On the classical semitranslation plane of order 16

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ABSTRACT. The plane in the title is investigated from the combinatorial point of view. Its Baer subplanes are classified and their distribution is studied. Properties of the Fano subplanes are shown. Blocking sets of Rédei type are constructed. Finally, hyperovals and complete 14-arcs are considered and classified.

1 Introduction

Semitranslation planes were introduced by T.G. Ostrom [11] who gave the following definition. A projective plane π of order q^2 is a semi-translation plane w.r.t. the line l if there exists a set S of $q+1$ points on l such that if $P \in S$, then π admits a group of elations of order q with centre P and axis l. Later, N.L. Johnson [7] gave an equivalent definition which enabled him to classify the semi-translation planes [8]. Namely, an affine plane of order q^2 is a semitranslation plane if it admits a group H of translations such that each orbit of H is the set of q^2 points of an affine Baer subplane. The plane is non-strict if its full translation group properly contains H. In the case H is the full translation group, then the plane is a strict semi-translation plane. Obviously, a proper non-strict semi-translation plane is neither a translation plane nor a dual translation plane. An infinite class of proper non-strict semitranslation planes of Lenz- Barlotti class I-1 can be obtained by deriving the dual Hall planes of even order 2^{2r} [9]. The so obtained planes admit two collineation groups of order 2^r which fix a Baer subplane. One of these groups fixes a Baer subplane π_0 pointwise and the other is a group of elations which fix π_0, i.e. a (P, l, π_0)-elation group. Furthermore, the full translation group of these derived planes has order 2^{2r+1}. There is one translation group, with a fixed centre, of order 2^{r+1}, the remaining translation groups with fixed centres have order 2^r or 2.

When $2^{2r} = 16$, the above mentioned construction yields the plane we call the classical semi-translation plane of order 16 and denote hereafter by π. Therefore, to construct π we start with the desarguesian plane PG(2,16), derive it to get the Hall plane Hl(16), dualize Hl(16) and derive again.

We investigate the distribution of the Baer subplanes of π which depends on the fact that π contains both a derivation set and a dual derivation set (sect. 4).

We show that a Fano subplane of π is either maximal, i.e. contained in no Baer subplane, or completes to a unique Baer subplane. Moreover, some of the Fano subplanes which extend to a unique Baer subplane give also rise to " 2-failed Baers" (sect. 3 and [4]) which are used to construct blocking sets of Rédei type having size 25. Again from 2–failed Baers, other blocking sets of Rédei type will be constructed which have size 27 and 30.

Finally, we investigate the hyperovals in π, whose structure is the same as that of the hyperovals in the exceptional semi-translation plane of order 16 [3, 10], and the complete 14-arcs of π. We observe that 14 is the maximum size for a complete arc in π other than a hyperoval. Complete k-arcs for $k = 10, 11, 12$ were already discussed in [2] ; some complete 13-arcs will be shown here.

2 Description of π

The plane π was obtained by the construction mentioned in sect. 1. We consider π both as a projective and an affine plane. Therefore, π has a distinguished line, its line at infinity, which we denote by $a0$, and the points on it are $A0, Aj$, $j = 1, 2, \dots, 16$, $A0$ being a special point.

The points $A0, A1, A2, A3, A4$ form the derivation set of π. The points of $\pi \setminus a0$ are denoted by $Bj, Cj, Dj, Fj, Hj, Kj, Lj, Mj, Nj, Pj, Rj, Sj, Tj, Wj$ Xj, Zj, $j = 1, 2, \dots, 16$.

The lines of π, other than $a0$, are denoted by the corresponding lower case letters and same subscripts. The lines on $A0$ are the lines aj ; $a1$ contains the affine points Bj, $a2$ the Cj's, $\dots, a16$ the Zj's. The affine lines on $A1$ are the lines bj. Each of these contains all affine points with subscript j ($b1$ contains the points $B1, C1, \dots, Z1$, and so on). The affine lines on $A2$ are the cj's, on $A3$ the dj's, $\dots$, on $A16$ the zj's. The affine points on such lines can be written down with the help of Table 1 (see Appendix) as we now show.

Since π has an involution σ which fixes $a0$ pointwise and acts on the finite points and lines by keeping the letters fixed and pairing off the subscripts as $(1,2)$, $(3,4)$, $(5,10)$, $(6,16)$, $(7,13)$, $(8,9)$, $(11,14)$, $(12,15)$, it suffices to write half of the lines on each Aj, $j = 2, \dots, 16$. Moreover, in Table 1 only the subscripts of the affine points of each line are given. The letters are to be inserted in alphabetic order.

E.g. from Table 1 we read

$h7$: $A5\,B7\,C1\,D8\,F6\,H13\,K12\,L16\,M15\,N9\,P4\,R2\,S5\,T14\,W10\,X3\,Z11$

and applying σ yields

$h13$: $A5\,B13\,C2\,D9\,F16\,H7\,K15\,L6\,M12\,N8\,P3\,R1\,S10\,T11\,W5\,X4\,Z14$.

3 Fano subplanes and blocking sets

First of all, we observe that π is generated by quadrangles and there are no forbidden vertices for generating quadrangles [2]. Some examples of generating quadrangles for π are:

$A0\,P7\,R14\,S7$	$A0\,M3\,F4\,S3$	$A0\,L14\,T14\,R13$	$A0\,H5\,P13\,K5$
$T11\,W11\,T2\,W10$	$A0\,A1\,M7\,Z4$	$L5\,L8\,M2\,M4$	$L5\,L8\,M4\,M6$
$A0\,D5\,X7\,Z5$	$L5\,L8\,M4\,M10$.		

When a quadrangle does not generate π, then it generates a Fano subplane of π. Since π is a derived plane, it contains Baer subplanes and so Fano subplanes. The obvious classification of Fano subplanes w.r.t. the distinguished points and lines of π shows that there is no forbidden situation. Thus, we choose to classify the Fano subplanes according to their being maximal or not. More precisely, it is well known that in PG(2,16), the desarguesian plane, every Fano subplane completes to (extends to, is contained in) a unique Baer subplane. (A Baer subplane will be briefly called a Baer; similarly, for Fano subplanes.) This is no longer true in the translation planes of order 16, which are classified in [1], where a Fano subplane can complete to more than one Baer and up to seven in the Johnson-Walker and in the Lorimer-Rahilly planes [5]. On the other hand, in the strict semitranslation plane of order 16 a Fano subplane is either maximal, i.e. is contained in no Baer, or completes to a unique Baer [3]. The same situation occurs in π.

