On Graphs With a Unique Least Common Multiple

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ABSTRACT. A graph G is H-decomposable if G can be decomposed into graphs, each of which is isomorphic to H. A graph G without isolated vertices is a least common multiple of two graphs G_1 and G_2 if G is a graph of minimum size such that G is both G_1-decomposable and G_2-decomposable. It is shown that two graphs can have an arbitrarily large number of least common multiples. All graphs G for which G and P_3 (and G and $2K_2$) have a unique least common multiple are characterized. It is also shown that two stars $K_{1,r}$ and $K_{1,s}$ have a unique least common multiple if and only if r and s are not relatively prime.

1 Introduction

A nonempty graph G is *decomposable* into the subgraphs $G_1, G_2, \ldots, G_n$ if no graph G_i $(1 \leq i \leq n)$ has isolated vertices and $\{E(G_1), E(G_2), \ldots, E(G_n)\}$ is a partition of $E(G)$. If $G_i \cong H$ for each i $(1 \leq i \leq n)$, then we say that G is *H-decomposable* and H *divides* G, and write $H|G$. All terms and notation not defined or described here may be found in [3].

Let G_1 and G_2 be two graphs without isolated vertices. A graph G without isolated vertices is called a *least common multiple* of G_1 and G_2 if G is a graph of minimum size such that $G_1|G$ and $G_2|G$. A graph H without isolated vertices is a *greatest common divisor* of G_1 and G_2 if G is a graph of maximum size such that $G|G_1$ and $G|G_2$. These concepts were introduced in [1]. That every two graphs have a greatest common divisor is

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evident. While it is probably not clear that every two graphs have a least common multiple, this is in fact the case and was verified in [1]. The size of a least common multiple of a star and a cycle was also considered in [1]. In [5] it was shown that every least common multiple of two connected graphs is connected and that every least comon multiple of two 2-connected graphs is 2-connected, but this cannot be extended to k-connected graphs for $k \geq 3$. The size of a least common multiple of K_3 (or K_4) and a path is also discussed in [5]. In [2] a (connected) graph without isolated vertices and having size at least 2 is called a *prime* (*prime-connected*) graph if its only (connected) divisors are K_2 and itself. A divisor that is prime is called a *prime divisor.* In [2] prime trees and prime-connected trees were studied as were graphs with a specified number of prime divisors. Many of these concepts were studied further by Saba in [7].

For the graphs $G_1 \cong P_5$ and $G_2 \cong K_{1,4}$, the graphs H, H', and H'' of Figure 1 are the least common multiples of G_1 and G_2, while $G \cong K_{1,2}$ is the unique greatest common divisor of G_1 and G_2.

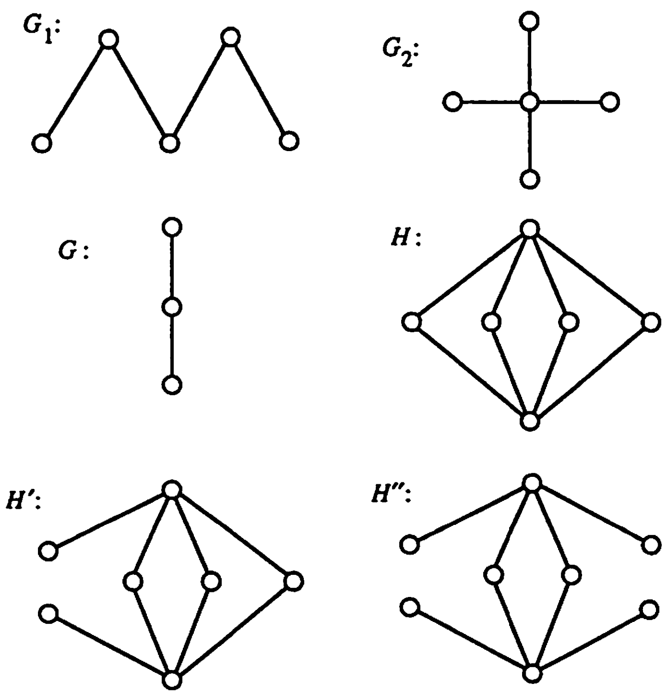

Figure 1

The problem of determining the greatest common divisors of two given graphs G_1 and G_2 appears to be considerably easier than the problem of determining the least common multiples of G_1 and G_2. This is probably due to the facts that a greatest common divisor has a smaller size (often considerably smaller) than a least common multiple and that each greatest common divisor is a subgraph of both G_1 and G_2. Although the graphs G_1 and G_2 of Figure 1 have a unique greatest common divisor, any number of greatest common divisors is possible. For an integer $n \geq 2$ and a graph G, the graph nG consists of n pairwise disjoint copies of G.

