Uniform Restricted Resolvable Designs with $r = 3$

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Abstract

A Restricted Resolvable Design $R_r RP(p, k)$ is a resolvable design on p points with block sizes r and $r + 1$ in which each point appears k times. An RRP is called uniform if all resolution classes consist of the blocks of the same size.

We show that a uniform $R_3RP(p, \frac{p}{2} - 2)$ exists for all $p \equiv 12 \bmod 24$, $p \neq 12$ except possibly when $p = 84$ or 156. We also show that if $g \equiv 3 \bmod 6$ $g \notin \{3, 21, 39\}$ and $p \equiv 4g \bmod 8g$ then there exists an $R_3RP(p, \frac{p}{2} - (r + 1))$ for all

- $r \leq \frac{p-4g}{8g}$ if $\frac{p}{4g}$ is a prime power congruent to 1 mod 6;
- $r \leq \frac{p}{4gq}$ where q is the smallest proper factor of $\frac{p}{4g}$ if $\frac{p}{4g}$ is composite and there exists an $RT(g, \frac{p}{4gq})$.

1 Introduction

A *Pairwise Balanced Design* $PBD(v, K, \lambda)$ is a pair $(X, \mathcal{B})$ where $|X| = v$ and $\mathcal{B}$ is a collection of subsets of X, called blocks. Each block has size $k \in K$ and each pair of points of X occurs exactly λ times in the blocks. If the blocks can be partitioned into *parallel classes* such that each parallel class contains each member of X exactly once the design is called *resolvable* and we write $RPBD(v, K, \lambda)$. In this paper we only consider designs with $\lambda = 1$ and so we will usually omit the λ. A parallel class is called *uniform* if it contains blocks of only one size.

Definition 1.1 *A Restricted Resolvable Design $R_r RP(p, k)$ is a resolvable pairwise balanced design on p points with block sizes r and $r + 1$ such that each point appears k times in the design. We call the design uniform if all the parallel classes are uniform and non-uniform otherwise.*

The reason for transposition of the usual roles of r and k is due to the relation of these designs to the so called $g^{(k)}(v)$ problem. $g^{(k)}(v)$ is defined to be the smallest number of blocks necessary to create a design on v points in which the largest block is of size k. Stinson [15] has shown that the optimal arrangement can be achieved by adding a block of size k at "infinity" to an $R_rRP(v-k,k)$, where $r = \lfloor \frac{v-1}{k} \rfloor$.

All the results in this paper are for uniform $R_rRP(p,k)$. Note that if an $R_rRP(p,k)$ design is uniform then there are $p - k(r-1) - 1$ parallel classes of size $r+1$ and $\frac{(r+1)(p-1)}{r-1} - kr$ parallel classes of size r. In particular if $r = 3$ this becomes $2p - 3k$.

The existence problem for $R_1RP(p,k)$ was solved in 1981 by Pullman [4] and Stanton et.al. [14]. Recently Rees [10, 12, 13] has solved the spectrum of $R_2RP(p,k)$. We will be concerned here with $R_3RP(p,k)$ designs, i.e. designs with block sizes three and four. In the rest of this section we present definitions and known results for designs which we shall use. In the next section we present a frame construction for R_3RP designs and two Harrison type constructions. In the final section we derive the results. We first derive the necessary conditions for the existence of an $R_3RP(p,k)$.

Theorem 1.1 *The necessary conditions for the existence of an $R_3RP(p,k)$ are that $\lfloor \frac{p-1}{3} \rfloor \leq k \leq \lfloor \frac{p-1}{2} \rfloor$ and $p(6k-p+1) \equiv 0 \bmod 12$.*

If the $R_3RP(p,k)$ is uniform the second condition reduces to $p \equiv 0 \bmod 12$ and we have that if $k < \frac{p}{2} - 1$ then $p > 16$.

Proof:

Consider a point, suppose it is in s blocks of size 3 and t blocks of size 4, thus $k = s + t$ and $p - 1 = 2s + 3t$, the first inequality follows. We now count blocks, there are $\frac{ps}{2} + \frac{pt}{3} = \frac{p}{12}(6k - p + 1)$ of them. Thus $p(6k-p+1) \equiv 0 \bmod 12$.

If the design is uniform then p must be divisible by both 3 and 4, thus $p \equiv 0 \bmod 12$. If $k < \frac{p}{2} - 1$ this implies that there is more than one parallel class of size 4. After we place the first parallel class of size four, if $p < 16$, we will be unable to place another one without repeating edges contained in the first parallel class. □

We will use the idea of a group divisible design.

Definition 1.2 *A group divisible design K-GDD of type $(g_1)^{u_1} \ldots (g_s)^{u_s}$ is a triple $(X, \mathcal{B}, \mathcal{G})$ where X is a set of $v = \sum g_i u_i$ points. $\mathcal{G}$ is a collection of subsets of X called groups, there are exactly u_i groups of size g_i. The groups exactly partition the points of X between them. $\mathcal{B}$ is a collection of blocks with sizes from the set K such that each pair of points of X, not both in the same group, appears exactly once in the blocks.*

If the blocks of a group divisible design may be resolved into resolution classes we call it a resolvable group divisible design and denote it by K-RGDD of type $(g_1)^{u_1} \dots (g_s)^{u_s}$.