Examples of maximal Fano subplanes of π are the following (only points and lines are given, the incidences can be obtained from Table 1):

points							*lines*						
$A0$	$A5$	$A14$	$C1$	$C3$	$H5$	$H13$	$a0$	$a2$	$a5$	$h7$	$h10$	$w9$	$w14$
$A0$	$A2$	$A5$	$C2$	$C3$	$H5$	$H7$	$a0$	$a2$	$a5$	$c1$	$c4$	$h10$	$h13$
$L16$	$M14$	$N1$	$N8$	$N10$	$R12$	$T6$	$a9$	$c9$	$d5$	$l16$	$p15$	$s14$	$w6$
$A7$	$L14$	$M6$	$N5$	$N9$	$N13$	$P7$	$a9$	$l2$	$l6$	$l15$	$m8$	$t12$	$w11$
$A0$	$D3$	$D5$	$F3$	$F4$	$M3$	$M8$	$a3$	$a4$	$a8$	$b3$	$c1$	$h14$	$k9$
$A0$	$L8$	$L14$	$R14$	$R16$	$T4$	$T14$	$a7$	$a11$	$a13$	$b14$	$d1$	$w9$	$z3$
$A0$	$A9$	$A10$	$D4$	$D8$	$K5$	$K6$	$a0$	$a3$	$a6$	$n3$	$n9$	$p2$	$p11$
$A0$	$L5$	$L8$	$M1$	$M4$	$W2$	$W11$	$a7$	$a8$	$a14$	$d12$	$f7$	$l16$	$m10$
$A0$	$D3$	$D15$	$L5$	$L8$	$M4$	$M7$	$a3$	$a7$	$a8$	$c6$	$d12$	$h15$	$l16$
$A0$	$A1$	$A5$	$L15$	$L16$	$M15$	$M16$	$a0$	$a7$	$a8$	$b15$	$b16$	$h5$	$h7$
$A0$	$A1$	$A5$	$H2$	$H5$	$P2$	$P5$	$a0$	$a5$	$a10$	$b2$	$b5$	$h3$	$h10$

A Fano subplane of π which is not maximal completes to one Baer only. However, there are Fano subplanes which besides completing to a Baer also complete to some 2−failed Baers. A failed Baer [4] in a plane of order 16 is an affine plane of order 4 with one missing parallel class of lines and whose line at infinity contains four points only. In [4] it was shown that in several non-desarguesian translation planes of order 16 failed Baers do exist; moreover, they can be extended to blocking sets of Rédei type by adding three points on the line at infinity (which contains the line at infinity of the failed Baer). Recall that a blocking set in a plane of order q is of Rédei type if it has $q+m$ points and one m-secant. Furthermore, in PG(2,q) a blocking set of Rédei type either contains a Baer or has at least $q+q^{2/3}+1$ points. The above mentioned blocking sets of Rédei type in some translation planes of order 16 all have 23 points (and 23 is slightly smaller than the lower bound in PG(2,16)) and intersection numbers $(1,3,5,7)$ with a unique 7-secant. There are no failed Baers in π. However, π contains what we call 2–failed Baers, i.e. Baer subplanes with two missing parallel classes of lines. Therefore, a 2–failed Baer consists of 19 points, three of which on the line at infinity, and twelve lines (its line at infinity contains just three points). A 2–failed Baer can also be viewed as a 3–net on sixteen points to which three points at infinity have been added. It turns out that π contains at least three non-equivalent types of 2–failed Baers. Indeed, the 19–set of points of a 2–failed Baer can have different intersection numbers w.r.t. the lines of π. The 2–failed Baers in π give rise to blocking sets of Rédei type by adding some points on the line at infinity which, again, contains the points at infinity of the 2–failed Baer. In the case of π, the size of the blocking set constructed by starting with a 2–failed Baer depends on the type of the latter. The minimum possible size is 25 in which case the intersection numbers are (1,3,5,9). There are also 2–failed Baers which produce blocking sets with 27 points and intersection numbers (1,2,3,4,5,11), and a unique 11–secant. Finally, some 2–failed Baers in π yield blocking sets of size 30 and a unique 14-secant, the remaining intersection numbers being 1,2,3,5. Obviously, the most interesting blocking sets are those of size 25. Next, we provide some examples of Fano subplanes which extend to one Baer only and of Fano subplanes which both extend to a Baer and to some 2–failed Baers. In the latter case, we also give the points to be added to the 2–failed Baers in order to obtain the above mentioned blocking sets of Rédei type. Our last example is of a 2–failed Baer which yields a blocking set of size 27.

1. Fano: $A0\,A1\,A2\,B1\,B2\,C1\,C2$, $a0\,a1\,a2\,b1\,b2\,c1\,c2$;
   Baer with $A3\,A4\,B3\,B4\,C3\,C4\,Dj\,Fj$, $a3\,a4\,b3\,b4\,c3\,c4\,dj\,fj$,
   $j=1,2,3,4$.

2. Fano: $A0\,A1\,A2\,D5\,D10\,F5\,F10$, $a0\,a3\,a4\,b5\,b10\,c7\,c13$;
   Baer with $A3\,A4\,D7\,D13\,F7\,F13\,Bj\,Cj$, $a1\,a2\,b7\,b13\,c5\,c10\,dj\,fj$,
   $j=5,10,7,13$.

3. Fano: $A5\,A6\,A7\,B5\,C4\,D8\,F6$, $a0\,h5\,h7\,k5\,k10\,l5\,l13$;
Baer with $A0\,A8\,B7\,B10\,B13\,C1\,C2\,C3\,D9\,D11\,D14\,F12\,F15\,F16$, $a1\,a2\,a3\,a4\,h10\,h13\,k7\,k13\,l7\,l10\,m5\,m7\,m10\,m13$.

4. Fano: $A6\,B5\,B7\,B10\,C2\,D8\,F6$, $a1\,h7\,k5\,k7\,k10\,l5\,m10$;
Baer with $A0\,A5\,A7\,A8\,B13\,C1\,C3\,C4\,D9\,D11\,D14\,F12\,F15\,F16$, $a0\,a2\,a3\,a4\,h5\,h10\,h13\,k13\,l7\,l10\,l13\,m5\,m7\,m13$.

5. Fano: $A0\,K4\,K10\,S10\,S14\,Z10\,Z12$, $a6\,a12\,a16\,b10\,h9\,p16\,t2$;
Baer with $K8\,K16\,S2\,S6\,Z1\,Z9\,D5\,D7\,D10\,D13\,X3\,X10\,X11\,X15$, $a3\,a15\,c7\,d5\,f13\,k14\,l11\,m8\,n15\,r6\,s12\,w3\,x1\,z4$.

6. Fano: $A0\,A1\,A12\,B7\,B12\,D7\,D12$, $a0\,a1\,a3\,b7\,b12\,s7\,s12$;
Baer with $A6\,A14\,B2\,B11\,D2\,D11\,Cj\,Fj$, $a2\,a4\,b2\,b11\,s2\,s11\,kj\,wj$, $j = 2, 7, 11, 12$.

7. Fano: $A0\,A9\,A10\,B3\,B4\,F7\,F13$, $a0\,a1\,a4\,n3\,n4\,p3\,p4$;
Baer with $A11\,A12\,B1\,B2\,F5\,F10\,C6\,C12\,C15\,C16\,D8\,D9\,D11\,D14$, $a2\,a3\,n1\,n2\,p1\,p2\,rj\,sj$, $\quad j = 1, 2, 3, 4$.

8. Fano: $A0\,A13\,A14\,B7\,B10\,C12\,C16$, $a0\,a1\,a2\,t7\,t10\,w7\,w10$;
Baer with $A15\,A16\,B5\,B13\,C6\,C15\,D1\,D2\,D3\,D4\,F8\,F9\,F11\,F14$, $a3\,a4\,t5\,t13\,w5\,w13\,xj\,zj$, $\quad j = 5, 10, 7, 13$.