Theorem 1. *For every positive integer n, there exist graphs G_1 and G_2 having exactly n greatest common divisors.*

Proof: For $n = 1$, the graphs G_1 and G_2 of Figure 1 have the desired property, so we may assume that $n \geq 2$. Define $G_1 = 2(n-1)P_3$ (a disconnected graph with $2(n-1)$ components each of which is a path of order 3) and $G_2 = 3(n-1)P_3$. For $i = 1, 2, \ldots, n$, define $H_i = (n-i)P_3 \cup 2(i-1)K_2$. (See Figure 2 for $n = 3$.)

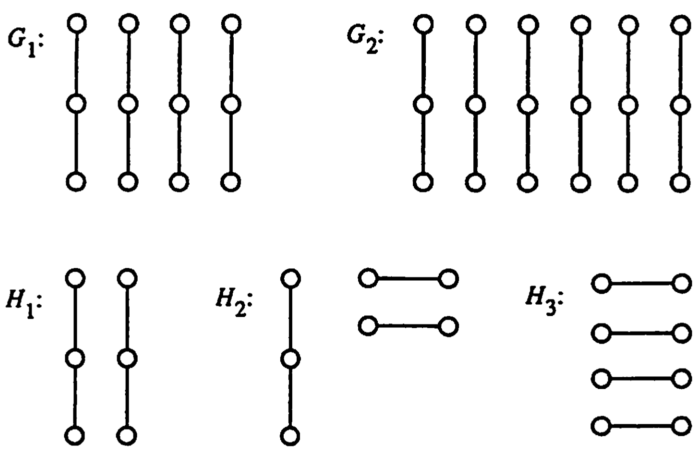

Figure 2

We show that the graphs $H_1, H_2, \ldots, H_n$ are the greatest common divisors of G_1 and G_2. Certainly, G_1 and G_2 are H_i-decomposable for $i = 1, 2, \ldots, n$. Observe that the size of G_1 is $4(n-1)$ and the size of G_2 is $6(n-1)$. Since each graph H_i has size $2(n-1)$ and $\gcd(4(n-1), 6(n-1)) = 2(n-1)$, the graphs H_i are greatest common divisors.

It remains to show that every greatest common divisor of G_1 and G_2 is H_i for some i $(1 \leq i \leq n)$. Let G be a greatest common divisor of G_1 and G_2. Since $G|G_1$, each component of G is P_3 or K_2. Thus $G = rP_3 \cup sK_2$ for nonnegative integers r and s for which $2r + s = 2(n-1)$. This implies that $G = H_i$ for some i $(1 \leq i \leq n)$. □

In addition, two graphs can have any number of least common multiples.

Theorem 2. *For every positive integer n, there exist two graphs having exactly n least common multiples.*

Proof: First observe that since $K_2|2K_2$, the graph $2K_2$ is the unique least common multiple of K_2 and $2K_2$. Moreover, $2K_3$ and the graph obtained by identifying vertices in two copies of K_3 are the only two least common multiples of $P_3 \cup K_2$ and K_3. Thus the result is true for $n = 1$ and $n = 2$. We assume, therefore, that $n \geq 3$. Consider the graph $2K_2$ and the star $K_{1,n-1}$. Since $K_{1,n-1}$ is not a multiple of $2K_2$ but $2K_{1,n-1}$ is, $2K_{1,n-1}$ is a least common multiple of $2K_2$ and $K_{1,n-1}$, and the size of every least common multiple of $2K_2$ and $K_{1,n-1}$ is $2(n-1)$; that is, every least common multiple of $2K_2$ and $K_{1,n-1}$ can be decomposed into two copies of $K_{1,n-1}$.

Observe that if H is a graph of size $2(n-1)$ that is $2K_2$-decomposable, then every vertex of H has degree at most $n-1$. Hence, if H is a least common multiple of $2K_2$ and $K_{1,n-1}$ that is decomposed into two copies H_1 and H_2 of $K_{1,n-1}$, then the vertex of degree $n-1$ in H_1 or H_2 cannot be the same in H as any vertex of the other. Thus H must be obtained by identifying a certain number m of end-vertices of H_1 with the same number of end-vertices of H_2. Since $0 \leq m \leq n-1$, there are n possibilities for m and, consequently, n possibilities for H. □

2 Graphs With a Unique Least Common Multiple

Although every two positive integers have a unique least common multiple, we have seen that such is not the case for graphs. Indeed, it seems to be commonplace for two graphs to have several least common multiples. That is, two graphs having a unique least common multiple appears to be the exception, not the rule. We now direct our attention to graphs having exactly one least common multiple. If G is the unique least common multiple of G_1 and G_2, then we write $\text{LCM}(G_1, G_2) = G$. Since K_2 divides every nonempty graph G, we have $\text{LCM}(G, K_2) = G$. We now consider the uniqueness of $\text{LCM}(G, P_3)$. The following result (see [3]) will be useful to us.