If $K = \{k\}$ then we write k-(R)GDD. The following was shown in [1, 5, 11].

Theorem 1.2 *A 3-RGDD of type g^u exists if and only if*

- $u \neq 2$
- $ug \equiv 0 \bmod 3$
- $g(u-1) \equiv 0 \bmod 2$
- $(g, u) \notin \{(2,6), (6,3)\}$

Thus, for every $u \equiv 0 \bmod 3$ there exists a 3-RGDD of type 4^u. We note that this implies the existence of an $R_3RP(4u, 2u-1)$, i.e. for all $p \equiv 0 \bmod 12$ there exists an $R_3RP(p, \frac{p}{2}-1)$.

One special case is when $v = gk$, in this case we have a resolvable transversal design, which we denote by $RT(k, g)$. These designs are a special case of *uniformly resolvable designs*.

Definition 1.3 *A uniformly resolvable design k-URD(v, g, r) is a resolvable design on v points each of whose resolution classes is uniform of size k or g. There are r resolution classes of size g in total.*

Danziger and Mendelsohn [2] have shown the following.

Theorem 1.3 *If $g \equiv 3 \bmod 6$ and n is odd then there exists a 3-URD(ng, g, r) for all*

- $r \leq \frac{n-1}{2}$ *if n is a prime power congruent to* $1 \bmod 6$;
- $r \leq \frac{n}{p}$ *if n is composite and p is any factor of n such that there exists an $RT(g, \frac{n}{p})$.*

We shall also use a special type of GDD called a frame.

Definition 1.4 *A K-frame of type $g_1^{u_1} \dots g_s^{u_s}$ is a group divisible design with groups G_i, where $|G_i| = g_i$. The blocks may be partitioned into holey parallel classes, each of which partitions the set $X \setminus G_i$ for some i. We call the groups holes, and say that the hole G_i is of degree d_i if there are d_i holey parallel classes which partition $X \setminus G_i$.*

For ease of notation we will sometimes denote a K-frame of type $g_1^{u_1} \dots g_s^{u_s}$ by the multiset with u_i copies of each g_i. We give the following definitions for particular types of frames.

Definition 1.5 *Given a K-frame of type* $g_1^{u_1} \dots g_s^{u_s}$

- *we call it uniform each partial parallel class is of only one block size;*
- *we call it completely uniform if for each hole* G_i *the resolution classes which span* $X \setminus G_i$ *are all of one block size;*
- *we call it pure if* $K = \{k\}$.

We note that the standard usage of the term *uniform* when applied to frames is that there is only one group size. However the term *uniform frame* is here reserved for another case in accordance with the general use of the term *uniform* in this paper. With this definition a *pure frame* is *completely uniform* and a *completely uniform* frame is *uniform*.

If a frame is uniform we may denote it as a frame of type

$$(g_1 \,;\, k_1^{n_{11}}, \dots, k_p^{n_{1p}})^{u_1} \dots (g_s \,;\, k_1^{n_{s1}}, \dots, k_p^{n_{sp}})^{u_s}$$

where there are u_i holes of size g_i missed by n_{ij} uniform partial resolution classes with blocks of size k_j. Thus the degree of each hole with size g_i is given by $d_i = \sum_j n_{ij}$. If the hole is completely uniform we will often omit the n_{ij}, in this case the n_{ij} are implied by the following corollary to a result due to Rees [8].

Theorem 1.4 *In a pure k-frame of type* $g_1^{u_1} \dots g_s^{u_s}$*, each hole has degree* $\frac{g_i}{k-1}$*. In a completely uniform K-frame of type* $(g_1 \,;\, k_1)^{u_1} \dots (g_s \,;\, k_s)^{u_s}$*, each hole has degree* $\frac{g_i}{k_i - 1}$

Stinson [16] has shown existence of pure Kirkman frames with block size 3.

Theorem 1.5 *There exists a pure* $\{3\}$*-frame of type* g^u *if and only if* $u \geq 4$ *and*

- $g \equiv 2$ *or* $4 \bmod 6$ *and* $u \equiv 1 \bmod 3$;
- $g \equiv 0 \bmod 6$, *all* $u \geq 4$.

We will also use a resolvable design with a hole.

Definition 1.6 *A resolvable PBD(*v, K, m*) with a hole of size* w *and degree* d *is a design on a* v*-set* X *with block sizes from* K *and a hole* G *of size* w *in which every pair of points in* X *not in the hole is covered exactly once by the blocks. Further the blocks may be partitioned into* $m + d$ *classes,* m *of which are resolutions of* X*,* d *of which are resolutions of* $X \setminus G$.

Rees and Stinson [6, 7, 9] have shown the following Theorem, which is a generalisation of the Doyen Wilson Theorem to the resolvable case.