9. Fano: $A0\,H2\,H7\,L7\,L16\,T7\,T9$, $a5\,a7\,a13\,b7\,l11\,r12\,t1$;
Baer with $H11\,H12\,L3\,L14\,T4\,T15\,D5\,D7\,D10\,D13\,P1\,P6\,P7\,P8$, $a3\,a10\,c10\,d13\,f5\,h9\,k14\,m8\,n16\,p15\,s6\,w4\,x2\,z3$.

10. Fano: $A0\,A1\,A5\,B6\,B8\,C6\,C8$, $a0\,a1\,a2\,b6\,b8\,h6\,h8$;
Baer with $A11\,A16\,B1\,B7\,C1\,C7\,Dj\,Fj$, $a3\,a4\,b1\,b7\,h1\,h7\,rj\,zj$, $j = 1, 6, 7, 8$.

Next, we show a Fano subplane which completes to one Baer and to four 2–failed Baers each of which yields a blocking set by adding some suitable points at infinity.
Fano: $A0\,A1\,A2\,P7\,P16\,S7\,S16$, $a0\,a10\,a12\,b7\,b16\,c3\,c14$;
Baer with $A3\,A4\,P3\,P14\,S3\,S14\,Bj\,Nj$, $b3\,b14\,c7\,c16\,dj\,fj$, $\quad j = 3, 7, 14, 16$.

1. 2–failed Baer with the points $A8\,P6\,P13\,S6\,S13\,Dj\,Mj, j = 6, 7, 13, 16$; by adding, to the Fano, $A3\,A5\,A9\,A10\,A11\,A12\,A13\,A14\,A15\,A16$ a blocking set is obtained with 30 points and intersection numbers (1,2,3,5,14), the unique 14-secant is $a0$.

2. 2–failed Baer with $A10\,P6\,P13\,S6\,S13\,Lj\,Zj$, $\quad j = 6, 7, 13, 16$; by adding (to the Fano) $A3\,A8\,A11\,A12\,A15$ we get a blocking set with 25 points and intersection numbers (1,3,5,9).

3. 2–failed Baer with $A13\,P5\,P15\,S5\,S15\,Hj\,Xj$, $j = 5, 7, 15, 16$; blocking set with 25 points and intersection numbers (1,3,5,9) by adding $A3\,A6\,A12\,A15\,A16$.

4. 2–failed Baer with $A12\,P5\,P15\,S5\,S15\,Cj\,Rj$, $j = 5, 7, 15, 16$; blocking set with 30 points and intersection numbers (1,2,3,5,14) by adding $A3\,A5\,A6\,A7\,A8\,A9\,A13\,A14\,A15\,A16$.

As we already mentioned, the 2–failed Baers, as well as the blocking sets they yield, are embedded in different ways in π. We now show this fact on the examples above.

The 2–failed Baer 1 consists of the 19 points $A0\,A1\,A8\,Dj\,Mj\,Pj\,Sj$, $j = 6, 7, 13, 16$. By adding $A2$, $A3$, $A5$, $A9, \dots, A16$ we get the blocking set. All finite lines on $A4$, $A6$, and $A7$ are tangent to the blocking set. The affine lines on $A0$, $A1$ and $A8$ are either 5–secant or tangent. The affine lines on $A5$ are either 3–secant or tangent; furthermore, the 16 affine points of the 2–failed Baer split into four quadruples each of which yields a Fano subplane together with $A0\,A1\,A5$. The affine lines on each of the remaining points at infinity are either tangent (four of them) or 3–secant (four lines) or 2–secant. The 3–secants come in pairs which form two Fano subplanes by adding their point at infinity and $A0, A1$.

The 2–failed Baer 4 has the same structure. Its points are $A0\,A1\,A12\,Cj\,Pj\,Rj\,Sj$, $j = 5, 7, 15, 16$. The affine lines on $A9$ are either 3–secants or tangents. The points at infinity on all tangents are $A4, A10, A11$. For the remaining points at infinity the situation is as in 1 (mutatis mutandis).

The 2–failed Baer 2 consists of the points $A0\,A1\,A10\,Lj\,Pj\,Sj\,Zj$, $j = 6, 7, 13, 16$, and the blocking set of size 25 is obtained by adding $A2$, $A3$, $A8$, $A11$, $A12$, $A15$. In this case, all finite lines on $A4$, $A5$, $A6$, $A7$, $A9$, $A13$, $A14$, and $A16$ are tangent. The affine lines on $A0$, $A1$ and $A10$ are either 5–secant or tangent. The finite lines on each of the remaining points at infinity are either 3–secant or tangent (eight of each). Moreover, the 16 affine points of such 2–failed Baer split into four quadruples each of which completes to a Fano subplane by adding $A0, A1$ and the point at infinity of the two 3–secants on which the four points lie.

The same situation occurs for the 2–failed Baer 3 whose points are $A0\,A1\,A13\,Hj, Pj, Sj, Xj$, $j = 5, 7, 15, 16$, and which yields a blocking set of size 25 by adding $A2$, $A3$, $A6$, $A12$, $A15$, $A16$. In this case all affine lines on $A4$, $A5$, $A7$, $A8$, $A9$, $A10$, $A11$, $A14$ are tangent to the blocking set. For the remaining lines the behaviours are as above (mutatis mutandis).

Finally, we show a third type of 2–failed Baer (which was not obtained by starting with a Fano subplane). The points of the 2–failed Baer are $A0\,A1\,A9\,Dj\,Hj\,Rj\,Zj$, $j = 8, 9, 11, 14$.

By adding $A5$, $A6$, $A7$, $A8$, $A13$, $A14$, $A15$, $A16$ we obtain a blocking set of size 27 and intersection numbers (1,2,3,4,5,11). This 2–failed Baer is different from the previous ones as four of the lines on $A8$ and $A15$ are 3–secant to the 2–failed Baer so they yield the 4–secants to the blocking set. The remaining lines behave as in the other blocking sets. Obviously, the unique 11–secant is $a0$.

4 Baer subplanes

First of all, we observe that the existence of Baer subplanes is guaranteed by the fact that π is a derived plane. The distinguished elements of π play a major role in the distribution of its Baers. Such distinguished elements are the following ones: the line at infinity $a0$, the distinguished point $A0$ on it, the derivation set $A0\,A1\,A2\,A3\,A4$, and the dual derivation set $a0\,a1\,a2\,a3\,a4$. This means there are four special lines, other than $a0$, on $A0$.

Obviously, a Baer either has five points on the line at infinity or is tangent to it and both situations do occur. As to the Baers with five points on $a0$, one of these is always $A0$. For the remaining points there are different situations. One is the following. The points on $a0 \setminus A0$ break into four quadruples, namely $A1\,A2\,A3\,A4, A5\,A6\,A7\,A8, A9\,A10\,A11\,A12, A13\,A14\,A15\,A16$, one of which is in the derivation set, and each of such quadruples together with $A0$ forms the set of points at infinity of some Baers. In particular, those Baers whose points at infinity other than $A0$ lie in a quadruple not in the derivation set have points on the lines of the dual derivation set. These Baers will be considered again together with the Baers tangent to $a0$.