Theorem A. *A connected graph is P_3-decomposable if and only if it has even size.*

Theorem 3. *A graph G of order p without isolated vertices and the graph P_3 have a unique least common multiple if and only if every component of G has even size or $G \cong K_p$, where $p \equiv 2$ or $3 \pmod 4$.*

Proof: If every component of G has even size, then it follows by Theorem A that $\text{LCM}(G, P_3) = G$. Suppose now that $G \cong K_p$, where $p \equiv 2$ or 3 (mod 4). Then G has odd size and G is not P_3-decomposable. Let H be the graph obtained by identifying a vertex of one copy of G with some vertex of another copy of G. Then H has even size and clearly is decomposable into two copies of G, so H is a least common multiple of G and P_3. The only other graph that can be decomposed into two copies of G is $2K_p$, which is not P_3-decomposable; therefore, $\mathrm{LCM}(G, P_3) = H$.

Conversely, suppose that G is a graph, not every component of which has even size and $G \not\cong K_p$, where $p \equiv 2$ or $p \equiv 3 \pmod 4$. Since K_p has even size if $p \equiv 0$ or $1 \pmod 4$, G is not complete. Suppose that G is connected (and so G has odd size). Let H' be the graph obtained by identifying a vertex of one copy of G with a vertex of another copy of G, and let H'' be the graph obtained by identifying each of two nonadjacent vertices of one copy of G with the corresponding vertices in another copy of G. Thus H' and H'' have different orders, and so $H' \not\cong H''$. Since H' and H'' have even size and can be decomposed into two copies of G, both H' and H'' are least common multiples of G and P_3.

Suppose now that G is disconnected with components $G_1, G_2, \ldots, G_k$ ($k \geq 2$). Let $v_{i,1}$ and $v_{i,2}$ ($i = 1, 2, \ldots, k$) be distinct vertices of G_i, and let $v'_{i,1}$ and $v'_{i,2}$ be corresponding vertices in another copy of G. Let H_1 be the disconnected graph obtained by identifying $v_{i,1}$ and $v'_{i,1}$ for each i ($1 \leq i \leq k$), and let H_2 be the connected graph obtained by identifying $v_{i,2}$ and $v'_{i,1}$ for $i = 1, 2, \ldots, k$ and $v'_{i,2}$ and $v_{i+1,1}$ for $i = 1, 2, \ldots, k-1$. Both H_1 and H_2 are least common multiples of G and P_3. □

We now consider the only other graph of size 2 without isolated vertices, namely $2K_2$. Ruiz [4] characterized the $2K_2$-decomposable graphs.

Theorem B. *Let G be a graph of size $q \geq 2$ without isolated vertices. Then G is $2K_2$-decomposable if and only if q is even and $\Delta(G) \leq q/2$, unless $G \cong K_3 \cup K_2$.*

We now determine all those graphs G for which $\mathrm{LCM}(G, 2K_2)$ is unique.

Theorem 4. *A nonempty graph G without isolated vertices and the graph $2K_2$ have a unique least common multiple if and only if $G \cong K_2$, $G \cong K_3$, or $2K_2|G$.*

Proof: First, it is clear that $\mathrm{LCM}(K_2, 2K_2) = 2K_2$ and if $2K_2|G$, then $\mathrm{LCM}(G, 2K_2) = G$. Suppose then that $G \cong K_3$. Since G is not $2K_2$-decomposable, any least common multiple of G and $2K_2$ contains at least two copies of G. However, $2K_3$ is both K_3-decomposable and $2K_2$-decomposable. Thus $2K_3$ is a least common multiple. Other than $2K_3$, only the graph H obtained by identifying vertices of two copies of K_3 can be decomposed into two copies of K_3. By Theorem B, H is not $2K_2$-decomposable however. Therefore, $\mathrm{LCM}(K_3, 2K_2) = 2K_3$.

We now consider the converse. Suppose that G is a graph without isolated vertices such that G is different from K_2 and K_3 and such that G is not $2K_2$-decomposable. We show that G and $2K_2$ do not have a unique least common multiple. Assume that G has size q, so $q \geq 2$. By hypothesis and Theorem B, either $G \cong K_3 \cup K_2$ or $\Delta(G) > q/2$. The graph $2G$ has size $2q$ and $\Delta(2G) \leq 2q/2 = q$, and clearly $2G \ncong K_3 \cup K_2$. Consequently, $2G$ is both G-decomposable and $2K_2$-decomposable, and so $2G$ is a least common multiple of G and $2K_2$.