Theorem 1.6 *For any* $v \equiv 3 \bmod 6$ *and* $w \equiv 3 \bmod 6$ *such that* $v \geq 3w$ *there is a resolvable PBD(*$v, \{3\}, \frac{v-w}{2}$*) with a hole of size* w *and degree* $\frac{w-1}{2}$

2 Constructions

In this section we introduce the basic constructions that we will use. First we introduce some constructions for producing frames, then we introduce a frame construction for restricted resolvable designs and finally we give two Harrison type constructions for these designs.

The following four frame constructions are well known. We are particularly interested in how blocks of different sizes are affected by these constructions. The first is the fundamental frame construction.

Theorem 2.1 (Fundamental Frame Construction (FFC)) *Let $(X, \mathcal{B}, \mathcal{G})$ be a group divisible design, and let $w(x) : X \to Z^+ \cup \{0\}$ ($w(x)$ is called a weighting), $d(x) : X \to Z^+ \cup \{0\}$ (d is called a degree function). Suppose that for each block B from the group divisible design there is a K-frame of type $\{w(x) \mid x \in B\}$ in which hole $w(x)$ has degree $d(x)$. Then there exists a K-frame of type $\{\sum_{x \in G_i} w(x) \mid G_i \in \mathcal{G}\}$, further each hole G_i has degree $\sum_{x \in G_i} d(x)$.*

Proof:

We give a sketch of the proof, for a complete proof see for example [1]. The idea is to expand each point, x, in the original GDD $w(x)$ times. On each block B place a K-frame of type $\{w(x) \mid x \in B\}$. Consider a point x from a group G. The set of points in the blocks of the GDD in which x appears is a resolution of $X \setminus G$. Thus the collection of all the partial resolution classes from the frames with hole $w(x)$ is a set of partial resolution classes of the whole set except for $\{\sum_{x \in G_i} w(x) \mid G_i \in \mathcal{G}\}$. □

We note that if for each point x, the blocks containing x get a uniform K-frame then the resulting K-frame will be uniform.

Another well known method for creating frames is inflation by transversal designs.

Theorem 2.2 (Inflation by Resolvable Transversals) *If there exists a K-frame of type $(g_1)^{u_1} \ldots (g_s)^{u_s}$ and for each $k \in K$ there exists an RT(k, n) then there exists a K-frame of type $(ng_1)^{u_1} \ldots (ng_s)^{u_s}$.*

Proof:

The idea is to expand each point n times, on the expanded blocks place an RT(k, n) in such a way that the groups are the expansion of each point, the result is the required design. □

We note that if there are m_i blocks of a given size k_i in the original frame then there will be nm_i blocks of size k_i in the new frame. Further if the original frame is uniform and there were r_i partial resolution classes of a given size k_i

then the new frame will also be uniform and there will be nr_i resolution classes of size k_i.

We also will use the following tripling construction due to Stinson [16].

Theorem 2.3 (Tripling Construction) *If there exists a uniform K-frame of type $(g_1)^{u_1}\dots(g_s)^{u_s}$ and for each $k \in K$ there exists an $RT(3,k)$ (i.e. there are no blocks of size 2 or 6) then there exists a $\{3\}\cup K$-frame of type $(3g_1)^{u_1}\dots(3g_s)^{u_s}$.*

Proof:

Expand each point of the K-frame 3 times. On each expanded block of size k_i place an RT$(3, k_i)$ in such a way that one resolution class of blocks of size 3 is the expansion of each point, remove this resolution class to obtain the design. □

We note that if the original frame is uniform and there are r_i partial resolution classes of size k_i ($k_i \neq 3$) then the resulting frame will also be uniform and there will be r_i partial resolution classes of size k_i. There will however be new parallel classes of size 3, and so a pure frame will not remain pure, unless $K = \{3\}$.

The advantage of this construction is that for all block sizes k_i, other than 3, if there are m_i blocks of this size in the original design then there will be m_i blocks of this size in the resulting design.

Theorem 2.4 *If there exists a 3-RGDD of type g^u then there exists a completely uniform $\{3,g\}$-frame of type $(2\,;\,3^1)^s(g-1\,;\,g^1)^r$, where $s = \frac{1}{2}(g(u-1))$.*

Proof:

Remove a point from the 3-RGDD of type g^u and identify the short block in each resolution class as the hole. □

We now give a frame construction for R_rRP designs.

Theorem 2.5 *If there exists a uniform $\{3,g\}$-frame of type $(g_1;\,3)^t(g_2;\,3^\rho,4^r)^1$ (where $\rho = \frac{1}{2}(g_2-3r)$) and w is such that $g_1+w \equiv 3 \bmod 6$, $2w \le g_1$, and there exists a uniform $R_3RP(g_2+w,k)$ ($k = \frac{g_2+w-r-1}{r-1}$) then there exists a uniform $R_3RP(g_1t+g_2+w, \frac{g_1t+g_2+w-r-1}{r-1})$.*

Proof:

The idea is to adjoin w points at infinity to the original frame, we then fill in each of the holes plus the points at infinity with a resolvable PBD$(g_1+w, \{3\}, \frac{g_1}{2})$ with a hole of size w and degree $\frac{w-1}{2}$ in such a way that the hole is exactly the w points at infinity. The last hole, of size g_2, we fill with the R$_3$RP(g_2+w,k).