To examine the other sets of points on $a0$ which are points at infinity of Baers having $a0$ as a line we have to consider the dual derivation set which is defined as follows. Firstly, one can view a usual derivation set as a set Δ of $q+1$ points at infinity (in a plane of order q^2) such that for any two points whose line hits Δ there is exactly one Baer on Δ containing those points. Then a dual derivation set Δ^* is a set of $q+1$ lines on a distinguished point, $A0$ say, such that for any two lines whose intersection point lies on a line of Δ^* there is a unique Baer having as lines those of Δ^* and the given lines.

π has such a dual derivation set and the related Baers all have five points on $a0$ and there are Baers two of whose points at infinity are in the derivation set and one of these is always $A0$. The other 5-tuples are those listed before and there are four Baers on each such 5-tuple. Furthermore, these Baers are either self-conjugate under σ or come in pairs of conjugate Baers under σ. There are four Baers which belong to both the derivation set and the dual derivation set and each of them is self conjugate under σ (cf. Sect. 2).

The Baers tangent to the line at infinity are all tangent to $a0$ at $A0$. Moreover, each of such Baers contains four points, other than $A0$, on one line of the dual derivation set. Each of these four affine lines contains four quadruples of points and there is a translation with centre $A0$ which maps one quadruple onto any other one on the same line. By adding $A0$ to the points of such a quadruple one gets five points which are the shared points of 16 Baers whose points partition the points of π minus the line on which these five points lie. These 16 Baers split into four sets of four parallel Baers each (i.e. sharing those five points only) which use the same lines. Only one of these four Baers has five points on $a0$ and is one of the Baers we mentioned before. The remaining Baers are all tangent to $a0$ at $A0$. Thus, when we consider all 16 Baers sharing the same five points on aj, $j = 1, 2, 3, 4$, we have four Baers with five points on $a0$ (and all four sets of the previously mentioned partition are used) and twelve Baers tangent to $a0$ at $A0$. The latter come in pairs of conjugate Baers under σ whereas the Baers with five points on $a0$ are self-conjugate under σ. Observe that the considered sets of points (on the lines of the dual derivation set) are not derivation sets. On each of them there are exactly sixteen Baers. The sets belonging to different lines on $A0$, and the Baers hanging on them, can be mapped, in pairs, one onto another one, by a translation with centre $A1$. However, not all translations with such a centre exist. Also, there is an involution which maps onto itself the common set of five points on aj and pairs off the Baers on those five points.

The distribution of the Baers of π we just described depends on the existence of a derivation set and a dual derivation set. However, some special involutions of π are also involved in such distribution. We mentioned and used the involution σ which acts by keeping letters fixed and pairing off subscripts of affine points and lines of π as $1\,2, 3\,4, 5\,10, 6\,16, 7\,13, 8\,9, 11\,14, 12\,15$. There is another involution, ω, which acts by keeping fixed the subscripts of affine points and lines and pairing off letters as $B\,C, D\,F, H\,P, K\,Z, L\,T, M\,N, R\,W, S\,X$.

Both σ and ω fix $a0$ pointwise. Moreover, σ fixes all lines on $A0$ $(aj\sigma = aj)$ and induces an involution on each such line. Similarly, ω fixes all lines on $A1$ $(bj\omega = bj)$ and induces an involution on each of such lines. σ and ω commute, so $\sigma\omega$ is an involution.

There is exactly one pencil of lines on which σ and ω act in the same way, namely the pencil with centre $A2 : cj\sigma = cj\omega$ which implies $cj\sigma\omega = cj$. This means that $\sigma\omega$ fixes each line on $A2$ and induces an involution on it.

Therefore, the derivation set contains a special triple, $A0\,A1\,A2$, to which the distinguished point $A0$ belongs, and a special pair, $A3\,A4$, which consists of the points at infinity of all hyperovals in π (cf. sect. 5). Observe that $< \sigma, \omega > \cong V_4$ acts as a Baer four–group on the Baers belonging to both the derivation set and the dual derivation set.

Next, we provide some examples of Baers and their distribution. (Some examples of Baers were already given in sect. 3.)

First of all, the four Baers which belong to both the derivation set and the dual derivation set are the following ones:

$A0\,Aj\,Bj\,Cj\,Dj\,Fj$, $a0\,aj\,bj\,cj\,dj\,fj$, $j = 1,2,3,4$; $A0\,Aj\,Bi\,Ci\,Di\,Fi$, $a0\,aj\,bi\,ci\,di\,fi$, $i = 5,10,7,13$;

$A0\,Aj\,Bi\,Ci\,Di\,Fi$, $a0\,aj\,bi\,ci\,di\,fi$, $i = 6,16,12,15$; $A0\,Aj\,Bi\,Ci\,Di\,Fi$, $a0\,aj\,bi\,ci\,di\,fi$, $i = 8,9,11,14$.

Each of such Baers is self-conjugate both under σ and under ω.

Next we illustrate the mentioned distribution of the Baers by taking the line $a3$ and writing down the sixteen Baers on $A0\,D5\,D7\,D10\,D13$. (We just list the points of such Baers; both the lines and the incidences are easily obtained from Table 1.)

$A0\,A1\,A2\,A3\,A4\,Bj\,Cj\,Dj\,Fj$, $j = 5,10,7,13$;

$A0\,A5\,A6\,A7\,A8\,B8\,B9\,B11\,B14\,C6\,C12\,C16\,C15\,D5\,D10\,D7\,D13\,F1\,F2\,F3\,F4$;

$A0\,A9\,A10\,A11\,A12\,B6\,B16\,B12\,B15\,C1\,C2\,C3\,C4\,D5\,D10\,D7\,D13\,F8\,F9\,F11\,F14$;

$A0\,A13\,A14\,A15\,A16\,B1\,B2\,B3\,B4\,C8\,C9\,C11\,C14\,D5\,D10\,D7\,D13\,F6\,F16\,F12\,F15$.

Notice that each of these four Baers is self-conjugate under σ. The remaining twelve Baers come in pairs of conjugate Baers under σ so that we just list half of them. Moreover, they are all tangent to $a0$ at $A0$.

$A0\,D5\,D10\,D7\,D13\,H2\,H7\,H11\,H12\,P1\,P6\,P7\,P8\,L3\,L7\,L14\,L16\,T4\,T7\,T9\,T15$

$A0\,D5\,D10\,D7\,D13\,M1\,M5\,M11\,M16\,N3\,N5\,N6\,N9\,R4\,R5\,R12\,R14\,W2\,W5\,W8\,W15$

$A0\,D5\,D10\,D7\,D13\,H4\,H8\,H10\,H16\,P3\,P10\,P11\,P15\,L1\,L9\,L10\,L12\,T2\,T6\,T10\,T14$

$A0\,D5\,D10\,D7\,D13\,M4\,M7\,M9\,M15\,N2\,N7\,N11\,N12\,R1\,R6\,R7\,R8\,W3\,W7\,W14\,W16$

$A0\,D5\,D10\,D7\,D13\,K3\,K5\,K6\,K9\,Z2\,Z5\,Z8\,Z15\,S1\,S5\,S11\,S16\,X4\,X5\,X12\,X14$

$A0\,D5\,D10\,D7\,D13\,K2\,K7\,K11\,K12\,Z3\,Z7\,Z14\,Z16\,S4\,S7\,S9\,S15\,X1\,X6\,X7\,X8$.