Now let H be the graph obtained by identifying vertices of minimum degree in two copies of G. Certainly, $H \ncong K_3 \cup K_2$. If $\delta(G) \leq q/2$, then $\Delta(H) \leq q$ and H is also a least common multiple of G and $2K_2$. Suppose, on the other hand, that $\delta(G) > q/2$. If G has order p, then the degree sum is $2q > pq/2$, that is, $p < 4$. So $p = 2$ or $p = 3$ and $\delta(G) > q/2$. Thus $G \cong K_2$ or $G \cong K_3$, contrary to hypothesis. □

Some of the techniques used in the previous two proofs suggest a new concept. Let H be a graph without isolated vertices. An *H-edge coloring* (or, more simply, an *H-coloring*) of a graph G is a coloring of the edges of G such that the subgraph induced by each color class is isomorphic to H. This, of course, is an alternative but equivalent way of describing an H-decomposition of G. Consider the disjoint sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ of $m + n$ distinct colors. For graphs G_1 and G_2 without isolated vertices, a *(G_1, G_2)-edge coloring* of a graph G is a mapping $f \colon E(G) \to A \times B$ such that the coloring induced by the ith coordinate ($i = 1, 2$) is a G_i-coloring of G. Hence, G is a common multiple of G_1 and G_2 if and only if there exists a (G_1, G_2)-coloring of G.

Let f be a (G_1, G_2)-coloring of G. For an edge e of G, the image $f(e)$ is an ordered pair of distinct colors. We denote by $C_f(e)$ (or $C(e)$ if f is clear from the context) the set of two colors assigned to e. If $E' \subseteq E(G)$, then

$$C_f(E') = \cup_{e \in E'} C_f(e),$$

and if $v \in V(G)$, then

$$C_f(v) = \cup C_f(uv),$$

where the last union is taken over all vertices u of G that are adjacent to v.

We now present a necessary condition for two graphs to have a unique least common multiple.

Theorem 5. *Let G_1 and G_2 be two graphs without isolated vertices. If H is the unique least common multiple of G_1 and G_2, then for every (G_1, G_2)-coloring f of H, neither of the following conditions holds:*

(i) for some vertex v of H, the edges incident with v can be partitioned into subsets E_1 and E_2 such that $C_f(v) = C_f(E_1) \cup C_f(E_2)$, where $C_f(E_1) \cap C_f(E_2) = \emptyset$;

(ii) if u and w are distinct vertices of G, then $C_f(u) \cap C_f(w) = \emptyset$.

Proof: Suppose, first, that there exists a (G_1, G_2)-coloring of H such that condition (i) holds. Let H' be the graph obtained from H by replacing v by two new vertices v_1 and v_2 such that the set of edges incident with v_i is E_i $(i = 1, 2)$. Since $C_f(E_1) \cap C_f(E_2) = \emptyset$, the (G_1, G_2)-coloring of H produces a (G_1, G_2)-coloring of H'; so the graph H' is a common multiple of G_1 and G_2. Since H and H' have the same size, H' is also a least common multiple of G_1 and G_2.

Now, suppose that there exists a (G_1, G_2)-coloring of H such that condition (ii) holds. Note that $uw \notin E(H)$. Let H'' be the graph obtained from H by identifying u and w. Since H and H'' have the same size and there exists a (G_1, G_2)-coloring of H'', the graph H'' is a least common multiple of G_1 and G_2. □

From Theorem 4, the graphs $G_1 \cong 2K_2$ and $G_2 \cong K_{1,3}$ do not have a unique least common multiple. Indeed, all of the graphs H_i $(1 \le i \le 4)$ of Figure 3 are least common multiples. A (G_1, G_2)-coloring of each graph H_i $(1 \le i \le 4)$ is also indicated in Figure 3. For each integer i $(1 \le i \le 3)$, the graph H_i satisfies condition (i) of Theorem 5 and the graph H_{i+1} is obtained from H_i by the construction given when condition (i) of Theorem 5 holds. Furthermore, for each integer i $(1 \le i \le 3)$, the graph H_{i+1} satisfies condition (ii) of Theorem 5 and the graph H_i is obtained from H_{i+1} by the construction given when condition (ii) of Theorem 5 holds.