More formally we will construct the design on $X = I_v \cup I_w$, where $v = g_1t+g_2$. We will denote the members of I_w by W and call them points at infinity.

Figure 1: The resolution classes A_{ij}

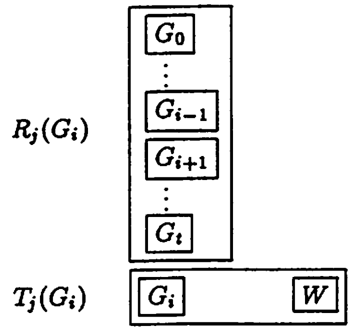
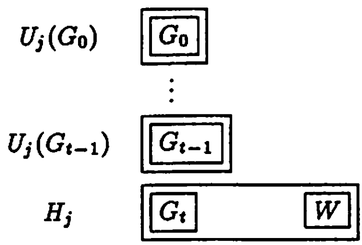


Figure 2: The resolution classes B_j

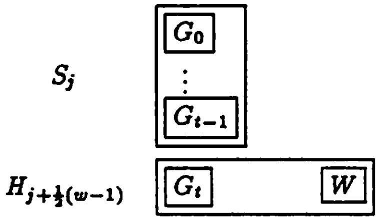


We start with the uniform $\{3, g\}$-frame of type $(g_1;\, 3)^t(g_2;\, 3^\rho, 4^r)^1$ with point set I_v. Let the holes of size g_1 be denoted by G_i, $i = 0, \ldots, t-1$ and the hole of size g_2 be denoted by G_t. For the holes of size g_1 denote the partial parallel classes which miss the hole G_i by $R_j(G_i)$, $i = 0, \ldots, t-1$, $j = 1, \ldots, \frac{g_1}{2}$. We note that by Theorem 1.4 the degree of the holes of size g_1 is $\frac{g_1}{2}$.

We denote the partial parallel classes of triples which miss the hole G_t of size g_2 by S_j, $j = 1, \ldots, \rho$ and those of size 4 by Q_j, $j = 1, \ldots, r$. By counting the degree of a vertex not in the hole of size g_2 we obtain $\rho = \frac{1}{2}(g_2 - 3r)$.

The conditions that $2w \leq g_1$ and $g_1 + w \equiv 3 \bmod 6$ implies, by Theorem 1.6, that there exists a resolvable PBD$(g_1 + w, \{3\}, \frac{g_1}{2})$ with a hole of size w and degree $\frac{w-1}{2}$. For each hole of size g_1, G_i, $i = 1, \ldots, t-1$, of the frame we construct such a design with point set $G_i \cup W$, with the hole of size w which exactly covers the points at infinity, W. This covers all edges in the holes of size g_1 from the frame. Denote the $\frac{g_1}{2}$ full resolution classes of this design by $T_j(G_i)$, $j = 1, \ldots, \frac{g_1}{2}$ and the $\frac{w-1}{2}$ partial ones by $U_j(G_i)$. Each of the collections of

Figure 3: The resolution classes C_j



blocks

$$A_{ij} = (R_j(G_i) \cup T_j(G_i)), \;\; i = 1, \ldots, t \;\; j = 1, \ldots, \frac{g_1}{2}$$

are resolutions of the set X.

For the hole of size g_2, G_t, we construct a uniform $\mathrm{R_3RP}(g_2 + w, \frac{g_2+w-r-1}{r-1})$ with point set $G_t \cup W$ and resolution classes of size 4 denoted by V_j, $j = 1, \ldots, r$ and those of size 3 by H_j, $j = 1, \ldots, \gamma$. ($\gamma = \frac{1}{2}(g_2 + w - 1 - 3r)$). This covers all the edges in the hole of size w and the hole of size g_2 from the frame. The collections of blocks

$$B_j = H_j \cup \bigcup_{i=0}^{t-1} U_j(G_i), \;\; j = 1, \ldots, \frac{w-1}{2}$$

are $\frac{w-1}{2}$ parallel classes of triples. This leaves exactly $\frac{1}{2}(g_2 - 3r)$ resolution classes of triples from the RRP, H_j, $j = \frac{w-1}{2}, \ldots, \gamma$, which we pair with the triples from the partial resolutions in the frame with hole G_t, S_j.

$$C_j = H_{j+\frac{1}{2}(w-1)} \cup S_j, \;\; j = 1, \ldots, \frac{1}{2}(g_2 - r(g-1)).$$

Finally we form the resolution classes of size 4, from the partial resolution classes from the frame and the resolution classes of size 4 from the RRP.