Observe that there are four Baers on the points $A0\,D5\,D10\,D7\,D13$ and on the lines $a3\,h9\,k14\,l11\,m8$ which partition the points on such lines. One of these Baers has $a0$ as a line and is the only one which is self-conjugate under σ. (Other quadruples of lines not in the dual derivation set with the same property are easily found by looking at the given examples.)

The translation with centre $A0$ which maps $D5$ onto $D8$, maps $D10$ onto $D9$, $D7$ onto $D14$ and $D13$ onto $D11$. Therefore, the above given partition by Baers is shifted along $a3$. Similarly, there are two other translations, again with centre $A0$, which act as $(D5, D10, D7, D13) \to (D1, D2, D4, D3)$, and $(D5, D10, D7, D13) \to (D6, D16, D12, D15)$. By applying such translations we obtain the Baers associated with the partition of $a3$.

Obviously, applying ω to the above given Baers (and to those obtained by the mentioned translations) yields the partition of $a4$ and the related Baers.

Finally, we can also use the translations with centre $A1$ (recall that π admits some translations with centre $A1$). E.g. there is a translation with centre $A1$ mapping $D5\,D7\,D10\,D13$ onto $B5\,B7\,B10\,B13$ (in the given order) which maps the Baer $A0\,D5\,D10\,D7\,D13\,H4\,H8\,H10\,H16\,P3\,P10\,P11\,P15$ $L1\,L9\,L10\,L12\,T2\,T6\,T10\,T14$ onto the Baer $A0\,B5\,B10\,B7\,B13\,H2\,H6\,H10$ $H14\,P1\,P9\,P10\,P12\,L3\,L10\,L11\,L15\,T4\,T8\,T10\,T16$.

Of course, the Baers in the partition above do not exhaust the Baers of π, and it seems worthwhile to give also some examples of Baers with two points in the derivation set. Such Baers use the lines in the dual derivation set:

$A0\,A1\,A5\,A11\,A16\,Bj\,Cj\,Dj\,Fj,\ j = 1, 6, 7, 8;$
$A0\,A1\,A5\,A11\,A16\,Bj\,Cj\,Dj\,Fj,\ j = 3, 10, 11, 15;$
$A0\,A1\,A5\,A11\,A16\,Bj\,Cj\,Dj\,Fj,\ j = 4, 5, 12, 14;$
$A0\,A1\,A5\,A11\,A16\,Bj\,Cj\,Dj\,Fj,\ j = 2, 9, 13, 16.$

Observe that these four Baers are parallel (i.e. share their points at infinity only), self-conjugate under ω and come in pairs of conjugate Baers under σ.

Another quadruple of Baers with two points in the derivation set is the following one:

$A0\,A4\,A5\,A9\,A14\,B1\,B9\,B10\,B12\,C3\,C7\,C14\,C16\,D4\,D6\,D11\,D13$
$F2\,F5\,F8\,F15$
$A0\,A4\,A5\,A9\,A14\,B3\,B7\,B14\,B16\,C1\,C9\,C10\,C12\,D2\,D5\,D8\,D15$
$F4\,F6\,F11\,F13$

and their conjugate Baers under σ. (Notice that the above written Baers are conjugate under ω.)

Obviously, if a Baer with $a0$ as a line is not self-conjugate under σ (ω or $\sigma\omega$), then its images under $< \sigma, \omega >$ provide a quadruple of Baers which share their points at infinity only (and, of course, as many lines).

Finally, it is clear that the existence of 2–failed Baers in π (cf. sect. 3) is a consequence of the distribution of its Baers.

5 Hyperovals

The hyperovals in π share a property with the hyperovals in the strict semitranslation plane constructed by N.L. Johnson [3]. More precisely, all hyperovals in π (as well as those in the other semitranslation plane) have the same points at infinity and such points belong to the derivation set. As a matter of fact, they are $A3$ and $A4$. Moreover, each hyperoval of π splits into four hyperovals, belonging to four distinct Baers on the derivation set, which share the points at infinity.

However, only some quadruples of Baers on the derivation set contain hyperovals which can be glued together to yield hyperovals in π. None of the involved Baers has lines in the dual derivation set and another quadruple of lines is forbidden. In other words, only eight lines on $A0$, aj for $j = 5, 10; 6, 16; 7, 13; 12, 15$, contain the Baers involved in the hyperovals. Also only specific quadruples of lines on $A1$ belong to these Baers.

Observe that all hyperovals in π are mapped onto themselves by σ and ω. We remark that the translation planes of order 16 [1] all contain quadruples of hyperovals which share either eight finite points or the two points at infinity [5, 6]. No such configuration exists in π. Next, we list some examples of hyperovals.

The following four hyperovals use points on the same lines on $A0$ and $A1$:

$H5$	$H10$	$P5$	$P10$	$L6$	$L16$	$T6$	$T16$	$K12$	$K15$	$Z12$	$Z15$
$S7$	$S13$	$X7$	$X13$	$A3$	$A4$						
$H16$	$H6$	$P6$	$P16$	$L5$	$L10$	$T5$	$T10$	$K7$	$K13$	$Z7$	$Z13$
$S12$	$S15$	$X12$	$X15$	$A3$	$A4$						
$H7$	$H13$	$P7$	$P13$	$L12$	$L15$	$T12$	$T15$	$K6$	$K16$	$Z6$	$Z16$
$S5$	$S10$	$X5$	$X10$	$A3$	$A4$						
$H12$	$H15$	$P12$	$P15$	$L7$	$L13$	$T7$	$T13$	$K5$	$K10$	$Z5$	$Z10$
$S6$	$S16$	$X6$	$X16$	$A3$	$A4$						

Each of these hyperovals splits into four hyperovals on $A3\,A4$ belonging to four distinct Baers. For the first hyperoval the four involved Baers which all share the points $A0\,A1\,A2\,A3\,A4$ have the following affine points and lines:

$\alpha 1$: $Hj\,Lj\,Pj\,Tj$; aj, bj, $j = 5, 10, 7, 13$, $c1\,c2\,c3\,c4$, $d6\,d16\,d12\,d15$, $f8\,f9\,f11\,f14$.

$\alpha 2$: $Kj\,Zj\,Sj\,Xj$; $a6\,a16\,a12\,a15$, bj, $c8\,c9\,c11\,c14$, $d1\,d2\,d3\,d4$, $f6\,f16\,f12\,f15$, $j = 5, 10, 7, 13$.

$\alpha 3$: $Hj\,Lj\,Pj\,Tj$; $a5\,a10\,a7\,a13$, bj, $c8\,c9\,c11\,c14$, $d5\,d10\,d7\,d13$ $f1\,f2\,f3\,f4$, $j = 12, 15, 6, 16$.

$\alpha 4$: $Kj\,Zj\,Sj\,Xj$, $a6\,a16\,a12\,a15$, bj, $c1\,c2\,c3\,c4\,d8\,d9\,d11\,d14$ $f5\,f10\,f7\,f13$, $j = 6, 16, 12, 15$.