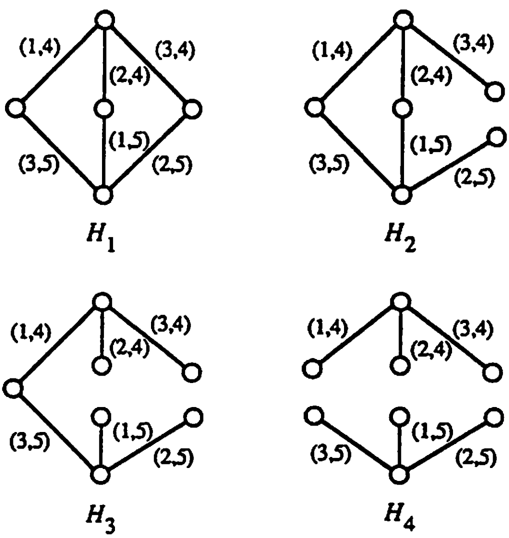

Figure 3

3 Stars With a Unique Least Common Multiple

We now consider the problem of determining for a given class of graphs those pairs of graphs in the class having a unique least common multiple. In particular, are there graphs G_1 and G_2 in the given class, neither of which is a multiple of the other, having a unique least common multiple? In the case of matchings (graphs of the type nK_2), there are no such graphs G_1 and G_2 having a unique least common multiple. The same is true for paths. We now consider this question for the class of stars. We first state two number-theoretic results.

Theorem C. *Let a and b be positive integers with $\gcd(a,b) = d$. There exist integers x and y that satisfy the equation $ax + by = c$ if and only if $d|c$.*

Theorem D. *Two integers a and b are relatively prime if and only if there exist integers x and y such that $ax + by = 1$. Moreover, if $a, b \geq 2$, then x can be chosen so that $0 < x < b$.*

Theorem C is well-known, as is the first statement in Theorem D. We verify the second statement in Theorem D. Let x be the smallest positive integer for which there exists an integer y such that $ax + by = 1$. Clearly, then, $y < 0$. We claim that $x < b$; for suppose, to the contrary, that $x \geq b$. When $x = b$, the equation $ax + by = 1$ becomes $ab + by = 1$; hence, $b|1$, which contradicts the fact that $b \geq 2$. Thus we may assume that $x > b$. Now

$$ax + by = ax - ab + ab + by = a(x - b) + b(a + y) = 1.$$

However, $x - b \geq 1$, contradicting the minimality of x and verifying the claim.

We are now prepared to present our next result. For integers $a, b \geq 2$, we denote the *double star* $S(a, b)$ as that tree containing only two vertices that are not end-vertices, one of degree a and one of degree b.

Theorem 6. *Let r and s be integers with $2 \leq r < s$. If $\gcd(r, s) = 1$, then the stars $K_{1,r}$ and $K_{1,s}$ do not have a unique least common multiple.*

Proof: Since $\gcd(r, s) = 1$, it follows that $\operatorname{lcm}(r, s) = rs$ and so $K_{1,rs}$ is a least common multiple of $K_{1,r}$ and $K_{1,s}$. By Theorem D, there exist integers x and y such that $rx + sy = 1$, where x may be chosen so that $0 < x < s$, implying that $xr < xs$. Clearly, $y < 0$; so $rx > sy$. Define $z = -y > 0$. Thus

$$rs - zs = rs + ys = rs + (1 - rx) = 1 + r(s - x) \geq 3.$$

Now let G be the double star $S(zs+1, rs-zs)$, which has size rs. We show that G is both $K_{1,r}$-decomposable and $K_{1,s}$-decomposable. Observe that at the vertex of degree $zs + 1$ ($= xr$) of G, there are x edge-disjoint copies of $K_{1,r}$, where the edge joining the vertices of G that are not end-vertices belongs to one of these copies of $K_{1,r}$. There are $rs - zs - 1$ ($= rs - rx$) unused edges of G at the vertex of degree $rs - zs$. Therefore, there are $s - x$ (> 0) edge-disjoint copies of $K_{1,r}$ at this vertex. Hence G is decomposable into s copies of $K_{1,r}$. Similarly, it can be shown that G is decomposable into r copies of $K_{1,s}$. Therefore, G is a least common multiple of $K_{1,r}$ and $K_{1,s}$. □

We now verify the converse of Theorem 6. For the purpose of doing this, we present the first of two additional number-theoretic lemmas.

Lemma 7. *Let $a_1, a_2, \ldots, a_k$ ($k \geq 2$) and $b_1, b_2, \ldots, b_n$ ($n \geq 2$) be (finite) sequences of positive integers satisfying the following conditions:*

(1) *there exists an integer $m \geq 2$ such that $m|a_i$ and $m|b_j$ for all i ($1 \leq i \leq k$) and j ($1 \leq j \leq n$);*

(2) *$a_i \leq n+k-1$ and $b_j \leq n+k-1$ for all i ($1 \leq i \leq k$) and j ($1 \leq j \leq n$);*

(3) *$\sum_{i=1}^{k} a_i = \sum_{j=1}^{n} b_j$.*

Then there exist proper nonempty subsets $A \subset \{1, 2, \ldots, k\}$ and $B \subset \{1, 2, \ldots, n\}$ such that $\sum_{i \in A} a_i = \sum_{j \in B} b_j$.