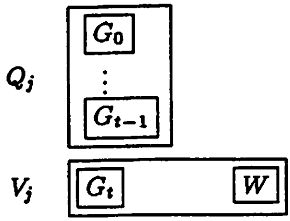
$$D_j = V_j \cup Q_j, \;\; j = 1, \ldots, r$$

□

We give the following two Harrison constructions. In these constructions each point in a *master* design is expanded and the resulting large blocks are filled by other designs called *ingredients*.

The following is a Harrison type construction which allows us to multiply p while leaving the number of blocks of size 4 constant.

Figure 4: The resolution classes D_j



Theorem 2.6 *If $n \equiv 0 \bmod 3$ and there exists an $R_3RP(p,k)$ $(p \neq 24)$ then there exists a $R_3RP(np, \frac{p(n-1)}{2}+k)$. Note that the number of parallel classes of size 4 remains the same.*

Proof:

By the conditions on n and Theorem 1.2 there is a 3-RGDD of type 4^n, we will use this design as the *master* design. It is well known that if $\frac{p}{4} \neq 2$ or 6 there exists an $\mathrm{RT}(3, \frac{p}{4})$. We will use an $\mathrm{RT}(3, \frac{p}{4})$ and the $\mathrm{R_3RP}(p,k)$ as *ingredient* designs.

Each point of the 3-RGDD of type 4^n is expanded $\frac{p}{4}$ times. On each of the expanded groups, of size p, we place the $\mathrm{R_3RP}(p,k)$. Each point occurs k times. On the expanded blocks, of size $\frac{3p}{4}$ we place an $\mathrm{RT}(3, \frac{p}{4})$ in such a way that the groups are the expansion of each point. Each point occurs $2(n-1)$ times in the 3-RGDD of type 4^n, each of these will occur $\frac{p}{4}$ times in the completed design. Thus this part contributes $\frac{p(n-1)}{2}$ occurrences of each point. □

Theorem 2.7 *If $p \equiv 4g \bmod 8g$ where $g \equiv 3 \bmod 6$ (which implies $\frac{p}{4}$ is odd) and there exists a $R_3RP(4g, 2g-2)$ and a 3-$URD(\frac{p}{4}, g, r)$ then there exists a $R_3RP(p, \frac{p}{2}-(r+1))$.*

Proof:

We will take the 3-$\mathrm{URD}(\frac{p}{4}, g, r)$ as our *master* design, we expand each point 4 times. Note that each point will occur r times in blocks of size g and $\frac{1}{2}(\frac{p}{4} - 1 - r(g-1))$ times in blocks of size 3.

Place copies of the $\mathrm{R_3RP}(4g, 2g-2)$ on the expanded blocks of size g in such a way that the expansion of a point is one of the parallel classes of size 4, remove this class from all of the copies to avoid duplication of edges. Thus each point occurs $r(2g-3)$ times. On the expansion of each point place a block of size 4, giving another parallel class of size 4.

Now place an $\mathrm{RT}(3,4)$ on the expanded blocks of size 3 in such a way that the groups are the expansion of each point. Each point occurs $2(\frac{p}{4}-1-r(g-1))$ times. Thus in total each point will occur $\frac{p}{2}-(r+1)$ times. □

3 Results

In this section we present the results. We first derive some frames to use in the construction Theorem 2.5, we then use this theorem to show that there exists an $R_3RP(p, \frac{p}{2} - 2)$ for all $p \equiv 12 \bmod 24$ with two possible exceptions. We use this result in conjuction with the construction Theorem 2.6 to derive the main result. We shall use the following result (see for example [8]).

Lemma 3.1 *If $t \equiv 1 \bmod 3$, $t \geq 4$ and $0 \leq u \leq \frac{4(t-1)}{3}$ then there exists a $\{4,5\}$-GDD of type $4^t u^1$. Further, each point in the group of size u appears only in blocks of size 5.*

Proof:

It is well known that there exists an PBD$(4t, \{4\}, 1)$ (see [3]), with $\frac{4t-1}{3}$ parallel classes of size 4. Add a group of size u at infinity to this design, to the first u parallel classes adjoin a distinct point from the group of size u to get blocks of size 5. By the conditions on u there is at least one parallel class of size four left untouched, take one of these parallel classes along with the group of size u as the groups of the new design. □

Lemma 3.2 *For all $t \equiv 1 \bmod 3$, $t \geq 4$, $0 \leq u \leq \frac{4(t-1)}{3}$ and $n \neq 2, 3, 6, 10$ there is a $\{3,4\}$-frame of type $(24n\,;\, 3^{12n})^t(9un\,;\, 4^{un}, 3^{3un})^1$*

Proof:

We will first construct a $\{3,4\}$-frame of type $(8n\,;\, 3^{4n})^t(3un\,;\, 4^{un})^1$, apply the *tripling construction* (Theorem 2.3) to this frame to get the required frame.