The four hyperovals in these Baers are:
$H5\,H10\,P5\,P10\,A3\,A4$, $K7\,K13\,Z7\,Z13\,A3\,A4$, $L12\,L15\,T12\,T15\,A3\,A4$, $S6\,S16\,X6\,X16\,A3\,A4$, respectively.

A similar partition holds for the other three hyperovals in π of the quadruple above.

Another quadruple of hyperovals is the following one and the related Baers can be easily found with the help of Table 1, as well as the hyperovals in such Baers.

$H11$	$H14$	$P11$	$P14$	$K8$	$K9$	$Z8$	$Z9$	$L3$	$L4$	$T3$	$T4$
$S1$	$S2$	$X1$	$X2$	$A3$	$A4$						
$H3$	$H4$	$P3$	$P4$	$L11$	$L14$	$T11$	$T14$	$K1$	$K2$	$Z1$	$Z2$
$S8$	$S9$	$X8$	$X9$	$A3$	$A4$						
$H1$	$H2$	$P1$	$P2$	$L8$	$L9$	$T8$	$T9$	$K3$	$K4$	$Z3$	$Z4$
$S11$	$S14$	$X11$	$X14$	$A3$	$A4$						
$H8$	$H9$	$P8$	$P9$	$L1$	$L2$	$T1$	$T2$	$K11$	$K14$	$Z11$	$Z14$
$S3$	$S4$	$X3$	$X4$	$A3$	$A4$						

6 Complete 14-arcs

All non-desarguesian planes of order 16 contain complete 14-arcs, whereas only the Hall plane [6] and the Johnson–Walker plane contain complete 16-arcs [5]. In π, as well as in the exceptional semitranslation plane of order 16 [3], all complete 14-arcs have two points on the line at infinity none of which in the derivation set.

There are some interesting configurations formed by complete 14-arcs which we now briefly describe. (By a 14–arc we always mean a complete one.)

There are quadruples of 14-arcs on the same finite 8-arc. These quadruples split into two pairs and the arcs in the same pair share also the two points at infinity. There are triples of 14–arcs on the same finite 8-arc and the three pairs of points at infinity of the arcs of the same triple are all distinct. However, there are also triples of 14–arcs on the same finite 8–arc with only five distinct points at infinity. We observe that the quadruples of 14–arcs we just mentioned behave as the quadruples of hyperovals which exist in all translation planes of order 16. Also, the pairs of points at infinity of the 14–arcs always come from those 4–sets which together with $A0$ form the points at infinity of Baers in the dual derivation set and a quadruple of 14–arcs uses four points all belonging to one of such sets. In order to classify the 14–arcs, one can look at the distribution of j–points, a j–point being a point off the arc on exactly j tangents. In π the same cases occur as in the strict semi-translation plane constructed by Johnson [4].
Next, some examples of 14–arcs are listed.

B1	B2	C1	C2	M3	M4	N3	N4	H11	H14	P11	P14	A5	A6
B1	B2	C1	C2	M3	M4	N3	N4	K8	K9	Z8	Z9	A5	A6
B1	B2	C1	C2	M3	M4	N3	N4	L11	L14	T11	T14	A7	A8
B1	B2	C1	C2	M3	M4	N3	N4	S8	S9	X8	X9	A7	A8

D7	D13	F7	F13	R12	R15	W12	W15	H8	H9	P8	P9	A9	A11
D7	D13	F7	F13	R12	R15	W12	W15	K1	K2	Z1	Z2	A9	A11
D7	D13	F7	F13	R12	R15	W12	W15	L11	L14	T11	T14	A10	A12
D7	D13	F7	F13	R12	R15	W12	W15	S3	S4	X3	X4	A10	A12

M6	M16	N6	N16	K5	K10	Z5	Z10	D12	D15	F12	F15	A9	A11
M6	M16	N6	N16	K5	K10	Z5	Z10	L3	L4	T3	T4	A13	A15
M6	M16	N6	N16	K5	K10	Z5	Z10	R11	R14	W11	W14	A5	A7

L6	L16	T6	T16	R5	R10	W5	W10	D7	D13	F7	F13	A10	A12
L6	L16	T6	T16	R5	R10	W5	W10	M11	M14	N11	N14	A9	A12
L6	L16	T6	T16	R5	R10	W5	W10	S1	S2	X1	X2	A6	A8

M8	M9	N8	N9	R6	R16	W6	W16	K5	K10	Z5	Z10	A5	A7
M8	M9	N8	N9	R6	R16	W6	W16	K7	K13	Z7	Z13	A6	A8
M8	M9	N8	N9	R6	R16	W6	W16	S1	S2	X1	X2	A5	A7
M8	M9	N8	N9	R6	R16	W6	W16	S3	S4	X3	X4	A6	A8

M8	M9	N8	N9	R5	R10	W5	W10	H6	H16	P6	P16	A9	A12
M8	M9	N8	N9	R5	R10	W5	W10	H12	H15	P12	P15	A10	A11
M8	M9	N8	N9	R5	R10	W5	W10	L1	L2	T1	T2	A9	A12
M8	M9	N8	N9	R5	R10	W5	W10	L3	L4	T3	T4	A10	A11

M8	M9	N8	N9	R7	R13	W7	W13	H1	H2	P1	P2	A9	A12
M8	M9	N8	N9	R7	R13	W7	W13	H3	H4	P3	P4	A10	A11
M8	M9	N8	N9	R7	R13	W7	W13	L6	L16	T6	T16	A10	A11
M8	M9	N8	N9	R7	R13	W7	W13	L12	L15	T12	T15	A9	A12

M8	M9	N8	N9	R12	R15	W12	W15	K1	K2	Z1	Z2	A5	A7
M8	M9	N8	N9	R12	R15	W12	W15	K3	K4	Z3	Z4	A6	A8
M8	M9	N8	N9	R12	R15	W12	W15	S5	S10	X5	X10	A6	A8
M8	M9	N8	N9	R12	R15	W12	W15	S7	S13	X7	X13	A5	A7

B8	B9	C8	C9	M11	M14	N11	N14	H3	H4	P3	P4	A5	A6
B8	B9	C8	C9	M11	M14	N11	N14	L3	L4	T3	T4	A7	A8
B8	B9	C8	C9	M11	M14	N11	N14	K1	K2	Z1	Z2	A5	A6
B8	B9	C8	C9	M11	M14	N11	N14	S1	S2	X1	X2	A7	A8

H3	H4	P3	P4	M11	M14	N11	N14	B8	B9	C8	C9	A5	A6
H3	H4	P3	P4	M11	M14	N11	N14	K6	K16	Z6	Z16	A15	A16
H3	H4	P3	P4	M11	M14	N11	N14	R5	R10	W5	W10	A9	A12

The last quadruple and the last triple show that a 14–arc can belong to both a quadruple and a triple.

By comparing the listed 14–arcs with the hyperovals in Sect. 5 it is clear that there are forbidden points for the hyperovals. Also the 14–arcs split into suitable hyperovals in three Baers.

Finally, we show the j–points and their distribution for the four 14–arcs of the first listed quadruple, which we denote by γ_j, $j = 1, 2, 3, 4$, in the above given order.