Proof: Assume, without loss of generality, that $n \geq k$. For each integer j with $1 \leq j \leq n-1$, let i_j denote the largest nonnegative integer such that

$$a_1 + a_2 + \cdots + a_{i_j} \leq b_1 + b_2 + \cdots + b_j, \tag{1}$$

where the left side of inequality (1) is defined as 0 if $i_j = 0$. The difference

$$d_j = (b_1 + b_2 + \cdots + b_j) - (a_1 + a_2 + \cdots + a_{i_j})$$

is a nonnegative integer and $m|d_j$. Moreover,

$$0 \leq d_j < a_{i_j+1} \leq n + k - 1 \tag{2}$$

for every integer j with $1 \leq j \leq n-1$. If $d_j = 0$ for some integer j ($1 \leq j \leq n-1$), then the lemma follows; thus it suffices to assume that $d_j > 0$ for all such integers j. Therefore, there are $\lfloor (n+k-2)/m \rfloor$ possible values for the integers $d_1, d_2, \ldots, d_{n-1}$.

We now show that two of the integers $d_1, d_2, \ldots, d_{n-1}$ are equal. We consider three cases.

Case 1. *Assume that $n \geq k+1$.* Since $m \geq 2$,

$$m(n-1) - (2n-3) \geq 2(n-1) - (2n-3) = 1 > 0.$$

Thus $2n-3 < m(n-1)$. Consequently,

$$\left\lfloor \frac{n+k-2}{m} \right\rfloor \le \frac{n+k-2}{m} \le \frac{2n-3}{m} < n-1.$$

Case 2. *Assume that* $n = k$ and $m \ge 3$. Since $n \ge 2$,

$$3(n-1)-(2n-2) = n-1 > 0$$

so that $2n-2 < 3(n-1)$. Therefore,

$$\left\lfloor \frac{n+k-2}{m} \right\rfloor \le \left\lfloor \frac{2n-2}{3} \right\rfloor \le \frac{2n-2}{3} < n-1.$$

Case 3. *Assume that* $n = k$ *and* $m = 2$. Since $n = k$ and all integers a_i $(1 \le i \le k)$ are even, inequality (2) becomes $d_j < a_{i_j+1} \le 2n-2$. Hence there are $\lfloor (2n-3)/2 \rfloor$ pcssible values for $d_1, d_2, \ldots, d_{n-1}$. However,

$$\left\lfloor \frac{2n-3}{2} \right\rfloor = n-2 < n-1.$$

Therefore, in every case two of the integers $d_1, d_2, \ldots, d_{n-1}$ are equal, say $d_p = d_q$, where $1 \le p < q \le n-1$. Thus $i_q \ge i_p$ and

$$(b_1+b_2+\cdots+b_q)-(a_1+a_2+\cdots+a_{i_q}) = (b_1+b_2+\cdots+b_p)-(a_1+a_2+\cdots+a_{i_p})$$

or, equivalently,

$$b_{p+1}+b_{p+2}+\cdots+b_q = a_{i_p+1}+a_{i_p+2}+\cdots+a_{i_q}. \tag{3}$$

Since the left side of (3) is positive, so too is the right side, implying that $i_q > i_p$. Taking $A = \{i_p+1, i_p+2, \ldots, i_q\}$ and $B = \{p+1, p+2, \ldots, q\}$ completes the proof. □

The following result is well-known from number theory.

Lemma E. *If* b *and* d *are relatively prime positive integers and* $ab \equiv 0 \pmod{d}$, *then* $a \equiv 0 \pmod{d}$.

We need one additional graph-theoretic result before presenting the main result of this section.

Lemma 8. *Let* H *be a least common multiple of the stars* $K_{1,r}$ *and* $K_{1,s}$, *where* $2 \le r < s$ *and* $\gcd(r,s) \ne 1$. *For any* $K_{1,r}$*-decomposition and* $K_{1,s}$*-decomposition of* H, *each center of a subgraph in the* $K_{1,r}$*-decomposition must be a center of at least one subgraph in the* $K_{1,s}$*-decomposition.*

Proof: Let $\gcd(r,s) = m$ (> 1) and $\operatorname{lcm}(r,s) = M$ $(< rs)$. Suppose that there are s_1 subgraphs in the $K_{1,s}$-decomposition of H and r_1 subgraphs in the $K_{1,r}$-decomposition of H. Then $M = ss_1 = rr_1$. Let $v_1, v_2, \ldots, v_k$ $(1 \le k \le s_1)$ denote the vertices of H that are the centers of the s_1 subgraphs in the $K_{1,s}$-decomposition. Let $v \in V(H)$, where $v \ne v_i$ $(1 \le i \le k)$. Every edge in H must be incident with one of the vertices $v_1, v_2, \ldots, v_k$, so the neighborhood $N(v) \subseteq \{v_1, v_2, \ldots, v_k\}$. Consequently, $\deg v \le k \le s_1$.