To construct this frame we use the the *fundamental frame construction* (Theorem 2.1) with the group divisible design given in Lemma 3.1. We give each point in the groups of size four weight $2n$ and each point in the group of size u weight $3n$. A $\{3\}$-frame of type $(2n)^4$ exists by Theorem 1.5, we use frames of this type to cover the expanded groups of size 4.

We cover the groups of size 5 with a $\{3,4\}$-frame of type $(2n\,;\, 3^n)^4(3n\,;\, 4^n)^1$. To construct this frame we note that by Theorem 1.2 there exists a 3-RGDD of type 4^3 thus by Theorem 2.4 there is a $\{3,4\}$-frame of type $(2\,;\, 3^1)^4(3\,;\, 4^1)^1$. Apply *inflation by resolvable transversals* (Theorem 2.2) to get a $\{3,4\}$-frame of type $(2n\,;\, 3^n)^s(3n\,;\, 4^n)^1$. □

We now apply the frame construction Theorem, 2.5 using the frame above, with $g_2 = 9un$ to get the following.

Theorem 3.1 *If there exists an $R_3RP(v, \frac{v-2}{un-1} - 1)$, where $v = 9un + w$ and $w \equiv 3 \bmod 6$, $w \leq 12n$ then there exists an $R_3RP(p, \frac{p-2}{un-1} - 1)$, where $p = (72s + 9u + 24)n + w$, for all $s \geq \frac{u}{4}$ and $n \neq 2, 3, 6, 10$.*

Taking $u = 3$, $n = 1$ and $w = 9$ we get the following.

Theorem 3.2 *For all $p \equiv 60 \bmod 72$ there exists a uniform $R_3RP(p, \frac{p}{2} - 2)$.*

Proof:
If $p > 60$ this is the previous theorem with $u = 3$, $n = 1$ and $w = 9$, a uniform $R_3RP(36, 16)$ is given in the appendix. For the case where $p = 60$ a uniform $R_3RP(60, 28)$ is given in the appendix. □

We now use a similar method to show there exists a uniform $R_3RP(p, \frac{p}{2} - 2)$ for all $p \equiv 12 \bmod 72$, $p \neq 12$ except possibly when $p = 84$ or 156. We first note that there is no $R_3RP(12,4)$ by the necessary conditions for the existence of a uniform Restricted Resolvable Design (Theorem 1.1).

Lemma 3.3 *There exists a {3,4}-frame of type $(24\,;\,3^{12})^t(51\,;\,3^{21},4^3)^1$ for all $t \equiv 1 \bmod 3$ with $t > 4$.*

Proof:
We start with the GDD given in Lemma 3.1 with $u = 7$, i.e a {4,5}-GDD of type 4^t7^1. We identify three of the points in the group of size u, we call these points high weight points, the other four points we will call low weight points.

Apply the *fundamental frame construction* (Theorem 2.1) giving each point in the groups of size 4 weight 6. We also give the four low weight points weight 6. The three high weight points get weight 9.

By Theorem 1.5 there exists a {3}-frame of type 6^4 and a {3}-frame of type 6^5. We cover each block of size 4 with a {3}-frame of type 6^4. If a block of size 5 contains a low weight point we cover that block with a {3}-frame of type 6^5.

For the blocks of size 5 containing a high weight point we use a {3,4}-frame of type $(6\,;\,3)^4(9\,;\,3^3,4^1)^1$. To get this frame take $v = 12$ in Theorem 2.4 and apply the *tripling construction* (Theorem 2.3). This gives the required frame. □

We now apply the frame construction Theorem 2.5.

Theorem 3.3 *There exists a uniform $R_3RP(p, \frac{p}{2} - 2)$ for all $p \equiv 12 \bmod 72$, $p \neq 12$, except possibly when $p = 84, 156$.*

Proof:
A uniform $R_3RP(60, 28)$ is given in the appendix, use the {3,4}-frame of type $(24\,;\,3^{12})^t(51\,;\,3^{21},4^3)^1$ from the above lemma in the construction Theorem 2.5. Take $g_1 = 24$, $g_2 = 51$, $w = 9$ and $t = 1 + 3s$ to get a uniform $R_3RP(72s + 84, 36s + 40)$ for all $s > 1$. □

Finally we show there exists a uniform $R_3RP(p, \frac{p}{2} - 2)$ for all $p = 36 \bmod 72$.

Lemma 3.4 *There exists a {3,4}-frame of type* $(24\,;\,3^{12})^t(219\,;\,3^{105},4^3)^1$ *for all* $t \equiv 1 \bmod 3$ *with* $t \geq 28$.

Proof:

The proof is similar to the proof of Lemma 3.3. We start with the GDD given in Lemma 3.1 with $u = 35$, i.e a {4,5}-GDD of type $(4)^t(35)^1$. We identify three of the points in the group of size u, we call these points high weight points, the other 32 points we will call low weight points.

Apply the *fundamental frame construction* (Theorem 2.1) giving each point in the groups of size 4 weight 6. We also give the low weight points weight 6. The three high weight points get weight 9.