γ_1: 0–points: $A0\, A1\, A2\, B7\, B13\, C7\, C13\, D5\, D10\, F5\, F10\, K1\, K2\, K3\, K4$ $Z1\, Z2\, Z3\, Z4$
(these 19 points split into four Fano subplanes on $A0\, A1\, A2$);

6–points: $B5\, B10\, C5\, C10\, D7\, D13\, F7\, F13\, S1\, S2\, S3\, S4\, X1\, X2\, X3\, X4$ $S8\, S9\, X8\, X9$;

10–points: $K8\, K9\, Z8\, Z9$.

The remaining points are either 2– or 4–points.

γ_2: 0–points: $A0\, A1\, A2\, H1\, H2\, H3\, H4\, M7\, M13\, N7\, N13\, P1\, P2\, P3\, P4$ $R5\, R10\, W5\, W10$;

6–points: $L1\, L2\, L3\, L4\, L11\, L14\, M5\, M10\, N5\, N10\, R7\, R13$ $T1\, T2\, T3\, T4\, T11\, T14\, W7\, W13$;

10–points: $H11\, H14\, P11\, P14$.

Again, we do not list 2– and 4– points.

The same situation (involving different points) occurs for γ_3 and γ_4.
We observe that a 14–arc admits either 10–points or 8–points, never both (cf. [3]).
For completeness, we also mention a complete 13–arc and observe that all 13–arcs in π seem to be tangent to $a0$:
$M8\, M9\, N8\, N9\, K12\, K15\, Z12\, Z15\, H1\, H2\, P1\, P2\, A15$.

Appendix

Table 1

*c*1 :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
*c*3 :	3	4	1	2	13	15	10	11	14	7	8	16	5	9	6	12
*c*5 :	5	10	13	7	1	14	4	12	15	2	16	8	3	6	9	11
*c*6 :	6	16	15	12	14	1	8	7	13	11	10	4	9	5	3	2
*c*7 :	7	13	10	5	4	8	1	6	16	3	15	14	2	12	11	9
*c*8 :	8	9	11	14	12	7	6	1	2	15	3	5	16	4	10	13
*c*11 :	11	14	8	9	16	10	15	3	4	6	1	13	12	2	7	5
*c*12 :	12	15	16	6	8	4	14	5	10	9	13	1	11	7	2	3
*d*1 :	1	4	2	3	11	7	8	6	12	9	16	10	14	15	13	5
*d*3 :	3	2	4	1	8	10	11	15	16	14	12	7	9	6	5	13
*d*5 :	5	7	10	13	16	4	12	14	8	15	11	2	6	9	3	1
*d*6 :	6	12	16	15	10	8	7	1	4	13	2	11	5	3	9	14
*d*7 :	7	5	13	10	15	1	6	8	14	16	9	3	12	11	2	4
*d*8 :	8	14	9	11	3	6	1	7	5	2	13	15	4	10	16	12
*d*11 :	11	9	14	8	1	15	3	10	13	4	5	6	2	7	12	16
*d*12 :	12	6	15	16	13	14	5	4	1	10	3	9	7	2	11	8
*f*1 :	1	3	4	2	16	8	6	7	10	12	5	9	15	13	14	11
*f*3 :	3	1	2	4	12	11	15	10	7	16	13	14	6	5	9	8
*f*5 :	5	13	7	10	11	12	14	4	2	8	1	15	9	3	6	16
*f*6 :	6	15	12	16	2	7	1	8	11	4	14	13	3	9	5	10
*f*7 :	7	10	5	13	9	6	8	1	3	14	4	16	11	2	12	15
*f*8 :	8	11	14	9	13	1	7	6	15	5	12	2	10	16	4	3
*f*11 :	11	8	9	14	5	3	10	15	6	13	16	4	7	12	2	1
*f*12 :	12	16	6	15	3	5	4	14	9	1	8	10	2	11	7	13
*h*1 :	1	7	6	8	4	11	14	9	15	13	10	3	16	2	5	12
*h*3 :	3	10	15	11	2	8	9	14	6	5	7	1	12	4	13	16
*h*5 :	5	4	14	12	10	6	15	16	11	1	3	7	8	13	2	9
*h*6 :	6	8	1	7	16	5	13	10	2	14	9	12	4	15	11	3
*h*7 :	7	1	8	6	13	12	16	15	9	4	2	5	14	10	3	11
*h*8 :	8	6	7	1	14	3	4	2	10	16	15	11	13	9	12	5
*h*11 :	11	15	10	3	9	1	2	4	7	12	6	8	5	14	16	13
*h*12 :	12	14	4	5	15	7	10	13	3	8	11	6	1	16	9	2

$k1$:	1	13	15	14	7	10	12	2	5	4	6	16	8	11	9	3
$k3$:	3	5	6	9	10	7	16	4	13	2	15	12	11	8	14	1
$k5$:	5	3	9	6	2	1	11	13	4	10	8	14	16	15	12	7
$k6$:	6	9	3	5	11	14	2	15	8	16	4	1	10	13	7	12
$k7$:	7	2	11	12	3	4	9	10	1	13	14	8	15	16	6	5
$k8$:	8	16	10	4	6	15	5	9	12	14	7	13	1	3	2	11
$k11$:	11	12	7	2	15	6	13	14	16	9	10	5	3	1	4	8
$k12$:	12	11	2	7	9	8	3	16	14	15	1	4	13	10	5	6

$l1$:	1	10	12	9	15	3	5	16	8	11	2	14	4	7	6	13
$l3$:	3	7	16	14	6	1	13	12	11	8	4	9	2	10	15	5
$l5$:	5	2	8	15	14	7	3	9	6	12	13	16	10	1	11	4
$l6$:	6	11	4	13	1	12	9	3	5	7	15	10	16	14	2	8
$l7$:	7	3	14	16	8	5	2	11	12	6	10	15	13	4	9	1
$l8$:	8	15	5	2	10	11	12	13	1	3	9	4	14	6	7	16
$l11$:	11	6	13	4	7	8	16	5	3	1	14	2	9	15	10	12
$l12$:	12	9	1	10	4	6	11	2	7	5	16	13	15	8	3	14

$m1$:	1	5	16	11	9	15	4	13	2	6	14	7	10	12	3	8
$m3$:	3	13	12	8	14	6	2	5	4	15	9	10	7	16	1	11
$m5$:	5	1	11	16	6	8	10	2	13	9	12	3	4	14	7	15
$m6$:	6	14	2	10	5	4	16	11	15	3	7	9	8	1	12	13
$m7$:	7	4	9	15	12	14	13	3	10	11	6	2	1	8	5	16
$m8$:	8	12	13	3	2	10	14	16	9	7	4	6	15	5	11	1
$m11$:	11	16	5	1	4	7	9	12	14	10	2	15	6	13	8	3
$m12$:	12	8	3	13	7	1	15	9	16	2	5	11	14	4	6	10

$n1$:	1	16	11	5	8	13	15	4	6	7	9	2	3	10	12	14
$n3$:	3	12	8	13	11	5	6	2	15	10	14	4	1	7	16	9
$n5$:	5	11	16	1	15	2	8	10	9	3	6	13	7	4	14	12
$n6$:	6	2	10	14	13	11	4	16	3	9	5	15	12	8	1	7
$n7$:	7	9	15	4	16	3	14	13	11	2	12	10	5	1	8	6
$n8$:	8	13	3	12	1	16	10	14	7	6	2	9	11	15	5	4
$n11$:	11	5	1	16	3	12	7	9	10	15	4	14	8	6	13	2
$n12$:	12	3	13	8	10	9	1	15	2	11	7	16	6	14	4	5

