\ for\ j = 1,2\}$. Then $\mathcal{F} \cap G = \emptyset$.

Note: If we delete the two pendant edges labelled 4 from (T, λ) in Example 4 we obtain a good pair (T', λ') with associated subgroup $G = \{id, \sigma_{\{1,2\}}\}$ of index 4 in Σ_3 (see Figure 22).

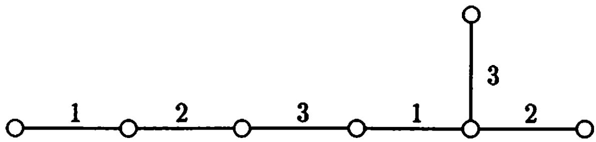


Figure 22. (T', λ')

Example 5 *Let T be the tree shown in Figure 23a. The $2-1$ labellings λ_1 and λ_2, shown in Figures 23b and 23c are both bad. For $\mathcal{F}(\lambda_1) \subset \Sigma_3\{id\}$ and $\mathcal{F}(\lambda_2) \subset \{\sigma_A \mid A \subset \{1,3,4\}, A \neq \emptyset\}$. However, the labelling λ_3 shown in Figure 23d is good. For each pendant edge labelled 4 is incident with and edge labelled 3. If we delete the 2 edges labelled 4 and their incident leaves, we have a pair (T', λ') (see Figure 23e) which has the associated good subgroup $G = \{id, \sigma_{\{1,2\}}\}$ of Σ_3. Thus by Proposition 7, λ_3 is a good $2-1$ labelling of T.*

We can now state the main result of this section. Its proof is quite long, requiring six lemmas, and so we omit it.

Theorem 1 *Let $n \geq 3$ and let Γ be a tree on $2n$ edges with diameter$(\Gamma) = 4$. Suppose $\Gamma \supset \Gamma_0$, $deg x_0 \leq n$, and $deg x_i \leq n-1$ for all vertices x_i adjacent to x_0. Then there exists a $2-1$ labelling λ of $E(\Gamma)$ which is E-good.*

Combining Theorem 1 and Proposition 5 we see that for $n \geq 3$, any tree on $2n$ edges with diameter 4 which has a $2-1$ embedding in Q_n also has a good $2-1$ embedding. We conjecture that this is true regardless of the diameter.

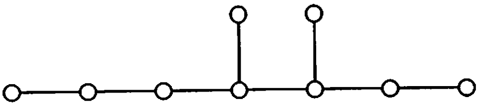


Figure 23a. T

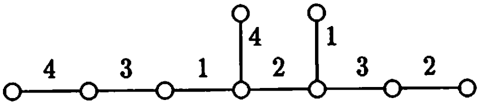


Figure 23b. (T, λ_1)

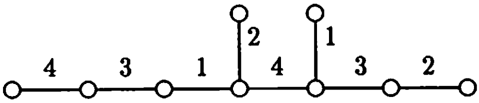


Figure 23c. $(T, \lambda_2$

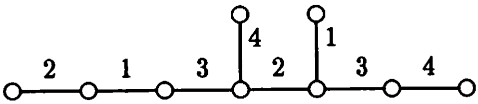


Figure 23d. (T, λ_3)

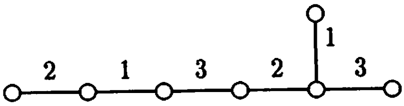


Figure 23e. (T', λ')

Conjecture 1 *If $n \geq 3$, any tree which has a $2-1$ embedding in Q_n also has a good $2-1$ embedding.*

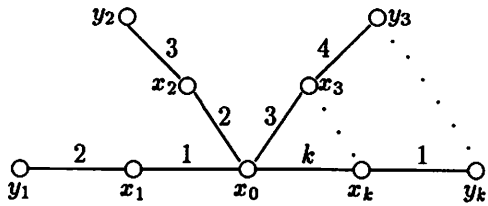


Figure 24. Γ

5 Paths on 2n Edges.

A path on $2n$ edges can *not* have a $2-1$ embedding in Q_n. For with each edge direction occurring twice, the image in Q_n would be a closed walk. We can ask instead whether there is an embedding of P_{2n} whose image is a fundamental set for Q_n. For n odd, this is impossible.

Proposition 8 *Let n be odd, and let P_k be a path on k edges in Q_n so that isomorphic copies of P_k yield an edge decomposition of Q_n. Then $k \leq n$.*

Proof: Q_n is n-regular. Since every non-terminal vertex of P_k has degree 2, and each vertex of Q_n has odd degree, each vertex of Q_n must be a terminal vertex of at least one copy of P_k. If Q_n had an edge decomposition by m copies of P_k, then there would be at most $2m$ terminal vertices in Q_n. Hence $2m \geq |V(Q_n)| = 2^n$ and so $m \geq 2^{n-1}$. Since $n \cdot 2^{n-1} = |E(Q_n)| = k \cdot m$, we find that $k = n \cdot 2^{n-1}/m \leq n$. □

Conjecture 2 *If $n \geq 4$ is even then some $P_{2n} \subset Q_n$ is a fundamental set for Q_n.*

Example 6 *We shall establish the conjecture for $n = 4$. Let P_8 be the path with initial vertex $\vec{0}$ and edge direction sequence* $1, 2, 1, 3, 2, 1, 2, 4$. *Let θ be the permutation* $(1,3)(2,4)$ *and let ρ_θ denote the automorphism of Q_4 defined by $x_1x_2x_3x_4 \mapsto x_{\theta(1)}x_{\theta(2)}x_{\theta(3)}x_{\theta(4)} = x_3x_4x_1x_2$. Then the subgroup $G = \langle \sigma_{\{1,2,3,4\}}, \rho_\theta \rangle$ of $Aut(Q_4)$ has order 4 and is $E(P_8)$-good. For $P_8 \cup \sigma_{\{1,2,3,4\}}(P_8)$ is the Hamiltonian cycle with edge direction sequence* $(1,2,1,3,2,1,2,4)^2$, *which, as was observed in* [2, Example (a)(i), Section 4], *is fundamental for Q_4 with associated subgroup $\langle \rho_\theta \rangle$ of $Aut(Q_4)$.*

References

[1] E. N. Gilbert, *Gray codes and paths on the n-cube*, Bell System Tech. J. 37, (1958) 815-826.

[2] M. Ramras, *Symmetric edge-decompositions of hypercubes*, Graphs and Combinatorics 7, (1991) 6587.