By Theorem 1.5 there exists a {3}-frame of type 6^4 and a {3}-frame of type 6^5. We cover each block of size four with a {3}-frame of type 6^4. If a block of size 5 contains a low weight point we cover that block with a {3}-frame of type 6^5.

For the blocks of size 5 containing a high weight point we use a {3,4}-frame of type $(6\,;\,3)^4(9\,;\,3^3,4^1)^1$. To get this frame take $p = 12$ in Theorem 2.4 and apply the *tripling construction* (Theorem 2.3). This gives the required frame. □

We now apply the frame construction Theorem 2.5.

Theorem 3.4 *There exists a uniform* $R_3RP(p, \frac{p}{2} - 2)$ *for all* $p \equiv 36 \bmod 72$.

Proof:

A uniform $R_3RP(228, 112)$ exists by Theorem 3.3, use the above frame in Theorem 2.5 with $g_1 = 24$, $g_2 = 219$, $w = 9$ and $t = 1 + 3s$, $s > 6$ to get a uniform $R_3RP(72s + 252, 36s + 124)$.

This gives the result except for the cases $p = 36$, 108, 180, 252, 324, 396, 468, 540, 612, 684, we deal with these cases individually, using Theorem 2.6.

A uniform $R_3RP(36, 16)$ is given in the appendix and so by Theorem 2.6 there exists a uniform $R_3RP(36n, 18n - 2)$ for all $n \equiv 0 \bmod 3$. This covers the cases $p = 36, 108, 324, 540$.

A uniform $R_3RP(60, 28)$ is also given in the appendix and so by Theorem 2.6 there exists an $R_3RP(60n, 30n - 2)$ for all $n \equiv 3 \bmod 6$. This covers the case $p = 180$.

By Theorem 3.2 there exists a uniform $R_3RP(204, 100)$ and a uniform $R_3RP(132, 64)$, thus by Theorem 2.6 there exists a uniform $R_3RP(396, 196)$ and a uniform $R_3RP(612, 304)$.

By Theorem 3.3 there exists a uniform $R_3RP(228, 112)$, thus by Theorem 2.6 there exists a uniform $R_3RP(684, 340)$

We may deal with the cases $p = 252, 468$ by using Theorem 2.7. There exists an RT(3,4), an $R_3RP(36, 16)$ is given in the appendix, by Theorem 1.3

there exists a 3-URD(63,9,1), this gives us an $R_3RP(252, 124)$. Similarly a 3-URD(117,9,1) gives us an $R_3RP(468, 232)$. □

Putting this together we get:

Theorem 3.5 *If $p \equiv 12 \bmod 24$, $p \neq 12$ then there exists a uniform $R_3RP(p, \frac{p}{2}-2)$ except possibly when $p = 84, 156$.*

We now apply Theorem 2.7 in conjuction with the Uniformly Resolvable Design result, Theorem 1.3, to get the following Theorem.

Theorem 3.6 *If $g \equiv 3 \bmod 6$ $g \notin \{3, 21, 39\}$ and $p \equiv 4g \bmod 8g$ then there exists an $R_3RP(p, \frac{p}{2} - (r+1))$ for all:*

- *$r \leq \frac{p-4g}{8g}$ if $\frac{p}{4g}$ is a prime power congruent to 1 mod 6;*
- *$r \leq \frac{p}{4gq}$ where q is the smallest proper factor of $\frac{p}{4g}$ if $\frac{p}{4g}$ is composite and there exists an $RT(g, \frac{p}{4gq})$.*

We note that even in the best of circumstances this approach can only hope to yield the very top of the spectrum. Even supposing we had the complete spectrum for 3-URD(v, g, r) with $g \equiv 3 \bmod 6$, then $r \leq \lfloor \frac{v-1}{g-1} \rfloor$. But $v = \frac{p}{4}$ so we can estimate the minimum k given by the theorem above as

$$k \approx \frac{p}{2} - \frac{\frac{p}{4} - 1}{g - 1} - 1 = \frac{p(2g-3) - 4(g-2)}{4(g-1)}.$$

This is still far from the theoretical minimum $\lfloor \frac{p}{3} \rfloor$. The value will be smaller for smaller values of g, but $g \geq 9$. Even in the best case, when $g = 9$, this reduces to $\frac{15p-28}{32}$.

A Appendix

In this appendix we give some specific designs. I would like to thank Terry Griggs for some useful conversations on the construction of these designs. These designs where found by a computer search on selected difference families.

Construction A.1 *There exists a uniform $R_3RP(36, 16)$.*

Proof:
We construct the design on $X = I_4 \times I_9$. We take as our three resolution classes of size four:

$$S_1 = \{(0,0), (1,0), (2,0), (3,0)\} \bmod (-, 9)$$

$$S_2 = \{(0,0),(1,1),(2,2),(3,3)\} \bmod (-,9)$$

$$S_3 = \{(0,3),(1,2),(2,1),(3,0)\} \bmod (-,9)$$

We note that there is a KTS(9). Place a copy of this design on each of the four 9 sets. Denote the parallel classes by R_{ij} where $i = 1,\ldots,4$ are the parallel classes of the KTS(9) and $j \in I_4$ denotes on which I_9 the parallel class is placed.