We claim that $s_1 < r$. Suppose, to the contrary, that $s_1 \ge r$. Since $mM = rs$, it follows that $mM = (M/r_1)(M/s_1)$, that is, $M = mr_1s_1$. Therefore,

$$M = mr_1s_1 \ge mr_1r = mM,$$

contradicting $m > 1$. Thus v is not the center of any subgraph $K_{1,r}$ in the $K_{1,r}$-decomposition of H. □

The hypothesis $\gcd(r, s) \ne 1$ is necessary in Lemma 8. For example, consider the graph H of size 12 in Figure 4. Then H is a least common multiple of $K_{1,3}$ and $K_{1,4}$. There is only one $K_{1,3}$-decomposition and only one $K_{1,4}$-decomposition of H. The vertex x is the center of one of these copies of $K_{1,3}$ but is the center of no copy of $K_{1,4}$.

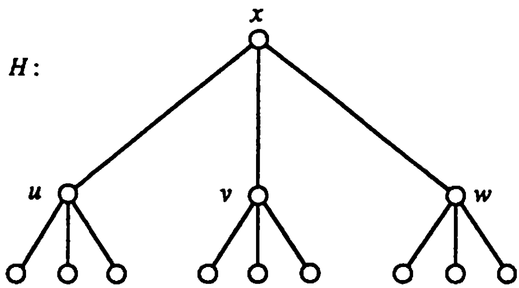

Figure 4

Theorem 9. *If $r \ge 2$ and $s \ge 2$ are integers with $\gcd(r, s) \ne 1$, then the stars $K_{1,r}$ and $K_{1,s}$ have a unique least common multiple.*

Proof: Suppose that $r < s$. Let $\gcd(r, s) = m$ (> 1) and $\operatorname{lcm}(r, s) = M$ $(< rs)$. Clearly, $K_{1,M}$ is a least common multiple of $K_{1,r}$ and $K_{1,s}$. We show that in fact $K_{1,M}$ is the only least common multiple of $K_{1,r}$ and $K_{1,s}$. The result is obvious if $r|s$ or equivalently, if $m = r$. Therefore, it suffices to assume that $m < r$. Suppose then, to the contrary, that H is a graph (without isolated vertices) that is not isomorphic to $K_{1,M}$ and is a least common multiple of $K_{1,r}$ and $K_{1,s}$. Let there be given a $K_{1,r}$-decomposition and a $K_{1,s}$-decomposition of H. By Lemma 8, every center of a subgraph in the $K_{1,r}$-decomposition of H must be a center of at least one subgraph in the $K_{1,s}$-decomposition of H.

Every edge of H belongs to a unique copy of $K_{1,s}$ in the $K_{1,s}$-decomposition of H. We construct a digraph D from H by directing each edge of H away from the center in the copy of $K_{1,s}$ to which the edge belongs. Therefore, the outdegree of every vertex of D is a multiple of s. Let $z_1, z_2, \ldots, z_g$ ($g \geq 2$) denote the vertices of positive outdegree in D. Hence the centers of the stars $K_{1,s}$ in the $K_{1,s}$-decomposition of H are $z_1, z_2, \ldots, z_g$. For each $i = 1, 2, \ldots, g$, assume that there are c_i (≥ 1) copies of $K_{1,s}$ with center z_i in the $K_{1,s}$-decomposition of H. Hence in D, od $z_i = c_i s$ for $i = 1, 2, \ldots, g$. The number of copies of $K_{1,s}$ in the $K_{1,s}$-decomposition is $M/s = r/m$. Therefore, $\sum_{i=1}^{g} c_i = r/m$.

Each arc of D is either directed from a vertex z_i $(1 \leq i \leq g)$ to a vertex z_j, $i \neq j$, (a *type 1 arc*) or is directed from a vertex z_i $(1 \leq i \leq g)$ to a vertex not among $\{z_1, z_2, \ldots, z_g\}$ (a *type 2 arc*). According to Lemma 8, finding a $K_{1,r}$-decomposition in H is equivalent to reversing the direction of some type 1 arcs of D so that the outdegree of every vertex in the resulting digraph is a multiple of r.

Now $\sum_{i=1}^{g} c_i s = (r/m)s = M$ is a multiple of r. Furthermore, since $m|s$, it follows that $m|c_i s$ for each i $(1 \leq i \leq g)$. However, none of the integers $c_i s$ is a multiple of r; for suppose, to the contrary, that $r|c_j s$ for some $j \in \{1, 2, \ldots, g\}$. Since $s|c_j s$, the integer $c_j s$ is a common multiple of r and s. Thus,

$$c_j s < (r/m)s = M \leq c_j s,$$

producing a contradiction. Hence when $c_i s$ $(1 \leq i \leq g)$ is divided by r, a nonzero remainder a'_i results; so $0 < a'_i < r$, for each i $(1 \leq i \leq g)$. Since $m|c_i s$ and $m|r$, we conclude that $m|a'_i$ for each i $(1 \leq i \leq g)$.