*p*1 :	1	12	9	10	13	16	3	5	11	14	15	8	6	4	7	2
*p*3 :	3	16	14	7	5	12	1	13	8	9	6	11	15	2	10	4
*p*5 :	5	8	15	2	4	9	7	3	12	16	14	6	11	10	1	13
*p*6 :	6	4	13	11	8	3	12	9	7	10	1	5	2	16	14	15
*p*7 :	7	14	16	3	1	11	5	2	6	15	8	12	9	13	4	10
*p*8 :	8	5	2	15	16	13	11	12	3	4	10	1	7	14	6	9
*p*11 :	11	13	4	6	12	5	8	16	1	2	7	3	10	9	15	14
*p*12 :	12	1	10	9	14	2	6	11	5	13	4	7	3	15	8	16
*r*1 :	1	6	8	7	12	9	11	14	13	3	4	15	5	16	2	10
*r*3 :	3	15	11	10	16	14	8	9	5	1	2	6	13	12	4	7
*r*5 :	5	14	12	4	9	16	6	15	1	7	10	11	2	8	13	3
*r*6 :	6	1	7	8	3	10	5	13	14	12	16	2	11	4	15	9
*r*7 :	7	8	6	1	11	15	12	16	4	5	13	9	3	14	10	2
*r*8 :	8	7	1	6	5	2	3	4	16	11	14	10	12	13	9	15
*r*11 :	11	10	3	15	13	4	1	2	12	8	9	7	16	5	14	6
*r*12 :	12	4	5	14	2	13	7	10	8	6	15	3	9	1	16	11
*s*1 :	1	15	14	13	3	2	10	12	4	16	7	5	9	8	11	6
*s*3 :	3	6	9	5	1	4	7	16	2	12	10	13	14	11	8	15
*s*5 :	5	9	6	3	7	13	1	11	10	14	2	4	12	16	15	8
*s*6 :	6	3	5	9	12	15	14	2	16	1	11	8	7	10	13	4
*s*7 :	7	11	12	2	5	10	4	9	13	8	3	1	6	15	16	14
*s*8 :	8	10	4	16	11	9	15	5	14	13	6	12	2	1	3	7
*s*11 :	11	7	2	12	8	14	6	13	9	5	15	16	4	3	1	10
*s*12 :	12	2	7	11	6	16	8	3	15	4	9	14	5	13	10	1
*t*1 :	1	9	10	12	2	5	16	3	14	8	13	11	7	6	4	15
*t*3 :	3	14	7	16	4	13	12	1	9	11	5	8	10	15	2	6
*t*5 :	5	15	2	8	13	3	9	7	16	6	4	12	1	11	10	14
*t*6 :	6	13	11	4	15	9	3	12	10	5	8	7	14	2	16	1
*t*7 :	7	16	3	14	10	2	11	5	15	12	1	6	4	9	13	8
*t*8 :	8	2	15	5	9	12	13	11	4	1	16	3	6	7	14	10
*t*11 :	11	4	6	13	14	16	5	8	2	3	12	1	15	10	9	7
*t*12 :	12	10	9	1	16	11	2	6	13	7	14	5	8	3	15	4

$w1$:	1	14	13	15	6	12	2	10	16	5	3	4	11	9	8	7
$w3$:	3	9	5	6	15	16	4	7	12	13	1	2	8	14	11	10
$w5$:	5	6	3	9	8	11	13	1	14	4	7	10	15	12	16	2
$w6$:	6	5	9	3	4	2	15	14	1	8	12	16	13	7	10	11
$w7$:	7	12	2	11	14	9	10	4	8	1	5	13	16	6	15	3
$w8$:	8	4	16	10	7	5	9	15	13	12	11	14	3	2	1	6
$w11$:	11	2	12	7	10	13	14	6	5	16	8	9	1	4	3	15
$w12$:	12	7	11	2	1	3	16	8	4	14	6	15	10	5	13	9
$x1$:	1	11	5	16	14	4	13	15	7	2	8	6	12	3	10	9
$x3$:	3	8	13	12	9	2	5	6	10	4	11	15	16	1	7	14
$x5$:	5	16	1	11	12	10	2	8	3	13	15	9	14	7	4	6
$x6$:	6	10	14	2	7	16	11	4	9	15	13	3	1	12	8	5
$x7$:	7	15	4	9	6	13	3	14	2	10	16	11	8	5	1	12
$x8$:	8	3	12	13	4	14	16	10	6	9	1	7	5	11	15	2
$x11$:	11	1	16	5	2	9	12	7	15	14	3	10	13	8	6	4
$x12$:	12	13	8	3	5	15	9	1	11	16	10	2	4	6	14	7
$z1$:	1	8	7	6	10	14	9	11	3	15	12	13	2	5	16	4
$z3$:	3	11	10	15	7	9	14	8	1	6	16	5	4	13	12	2
$z5$:	5	12	4	14	3	15	16	6	7	11	9	1	13	2	8	10
$z6$:	6	7	8	1	9	13	10	5	12	2	3	14	15	11	4	16
$z7$:	7	6	1	8	2	16	15	12	5	9	11	4	10	3	14	13
$z8$:	8	1	6	7	15	4	2	3	11	10	5	16	9	12	13	14
$z11$:	11	3	15	10	6	2	4	1	8	7	13	12	14	16	5	9
$z12$:	12	5	14	4	11	10	13	7	6	3	2	8	16	9	1	15

References

[1] U. Dempwolff and A. Reifart, The classification of the translation planes of order 16, I, *Geom. Ded.* **15** (1983), 137–153.

[2] M. J. de Resmini, Some combinatorial properties of a semitranslation plane, *Congressus Numerantium*, **59** (1987), 5–12.

[3] M. J. de Resmini, On an exceptional semitranslation plane, in "Advances in Finite Geometries and Designs", Oxford U.P., 1991, 141–162.

[4] M. J. de Resmini, Failed Baers and blocking sets, *Mitt. Math. Semin. Univ. Giessen*, **201** (1991), 45–48.

[5] M. J. de Resmini, On the Johnson–Walker plane, *Simon Stevin* **64** (1990), 113–139.

[6] M. J. de Resmini, Some remarks on the Hall plane of order 16, *Congr. Num.* **70** (1990), 17–27.

[7] N. L. Johnson, Non-strict semitranslation planes, *Arch. Math.* **20** (1969), 301–310.

[8] N. L. Johnson, A classification of semitranslation planes, *Canad. J. Math.* **21** (1969), 1372–1387.

[9] N. L. Johnson, On non–strict semitranslation planes of Lenz–Barlotti class I-1, *Arch. Math.* **21** (1970), 402–410.

[10] N. L. Johnson, A note on semitranslation planes of class I-5a, *Arch. Math.* **21** (1970), 528–532.

[11] T. G. Ostrom, Semi–translation planes, *Trans. Amer. Math. Soc.* **111** (1964), 1–18.