We form the resolution classes of triples as follows:

$$\{(0,0),(1,2),(2,4)\} \bmod (-,9) \cup R_{13}$$
$$\{(0,0),(1,3),(2,6)\} \bmod (-,9) \cup R_{23}$$
$$\{(0,0),(1,5),(2,1)\} \bmod (-,9) \cup R_{33}$$

$$\{(0,0),(1,4),(3,5)\} \bmod (-,9) \cup R_{12}$$
$$\{(0,0),(1,6),(3,1)\} \bmod (-,9) \cup R_{22}$$
$$\{(0,0),(1,7),(3,4)\} \bmod (-,9) \cup R_{32}$$

$$\{(0,0),(2,5),(3,7)\} \bmod (-,9) \cup R_{11}$$
$$\{(0,0),(2,3),(3,8)\} \bmod (-,9) \cup R_{21}$$
$$\{(0,0),(2,8),(3,2)\} \bmod (-,9) \cup R_{31}$$

$$\{(1,0),(2,4),(3,8)\} \bmod (-,9) \cup R_{10}$$
$$\{(1,0),(2,6),(3,3)\} \bmod (-,9) \cup R_{20}$$
$$\{(1,0),(2,7),(3,5)\} \bmod (-,9) \cup R_{30}$$

The final resolution class of triples is given by

$$\bigcup_{i=0}^{4} R_{4i}$$

□

Construction A.2 *There exists a uniform $R_3RP(60,\ 28)$.*

Proof:

We construct the design on $X = I_4 \times I_{15}$. We take as our three resolution classes of size four:

$$S_1 = \{(0,0),(1,0),(2,0),(3,0)\} \bmod (-,15)$$

$$S_2 = \{(0,0),(1,1),(2,2),(3,3)\} \bmod (-,15)$$

$$S_3 = \{(0,3),(1,2),(2,1),(3,0)\} \bmod (-,15)$$

We note that there is a KTS(15). Place a copy of this design on each of the four 15 sets. Denote the parallel classes by R_{ij} where $i = 1,\ldots,7$ are the parallel classes of the KTS(15) and $j \in I_4$ denotes on which I_{15} the parallel class is placed.

We form the resolution classes of triples as follows:

$$\{(0,0),(1,2),(2,4)\} \bmod (-,15) \cup R_{13}$$
$$\{(0,0),(1,3),(2,6)\} \bmod (-,15) \cup R_{23}$$
$$\{(0,0),(1,4),(2,8)\} \bmod (-,15) \cup R_{33}$$
$$\{(0,0),(1,5),(2,10)\} \bmod (-,15) \cup R_{43}$$
$$\{(0,0),(1,7),(2,14)\} \bmod (-,15) \cup R_{53}$$
$$\{(0,0),(1,8),(2,1)\} \bmod (-,15) \cup R_{63}$$

$$\{(0,0),(1,6),(3,7)\} \bmod (-,15) \cup R_{12}$$
$$\{(0,0),(1,10),(3,13)\} \bmod (-,15) \cup R_{22}$$
$$\{(0,0),(1,9),(3,14)\} \bmod (-,15) \cup R_{32}$$
$$\{(0,0),(1,12),(3,1)\} \bmod (-,15) \cup R_{42}$$
$$\{(0,0),(1,13),(3,4)\} \bmod (-,15) \cup R_{52}$$
$$\{(0,0),(1,11),(3,5)\} \bmod (-,15) \cup R_{62}$$

$$\{(0,0),(2,9),(3,12)\} \bmod (-,15) \cup R_{11}$$
$$\{(0,0),(2,10),(3,14)\} \bmod (-,15) \cup R_{21}$$
$$\{(0,0),(2,6),(3,11)\} \bmod (-,15) \cup R_{31}$$
$$\{(0,0),(2,11),(3,8)\} \bmod (-,15) \cup R_{41}$$
$$\{(0,0),(2,12),(3,10)\} \bmod (-,15) \cup R_{51}$$
$$\{(0,0),(2,13),(3,7)\} \bmod (-,15) \cup R_{61}$$

$$\{(1,0),(2,9),(3,12)\} \bmod (-,15) \cup R_{10}$$
$$\{(1,0),(2,10),(3,14)\} \bmod (-,15) \cup R_{20}$$
$$\{(1,0),(2,6),(3,11)\} \bmod (-,15) \cup R_{30}$$
$$\{(1,0),(2,11),(3,8)\} \bmod (-,15) \cup R_{40}$$
$$\{(1,0),(2,12),(3,10)\} \bmod (-,15) \cup R_{50}$$
$$\{(1,0),(2,13),(3,7)\} \bmod (-,15) \cup R_{60}$$

The final resolution class of triples is given by

$$\bigcup_{i=0}^{4} R_{7i}$$

□

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