Since there is also a $K_{1,r}$-decomposition of H, it is possible to change the directions of some arcs joining pairs of vertices in $\{z_1, z_2, \ldots, z_g\}$ so that the outdegree of each vertex in the resulting digraph is a multiple of r. Hence the outdegrees of some of the vertices $z_1, z_2, \ldots, z_g$ must decrease; while the outdegrees of the remaining vertices must increase. Suppose that the outdegrees of $z_1, z_2, \ldots, z_k$ decrease, while those of $z_{k+1}, z_{k+2}, \ldots, z_g$ increase. Let $g - k = n$. Neither k nor n equals 1; for suppose, to the contrary, that $k = 1$, say. Then z_1 has its outdegree decreased by exactly $g - 1$, while the outdegree of each of $z_2, z_3, \ldots, z_g$ increases by exactly 1. This implies that $r - a'_i = 1$ for each i $(2 \leq i \leq g)$, contradicting $m|(r - a'_i)$ since $m \geq 2$. Thus $k \geq 2$, and, similarly, $n \geq 2$.

Denote the numbers $r - a'_{k+1}, r - a'_{k+2}, \ldots, r - a'_g$ by $b'_1, b'_2, \ldots, b'_n$, respectively. For each i $(1 \leq i \leq k)$, let a_i denote the outdegree decrease of z_i. Thus $a_i \geq a'_i$ and $a_i \equiv a'_i \pmod{r}$. Furthermore, for each j $(1 \leq j \leq n)$, let b_j denote the outdegree increase of z_{k+j}. Then $b_j \geq b'_j$ and $b_j \equiv b'_j$ (mod r). Since the total outdegree decrease equals the total outdegree increase,

$$a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_n.$$

Since no outdegree can be decreased or increased by more than $g - 1$ and $g - 1 = n + k - 1$, we see that the sequences $a_1, a_2, \ldots, a_k$ and

$b_1, b_2, \ldots, b_n$ satisfy the hypotheses of Lemma 7. Consequently, there exist proper nonempty subsets $A \subset \{1, 2, \ldots, k\}$ and $B \subset \{1, 2, \ldots, n\}$ such that $\sum_{i \in A} a_i = \sum_{j \in B} b_j$. Let $A' = \{1, 2, \ldots, k\} - A$ and $B' = \{1, 2, \ldots, n\} - B$. Then it also follows that $\sum_{i \in A'} a_i = \sum_{j \in B'} b_j$.

For a set C of positive integers and a fixed positive integer d, define the set

$$C + d = \{c + d | c \in C\}.$$

Since $\sum_{i \in A} a_i = \sum_{j \in B} b_j$, it follows that

$$\sum_{i \in A} a_i = \sum_{j \in B+k} (r - a_j),$$

or, equivalently,

$$\sum_{i \in A \cup (B+k)} a_i \equiv 0 \pmod{r}.$$

Since $a_i \equiv c_i s \pmod{r}$ for each i $(1 \leq i \leq g)$, we have

$$\sum_{i \in A \cup (B+k)} c_i s \equiv 0 \pmod{r}.$$

From this, it follows that

$$\sum_{i \in A \cup (B+k)} c_i (s/m) \equiv 0 \pmod{r/m},$$

or

$$\left(\sum_{i \in A \cup (B+k)} c_i \right) (s/m) \equiv 0 \pmod{r/m}.$$

Since the integers s/m and r/m are relatively prime, it follows by Lemma E that

$$\sum_{i \in A \cup (B+k)} c_i \equiv 0 \pmod{r/m}.$$

Similarly,

$$\sum_{i \in A' \cup (B'+k)} c_i \equiv 0 \pmod{r/m}.$$

Neither $\sum_{i \in A \cup (B+k)} c_i$ nor $\sum_{i \in A' \cup (B'+k)} c_i$ is 0, however; hence

$$\sum_{i=1}^{g} c_i = \sum_{i \in A \cup (B+k)} c_i + \sum_{i \in A' \cup (B'+k)} c_i \geq \frac{r}{m} + \frac{r}{m} = \frac{2r}{m}.$$

This contradicts the fact that $\sum_{i=1}^{g} c_i = r/m$. □

Combining Theorems 6 and 9, we have the following result.

Corollary 10. *Let $r \geq 2$ and $s \geq 2$ be integers. The stars $K_{1,r}$ and $K_{1,s}$ have a unique least common multiple if and only if r and s are not relatively prime.*

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